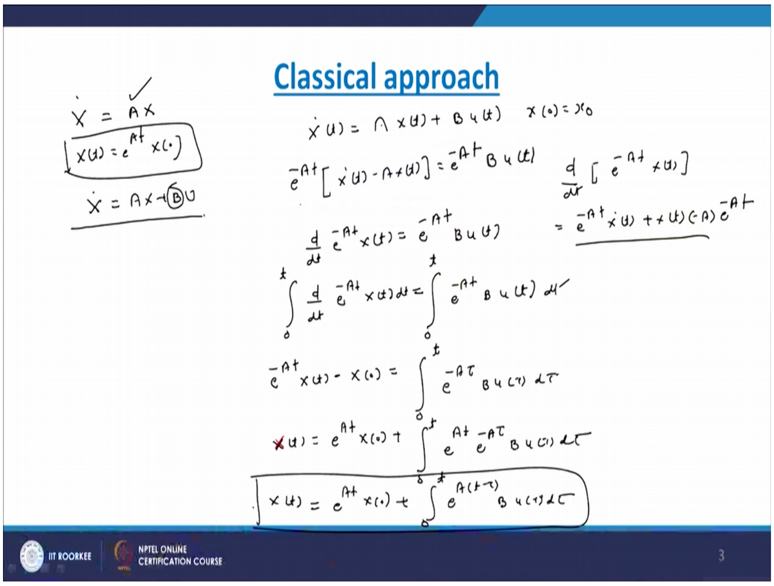
Advanced Linear Continuous Control Systems
Dr. Yogesh Vijay Hote
Department of Electrical Engineering
Indian Institute of Technology, Roorkee

Lecture – 23
Solution of State Equation (Forced System)

When I we start with the Solution of State Equation, that is for Forced System. In this we will study classical approach, then Laplace transform approach, and then we will see one example now about the classical approach. Last time we have seen the state equation for unforced system.

(Refer Slide Time: 00:50)



That is last time we have seen X dot equal to A X, is a plant and X of t equal to e raised to A t X of 0. This part we have seen last time. So, this is unforced system. Now he today we have to see the forced system there is a force is applied, therefore the plant which we are taking there is X dot equal to A X plus BU. Now for this plant we required a state equations. As I told last time the state equation plays very important role, because and the states are playing a important role in the completes part of the system how the state changes from initial time to final time and as well as I did in to infinity that information is very much important.

And in case of unforced system we have seen that the complete state equation it depends on A matrix. But now we have state space model in terms of A X plus BU definitely their state equation depends upon the B matrix. Now let us see when we derive the state equation in this case can we get the term of B or not. Now we start developing the state equation for a forced system. So, this forced system can be written as X dot t equal to A X of t plus B into u of t where X of 0 equal to x 0.

Now, how will you proceed further this. Because finally, we want X of t; so what we can do. We will multiply this equation by raised to minus A t two both side. So, multiply this equation both side by raised to minus A t. So, we will get raised to minus A t X dot t minus A X of t equal to e raised to minus A t B into u of t. If you make differentiation of e raised to minus A t into X of t. Now, we will differentiate this equation and we will see the result and then we correlate this with these equations. So, you differentiate these equations, so you will get raised to minus A t into X dot t plus X of t into minus A into minus A of t; that is what we have done raised to minus A t.

The first term constant, the differentiation of these X of t in dot t plus first term then differentiation of raised to minus A t is raised to minus A t and this constant term minus A we have multiplied by this. And they we will find that this equation and this equations both are the same thing. Therefore, next step we can write down as the differentiation of e raised to minus A t into X of t equal to e raised to minus A t in to B into u of t. Now, this equation you have done. That means, what is what we are done here: first of all we have multiply e raised to A t both sides, so here e raised to A t, here u raised to minus A t, A BU of t. Then this equation we have written like this. And finally, this differentiation has come raised to minus A t B into u of t.

Now, our aim is to determine X of t. Now here differentiation has come, so the best option is that we integrate this equation. So, if you integrate equation definitely some term of X of t you will get therefore, integrating both sides from 0 to t. So, if you integrate both sides 0 to t that is differentiation of d of dt e raised to minus A t into X of t equal to 0 to t raised to minus A t B into u of t.

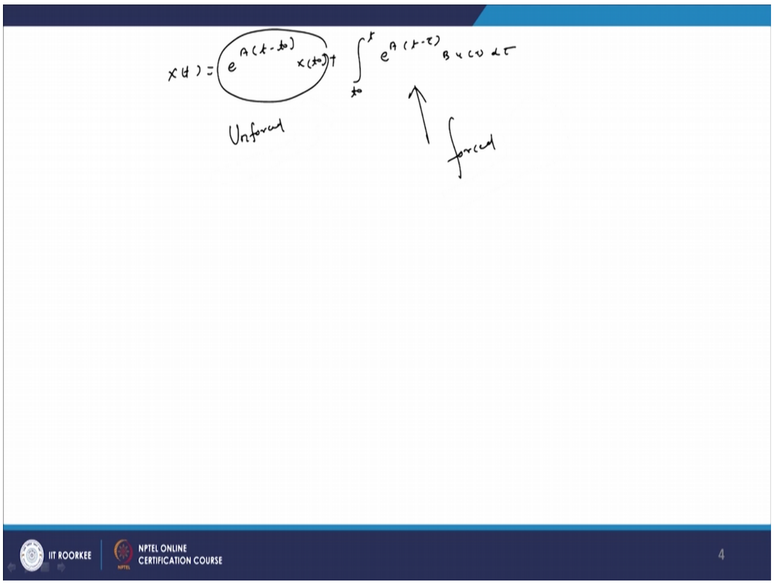
Now you will find, we get differentiation one integration, so integration and differentiation is there, but it will nullified. That is, remaining is e raised to 18 to X of t, but the limit will come that is 0 to t; that means, that this equation can be written as e raised to minus A t X of t minus X of 0 0 to t e raised to minus A tau B u tau d tau.

So, now we got e raised to A t X of t minus X of 0, 0 t e raised to A tau B tau d tau. Now, how to move further? So, what we had done; we will remove this X of 0 on the right hand side and then try to use the concept of raised to minus A t in the right hand side and see what result will come.

Now, after this equation what we can do we will multiply it by e raised to A t plus A t both side. So what happened, this full term will be nullified. And therefore we can write down the equation as: X of t equal to e raised to A t two X naught plus 0 to t e raised to A t e raised to minus A tau B of u tau d tau. And therefore, finally, this X of t equal to e raised to A t X naught plus 0 to t e raised to A t minus tau B u tau d tau.

So, here we have got these state equations that this is the state equation. The state starts from 0 up 2 time t and we have got is equations, and finally we found at that this state equation depend upon A as well as the B. But now sometimes is found at we can to start the state from t 0 not 0 actually t naught; t naught to t then how to write down this equations. Now, we write down the equations.

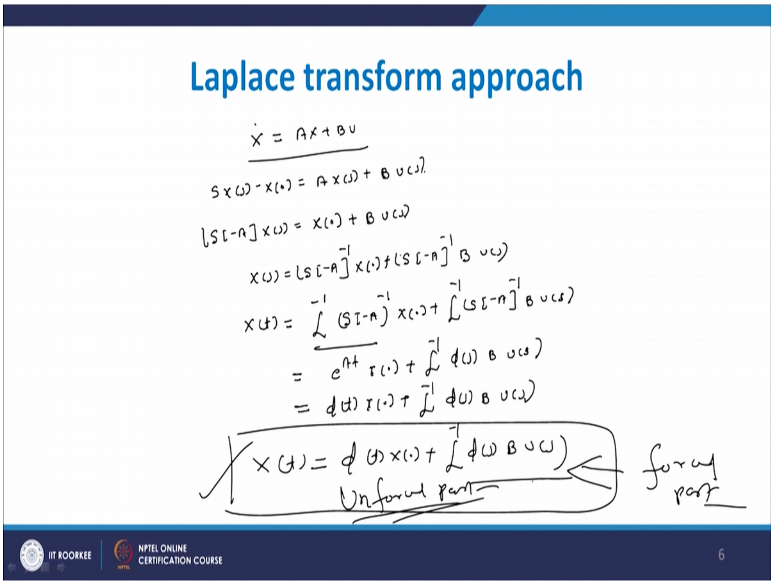
(Refer Slide Time: 07:47)



So, if the initial state start from t equal to t naught, we can write down the equation as: X of t equal to e raised to A t minus t naught plus t 0 to t e raised to A t minus tau B u tau and d tau; so here limit from t 0 to t. And you will find that this particular portion there is called unforced response, and this particular portion is called forcible response. And complete state equation depends upon A matrix B matrix input. And therefore, this is the state equation we have determined.

So, this is one classical approach which can be used, but as I told you that as the order increases sometimes the calculation wise is quite difficult. Therefore, we need an approach so that we can easily determine the X of t. So, based approach is the Laplace transform. Last time we have seen for a unforced system Laplace transform project. So, here also we can use this concept of Laplace transform approach for the force system. Now we start deriving the state equation using Laplace transform approach. That is, we can write down the equation as X dot equal to A X plus BU.

(Refer Slide Time: 09:11)



Now, what we do? We have take the Laplace transform for these equations. So, taking Laplace transform on both side we can write down as x X of s X naught A into X of s plus B into u of s. Now we can take X of s on left hand side and then we can get the equation as SI minus A into X of s to X naught plus B into u of s. Now, this X of s equal to SI minus A inverse X naught plus SI minus A inverse B into u of s. That is X of s equals two SI minus A inverse X of 0 SI minus A inverse B into u of s.

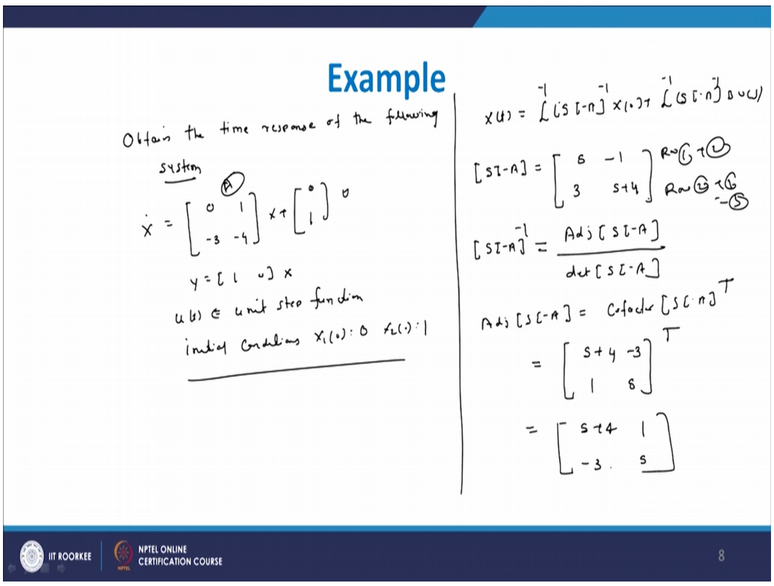
Now, we have got X of s, but we have state equation definitely it in terms of time domain. So, to get a time domain what we can do? We have to apply inverse Laplace transform to this equation. Now we apply inverse Laplace transfer on to these equations so you can get it as: X of t equal to Laplace inverse of SI minus A inverse X naught plus

Laplace inverse of SI minus A inverse B into u of s. And here we can write down this as e raised to A t into X naught plus Laplace inverse of phi of s into B u of s. That is here it came it as phi of t into X naught plus Laplace inverse of phi of s B into u of s. That means finally, this X of t equal to phi of t into X naught plus Laplace inverse of phi of s B into u of s.

So, you will find that even mathematically also or by deriving this equation we have reached easily the X of t. And this X of t is based on the Laplace inverse of SI minus A. This concept of SI minus A inverse we use in a diagonalizations. So, it is useful in a various type of modeling. And this getting a inverse we here used a do different approach: one is the classical approach where we had to take adjoint and determinant, and another simple method I have to explain earlier that is about a liberal algorithm.

So, if we can you can use any of the method but, as the order increases it is better to go for the liberal algorithm to get the SI minus A inverse. But it will order is smallest a second third order, we can use the classical approach to get the SI minus A inverse. That means, finally, this X of t it depend upon this lab phi of t into x of x 0 and this phi is B in two u s and this portion it is consist as force part and that is the unforced part. So, now based on this concept now we try to solve an example and the see what value of X of t will get.

(Refer Slide Time: 13:22)



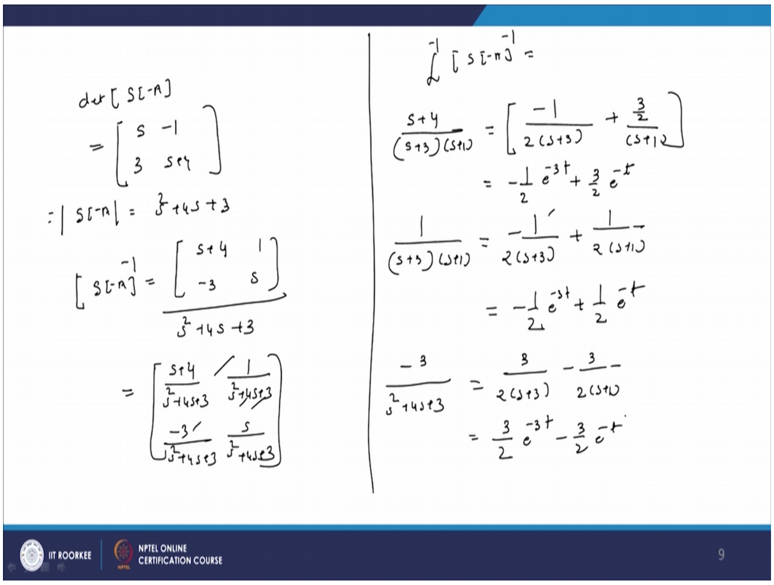
Now for example is where to obtain the time response; obtain the time response of the following system following system. And our example is X dot equal to 0 1 minus 3 minus 4 x 0 1 into u y equal to 1 0 x. And you are u of t; u of t is basically a unit step function. So, u of t is unit step function. And initial conditions initial conditions that is X 1 of 0 equals 0 and X 2 of 0 equals to 1, this is the initial condition. Now, this is your a state model and our purpose is to get the X of t. Now which method we can apply? So, you can use any of the method, but now we go for the Laplace transform approach.

So, we have seen earlier how to get the y of t or sorry; X of t how to: that is the X of t can be written as Laplace inverse of SI minus A inverse X naught plus Laplace inverse of SI minus A inverse B into u of s. That is basic calculation involvement here SI minus A inverse; so A is this one A matrix is 0 1 minus 3 4. So, first of all write down SI minus A. So, we write down this SI minus A equal to S minus 1 3 S plus 4: SI minus A. Now, to get the SI minus A inverse we can write down this as: SI minus A inverse that is equal to adjoint of SI minus A divided by the determinant of SI minus A.

You know, how do you define the adjoint of SI minus A. So, adjoint of SI minus A equal to cofactor of SI minus A transpose. First of all we determine the adjoint of SI minus A. So, here we can write down as SI minus A that is just see here: to get the cofactor so here for s we will get S plus 4, then for this S plus 4 we can write down S.

Then for this minus 1 position; so here it is in the odd term: odd means your first row first and column second to whatever elements will get you have at a connect a minus sign, so you here minus 1; with that is why we will get 3. So, we have to use a sign of minus 3. And here, for this portion for the first three the minor is this minus 1; again it is odd term that is row, row is 2 column is 1. So, it is three. So, odd is there we had to use connect a minus sign. So, here we will get minus 1 so minus of minus 1 equals to plus 1. So, we will get it is like this, this is transpose. And therefore, we can write down this equation as S plus 4 1 minus 3.

(Refer Slide Time: 17:44)



Now, coming to the determinant; the determinant of SI minus A so just see here, it is S minus 1 and here S 3 S 4 4 3 S plus 4. Now, the determinant of SI minus A equal to S square plus 4 s plus 3. Therefore finally, the universe that is SI minus A inverse we can write down as: S plus 4 minus 3 1 S divided by S square plus 4 S plus 3. And finally, this equation can be written as S plus 4 divided by S square plus 4 S plus 3 1 upon S square plus 4 S plus 3 minus 3 divided by S square plus 4 S plus 3, and here S divided by S square plus 4 S plus 3.

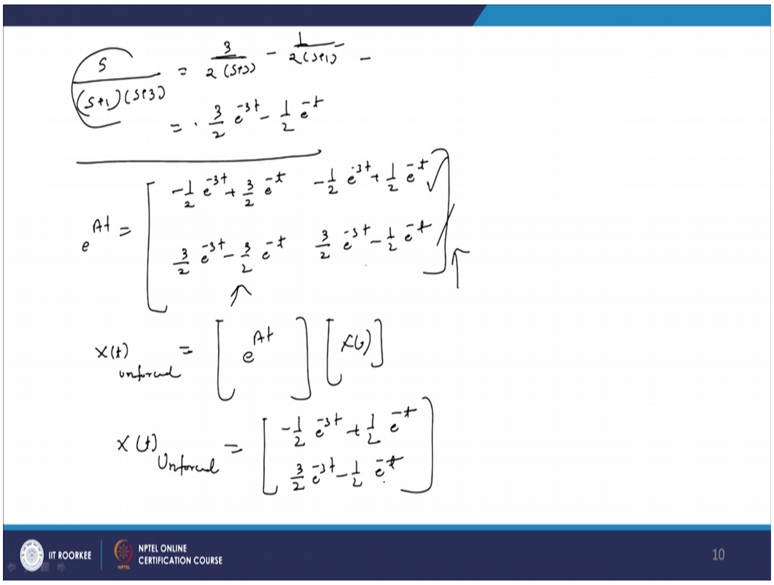
Now this we have written. And now, our main purpose is to get the this SI minus A inverse X of t 0; that is Laplace inverse. Therefore, the equation we have got we have to use the Laplace inverse; that means, the Laplace inverse of SI minus A inverse. So, these equations we will get in terms of X of t. So, if you see this S plus 4 S plus 4 S plus 3 it is not possible to calculate the inverse directly. Therefore, this equation we have to convert into partial fraction expansion. Last time you have seen how to convert into partial fraction expansion, and then we can go for the Laplace inverse.

So, particularly if you say S plus 4 divided by S plus 3 S plus 1; if you convert this into a partial fraction expansion we will get the result as minus 1 divided by 2 S plus 3 plus 3 by 2 divided by S plus 1. And now, for this particular portion we have to calculate the inverse, and more or the inverse of this is not dependent upon this, therefore individually we can calculate the inverse of this. So, Laplace inverse of this we can write down as: minus 1 by 2 e raised to minus 3 t 3 by 2 e raised to minus t.

Now, this term we have taken into constructions, now about this term this and this. So, we can write down for all these terms the Laplace inverse as: 1 by S plus 3 S plus 1. This can be written as: minus 1 by 2 S plus 3 plus 1 by 2 S plus 1. And after solving this we will get 1 by 2 and here 1 by S plus 3 e raised to minus 3 t 1 by 2 e raised to minus t. That is this we have coming in to partial fraction expansion like this and then we have take a Laplace inverse and we have got this. Now next, so after completion of this part we can write 3 upon S square plus 4 S plus 3 at this part. This S plus 3 S plus 1 is same thing. So, here is S 4 plus 4 S plus 3 here minus 3 like this.

So, for this you can get the result as 3 by 2 S plus 3 minus 3 divided by 2 S plus 1. And after solving this we will get 3 by 2 e raised to minus 3 t minus 3 by 2 e raised to minus t. So, we have taken care of this portion, this portion and finally, we want the Laplace inverse of this particular part.

(Refer Slide Time: 22:26)



Therefore, we can write down this as S divided by S plus 1 S plus 3. So, if you convert these into partial fraction expansion you will get 3 by 2 S plus 3 minus 1 by 2 S plus 1. And if you solve it you will get 3 by 2 into e raised to minus 3 t minus 1 by 2 e raised to minus t.

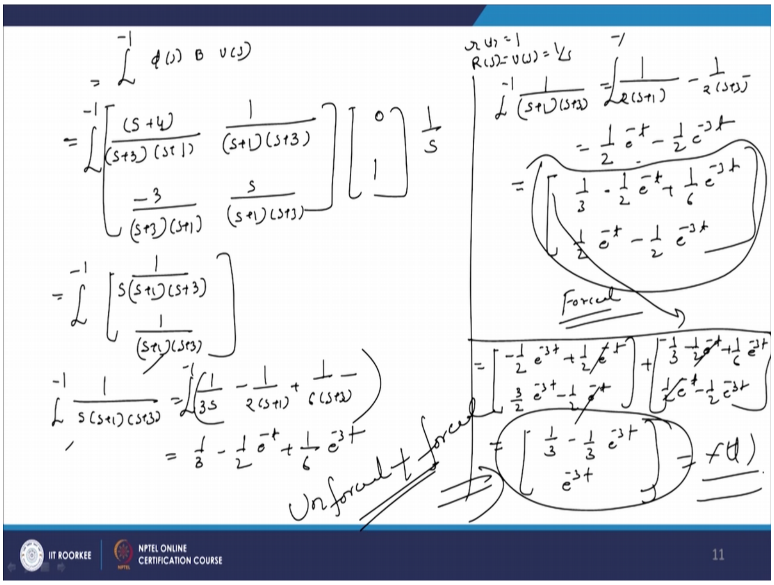
So, all these individual transfer functions, we have taken into consideration, we had n partial fraction expansion, and then we are applied the inverse Laplace. And now we have got e raised to A t. But, our main problem is concerned with this SI minus inverse BU of S. So, this part we have got and, but for this particular part we have to multiplied by X naught and then we think about this portion.

So finally, all these four transfer function we can write in this final matrix form as minus 1 by 2 e raised to minus 3 t plus 3 by 2 into e raised to minus t 3 by 2 into e raised to minus 3 t minus 3 by 2 into e raised to minus t. Then here minus 1 by 2 e raised to minus 3 t plus 1 by 2 e raised to minus t 3 by 2 into e raised to minus 3 t minus 1 by 2 e raised to minus t; now, we can cross check it. See here this portion is like this, so we have return here. Now coming to this portion minus 3 upon S square plus 4 S plus 3 that is 3 by 2 e raised to minus 3 t minus 3 by 2 into e raised to A t; so you see here you can written here.

Then coming into this portion, first part: S plus 4 divided by S square plus 4 S plus 3 that is this 1 minus 1 by 2 e raised to 3 t plus 3 by 2 into e raised to minus t, so we have written here. And this one, 1 upon S square plus 4 S plus 3 we have written as minus 1 by 2 e raised to minus 3 t plus 1 by 2 e raised to minus t. So, we have written here. Now, we want X of t X of t that is unforced this portion. That is this part, that is e raised to A t this portion and we have X naught the initial conditions. So, initial condition is 0 1.

So, if you multiply these conditions here. So, what happen if you multiply this with this; so this portion left side portion that is B neglected because 0 will come. So, remaining will be the terms on this, this one that is this element and this element. So finally, the X of t for unforced system; we can write down as: minus 1 by 2 e raised to minus 3 t plus 1 by 2 e raised to minus t 3 by 2 e raised to minus 3 t minus 1 by 2 e raised to minus t. So, we have done the part conserve with the unforced part, now coming with the part concern with the force part. Now would force part we have Laplace inverse of phi of s in to B u of s.

(Refer Slide Time: 25:59)



This is the first part. Now, phi of s it is s, now we will write in terms of s domain itself. So, Laplace inverse of S plus 4 divided by S plus 3 S plus 1 1 divided by S plus 1 S plus 3 minus 3 divided by S plus 3 S plus 1 and here S divided by S plus 1 S plus 3. And what is the value of B? B you see, the value of B is 0 1. So, we multiply this by 0 1. And now what is the input? Input unit step. So, unit step you know r t equals to 1 that is unit step input r of t equals to 1 if you make r of s. Or here you have take we have considering input as u of s, so r of s into u of s equals to 1 by s; therefore, here u of s if you multiplied by 1 by s.

Now if first of all we multiply all these equations and after multiplications we have inverse Laplace inverse; that means, we have root take inverse that is Laplace inverse means it is a inverse Laplace; that means, we can get the equation in terms of X of t. So, first of all we multiply these equations.

So here, Laplace inverse of now if you multiply it; so again the same thing; So, this multiplied by this, so here we will get the term of 1 by S plus 1 S plus 3, and here also we get the term are S plus 3. And now it is multiplied by S, so here this particular portion order is to the order will increase to third, but whereas this s will cancel with this. And therefore, we can get the equation as 1 by S plus 1 S plus 3 S and here we will get as S plus 1 S plus 3.

Now, we have inverse Laplace inverse. Now again we want a time domain equations. So, if you take Laplace inverse of 1 by S S plus 1 S plus 3. This can be written as 1 by 3 s minus 1 by 2 S plus 1 plus 1 by 6 S plus 3. And then we apply Laplace inverse for this. So, what happens? Inverse for all, so we can get as 1 by 3 minus 1 by 2 e raised to minus t plus 1 by 6 into e raised to minus 3 t. So, we have taken the Laplace inverse of this, now coming into the Laplace inverse of 1 upon S plus 1 S plus 3.

So, this can be written as 1 by 2 S plus 1 minus 2 S plus 3. And that is a inverse Laplace inverse. So, we can get this equation as 1 by 2 e raised to minus t minus 1 by 2 e raised to minus 3 t. And therefore, final equation it becomes 1 by 3 minus 1 by 2 e raised to minus t plus 1 by 6 e raised to minus 3 t. And about this portion, this one is we got this. So, we can write down as 1 by 2 e raised to minus t 1 by 2 e raised to minus 3 t. Now these part, this portion, this is the forced part. And here earlier we have got unforced part, and that complete state equation is a combination of unforced and force. Now all these elements we have to add it and we see what final results we will get.

So, the final results which will get that is: if you add this that is a first of all we write the unforced part. So, unforced part is 1 by 2 e raised to minus 3 t plus 1 by 2 e raised to minus t and here 1 3 by 2 e raised to minus 3 t minus 1 by 2 e raised to minus t. Plus, here same this part: this is 1 by 3 minus 1 by 2 e raised to minus t 1 by 6 e raised to minus 3 t and here 1 by 2 e raised to minus t 1 by 2 e raised to minus 3 t.

Now, this is there. And we have to add this element. So, you will find that if you add this particular element; so what will get the results which will get it as 1 by 3. And here this portion will cancelled we will get 1 by 3 1 by 3 e raised to minus 3 t. And about this portion say, 1 by 2 e raised to A t this will cancel. And here we will get as 3 by 2 minus 1 by 2; that means, you will get e raised to minus 3 t.

So, this is the complete response, this part is nothing but the combination of unforced plus forced. And this we got final X of t of a state space system when we are we put a force input. So, we will find at whenever a system is there very is involvement of A and B the total response is depend upon unforced part as well as the forced for part. But, whenever we calculate when whenever we have given a system in terms of unforced part then the X dot equal to A X that is there is only concerned with the e raised 2 A t and X naught. But whenever a force is concerned we have to take the care of the B.

And again if you see that as we are involving the forced part the calculations increases. And again we see that all these things is all the mathematics, but it is a very much applications in a controller design, controllvity, observity observal, so it has wide applications.

Now you see the references: Nagarat Gopal, D. Roy Choudhury, B. C. Kuo. And there are many of book as are available, you can use this book. As we have solve one example similarly you can solve many example. And the most important point the SI inverse, I have used the concept of the classical approach, but you can also go for the liberal algorithm to calculate the inverse.

Thank you.