**Advanced Linear Continuous Control Systems**
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**Lecture – 22**
**Solution of State Equation**

We will now start with the Solution of State Equation.

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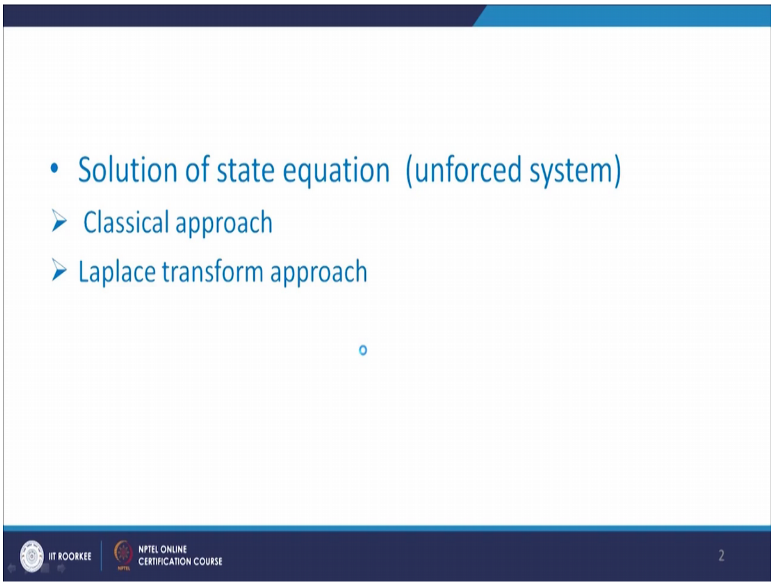


So, solution of state equations: so we are we have discussing unforced system. In this we will see a classical approach as well as Laplace transform approach. Now about the classical approach; so, before going for a classical approach we have to see why there is need for the state equations.

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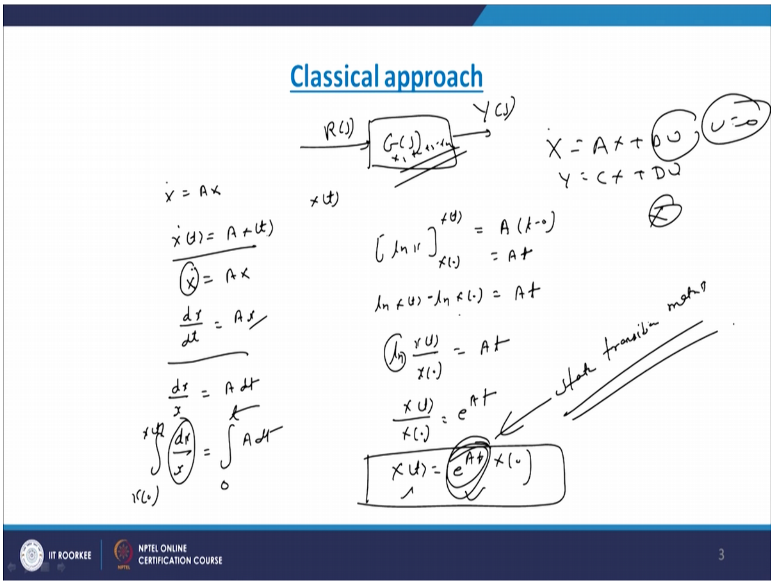


Because, if you see the classical control in classical control there is the involvement of only a plant G s and here R of s and Y of s. So, we are concerning with the output only, but here particularly in state space we are not only concerned with the input output, but also the internal state that is there is involvement of x 1, x 2, x 3, x n. So, these are all the states are important, they are govern the performance. And therefore, how the performance of these states will come that also need to know; that means, till now we have seen that we have to calculate stability of the system, we had to convert into different form, but along with this there is need to know the state equation of the system.

Now, how to calculate the state equations? Now the plant which we were taken a we have considered that is X dot equal to A X plus B U Y equal to C X plus D U. Now when we are concerned with the state equation; that means, we are only concerned with the X; so now here we are assuming the case as we are discussing unforced system that is we are taking in the case that is when this U equal to 0.

So, when U equals to 0 we have got a state model as X dot equal to A X. So, what is the state equations; that is we want to know X of t; that means, the equation is X dot of t equal to A X of t. So, now, here X dot we can write down as A X. Now this X dot we can write down as dx by dt into A into X. This is a first order differential equation. So, you have already solve this part in you will use in a 12th or maybe in the first year or second year.

Now here this we have to solve. So, how do you solve it? So, we can write down as dx by x A into dt. So now, we integrate these equations dx by x A into dt. Now here limit we taken for 0 to t and for this dx let us say x 0 to X of t. And now you know the rule of integrations; the rule of integration says the logarithm x of x X of 0 to X of t into A into t that is here t minus 0 there is equals to A into t; that means, the logarithmic of X of t minus logarithmic, natural logarithm of X of 0 into A into t. Now, here logarithmic of X of t minus divided by X of 0 equal to A into t. Now here, this has been changed like this X of t goes to X of 0 this l n can replace as e raised to A t. And now here X of t equals to e raised to A t into X of 0.

So, how you have got it? So, what we have done here X dot we have we have to write dx by dt equal to A X, then these x you have taken this side so dx by x a dt, then we are integrated. And here logarithm of x x 0 to x t into t and then we have solve it. And finally, we got X of t it goes to e raised to A t into X of 0. The e raised to A t: the problem is sometimes that how to determine the e raised to A t that is called a state transition matrix. This is called state transition matrix. So, if you see, if afterward that how to calculate the state transition matric by various methods.

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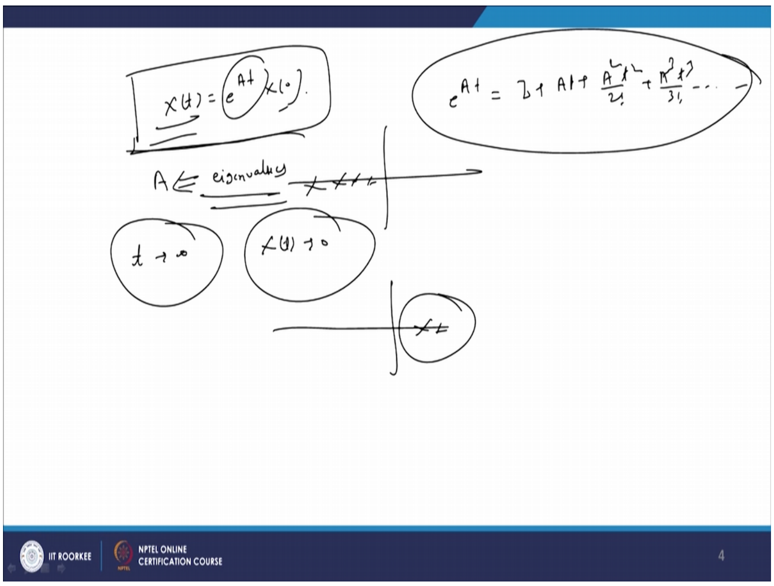


But now here we have got X of t equals to e raised to A t into x of X of 0; that is X of t equal to e raised to A t X of 0. So, we will see here this equation of X of t if A has eigenvalues let us say on the left hand side of this plane here.

So that means, this A we can write down in terms of lambda in terms of eigenvalue that comes a minus sign. So, at t in to infinity this X of t tending to 0. That is a stable system. But if same a has eigenvalue on the right side of this plane then this X of t may be the higher values, and that is it moves towards the instability. That means, from the X of t we can also get the information of the stability of the system. So, that is the state equation X of t.

And this equation we have determined the case when there is no U input is 0 and force system, there is no force is involved. So, this e of t we can calculate as I plus A to t plus A square t square factorial 2 A cube t cube factorial 3 like this; that is the e raised to A t. So, once you get e raised to A t then multiplied by X of 0 that is the initial state we can easily get the values of X of t. That is called state equations.

Now, we see another approach by which we can determine the state equation. Because, this earlier approach we have seen that we have to calculate e raised e of A t that is I we have to multiply a A square factorial 2. So, this is a quite a cumbersome process difficult; calculation wise difficult as the order increases calculation is more. And we have a very good tool a Laplace transform, this is used for our various applications. So, we will see again the Laplace transform helps in getting the state equation.

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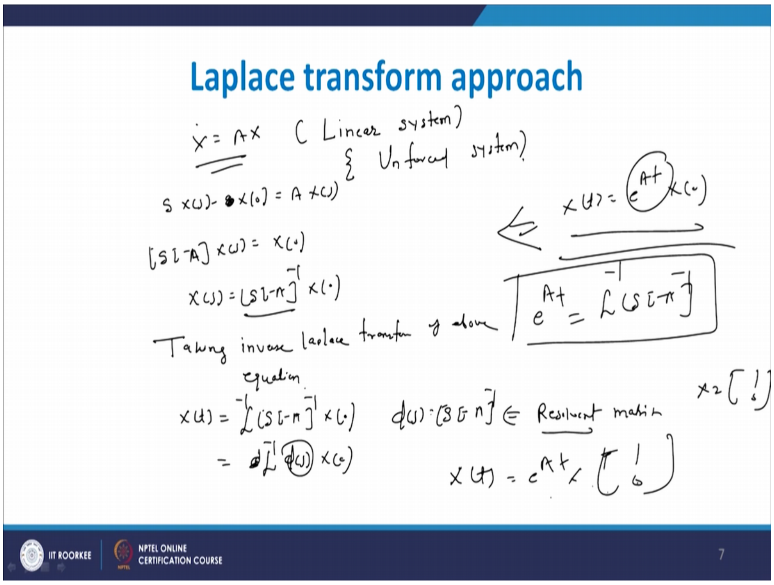


So, we start deriving it. So, our original equation is X dot equal to A X, this is a linear system. We can called as Linear System and now this is also called as Unforced System.

Now, our purpose is to get the X of t, so what we do. We can take the Laplace transform this is same this equations and see whether we have getting X of t or not. So, you taking Laplace transform the both sides, so here x X of s minus X of 0 A into X of s. Now here we take X of s left side, so here SI minus A into X of s sorry; into X of 0. Now here X of s to SI minus A inverse into X of 0. That is what we have done.

We have taken the Laplace transform S X of x minus X of 0 multiplied by A this is X of s. Now this X of s and X of s we have taken one side so we get an I we represent the identity matrix because we have to concerned with A; so SI minus A into X of s equal to X of 0.

Now, this SI minus A we have taken on right hand side, so the SI minus A inverse into X of 0. Now we have to take the inverse Laplace transform so, taking inverse Laplace transform of above equation. So, if we take the inverse Laplace transform equation, we will get it as X of t that is equal to Laplace inverse of SI minus A inverse into X of 0. And this we can write down as Laplace inverse of phi of s into X of 0 and this phi of s is nothing but SI minus A inverse. And it is called as Resolvent matrix; r e s o l v e n t - resolvent matrix phi s.

And earlier we have determined this X of t equal to into X of 0. And now this e of A t we can write on as Laplace inverse of SI minus A inverse. That means, this e raised to power A t which is very important in getting the state equation can be determined by a SI minus A inverse.

And hence, this X of t we have e raised to A t. And we have to see what are the initial conditions; let us say this initial condition may be 1 0 say 1 0 we have to consider and we have to multiplied by this and we can get the value of X of t. That is the state equation of the system. So, here state equations can be determined by a by means of Laplace transform or by means of conventional approach, but we will find that as the order increases this Laplace transform approach is quite convenient.

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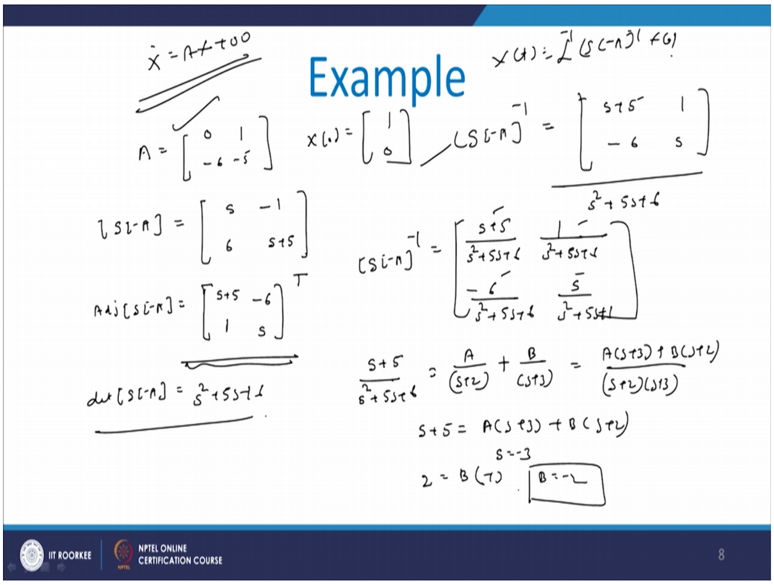


Now, we start solving an example. Let us say A matrix is given as 0 1 minus 6 minus 5, and X of 0 initial condition is given as 1 0. Actually your plot is like this X dot equals to A X and we are talking about state equation where there is no input. Therefore, this is 0 into U. Therefore, the example which I have taken it is only mentioning A and X of 0. Now we have to calculate the X of t and here we will we will going to use Laplace transform approach.

So, now the first step is we have to make take SI minus A, this is SI minus A equal to S minus 1 6 S plus 5. Then, we can write down as adjoint of SI minus A, because we need to determine the inverse. So, how to get the adjoint of SI minus A? So, you have to take the transpose of the cofactor. So, cofactors are S plus 5, here is S, here is minus 6, here is 1, this transpose divided by it determinant the determinant of this is adjoint.

Now we have to calculate the determinant: the determinant of SI minus A equal to S square plus 5 S plus 6. And our main purpose is to determined SI minus A inverse. Therefore, this SI minus A inverse equal to S plus 5 1 minus 6 S divided by the determinant S square plus 5 S plus 6. So, now this equation can be written as SI minus A inverse S plus 5 divided by S square plus 5 S plus 6, here 1 divided by S square plus 5 S plus 6 divided by minus 6 S square plus 5 S plus 6, and lastly S divided by S square plus 5 S plus 6.

But, our problem is basically we want X of t is equals to Laplace inverse of SI minus A inverse into X naught.; That means, we want final equation in time domain, that although we have got SI minus inverse but that is not sufficient, we want to determine the X of t. So, in order to do it we have to determine its inverse Laplace. So, here we have got four functions: one two three four, we have to consider this function individually and then try to determine the inverse.

So therefore, we take first function as S plus 5 S square plus 5 S plus 6. This equation can be written as A divided by S plus 2 plus B S plus 3. So now, we solve it. So, we will get S plus 2, S plus 3; now here A multiplied by S plus 3 B multiplied by S plus 2. See here these equations.

Now, you compare this, these and this we will write down as S plus 5 A S plus 3 plus B S plus 2. Suppose, if you replace S equals to minus 3. So, if you replace S equals to minus 3 this becomes 0 and we can get the value of B. So, we can write down as this is 2 B minus 1. So, we can get as B equal to minus 2. So, we have got B equals to minus 2 by means of this. Now, in the next step what we have to do; we have to replace X equals to minus 2 here. So, when we replace the x equals to minus 2; that means, the part of B will cancel you can get the values of A.

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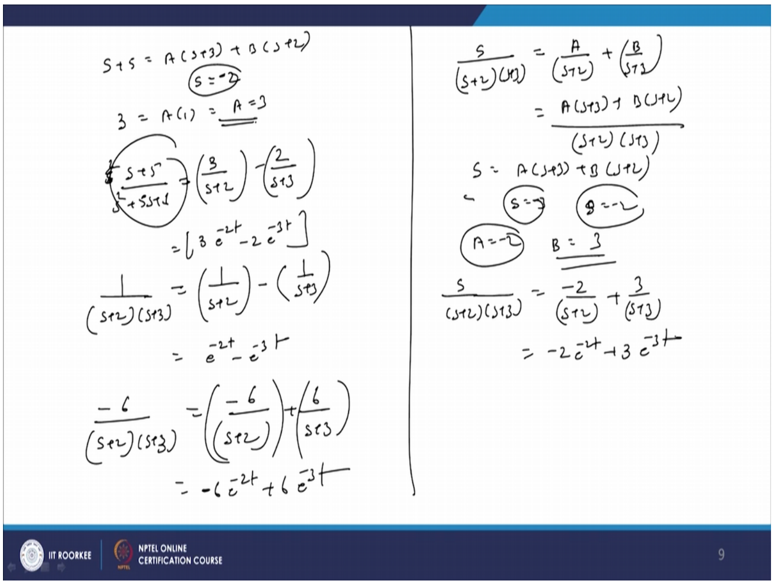


That means S plus 5 A S plus 3 to B S plus 2, so we replace S equals to minus 2. So, you can get it at 3 a into 1; so a equal to 3.

Now if you see this part, this particular part this will be retained in terms of A and B. So, here we will get the equation of say S square plus 5, so S plus 5 divided by S square plus 5 S plus 6. So, this is 3 by S plus 2 minus 2 S plus 3. Now if we take the inverse Laplace of this, so we get it as 3 into e raised to minus 2 t minus 2 e raised to minus 3 t.

So this part, this is we have determined for these particular functions. Now we can determined for this, this and this. So similarly, we can write down as 1 by S plus 2 S plus 3 that is equal to 1 by S plus 2 minus 1 by S plus 3. So, you will find that if you solve it 3 minus 2 1; 1 by S plus 2 and S plus 3 now we have to take the inverse Laplace transform for this, we will get e raised to minus 2 t minus e raised to minus 3 t. Similarly for this, this part we can write down as: minus 6 divided by S plus 2 S plus 3, we can write down as minus 6 S plus 2 plus 6 S plus 3. So, here we will get it as minus 6 e raised to minus 2 t plus 6 e raised to minus 3 t. And lastly this S divided by S plus 2 S plus 3.

So, we can write down as A divided by S plus 2 B S plus 3. Now, here we can write down as A into S plus 3 plus B into S plus 2 divided by S plus 2 S plus 3. Now if you compare it S A S plus 3 plus B S plus 2. So, after solving this equation that is replacing initially S equals to minus 3 and afterward the S equals to minus 2 here and we will get it as A you will get it as minus 2 and B you will get it as 3.

Therefore, this S divided by S plus 2 S plus 3 this can be written as 2 minus 2 divided by S plus 2 plus 3 S plus 3. And if you take the inverse of this, the inverse Laplace transform of this minus 2 e raised to minus 2 t plus 3 e raised to minus 3 t. So, now we have considered the all individual transfer functions and now we combined in one matrix and then we multiplied with the initial conditions.

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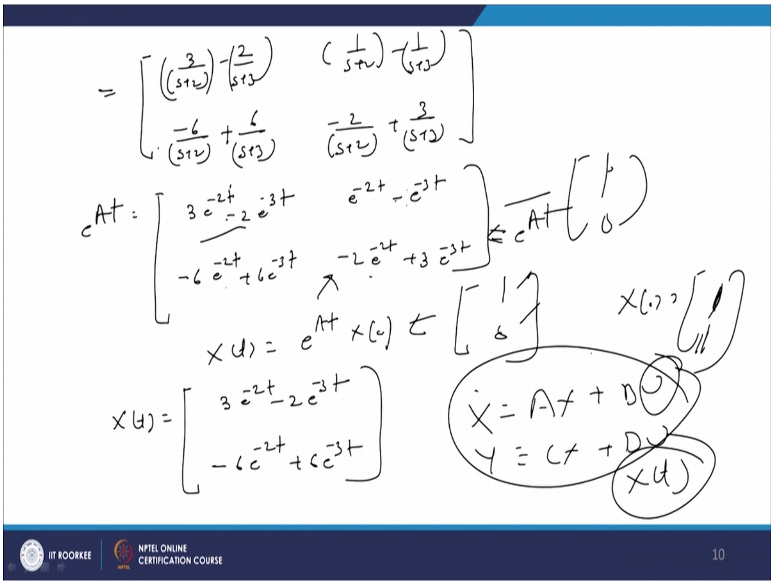


So, we can write down the complete matrix as: 3 S plus 2 2 S plus 3 then you have minus 6 S plus 2 plus 6 S plus 3. And here we can write down as 1 by S plus 2 minus 1 by S plus 3. Here minus 2 S plus 2 plus 3 S plus 3. And finally, we can write down as 3 into e raised to minus 2 t minus 2 it is 2 e raise to minus 3 t. Now about this, minus 6 e raised to minus 2 t plus 6 e raised to minus 3 t. Then here, e raised to minus 2 t minus e raised to minus 3 t. Then here lastly, this minus 2 divided by S plus 2 we can write down as minus 2 t plus 3 e raised to minus 3 t.

But here, this particular portion is nothing but e raised to A t. But now, the total final state equation is based on our initial condition that is e raised to e A t multiplied by X naught. Now here this X naught is given as 1 naught; 1 0 1 0. Now we have to multiply this part is 1 0. So, what will happened here in this case this element elements in the second column will not exist because of this initial condition it is finally we will get the value of X of t equal to 3 e raised to minus 2 t, minus 2 e raised to minus 3 t, minus 6 e raised to 2 t, plus 6 e raised to minus 3 t.

That means, in this example because of 1 0, because we multiplied this 1 0 with this; so this multiplied by this and this portion becomes 0, even this is 0 remaining only this part if you have a initial condition is like x 0 equals to 1 1 1 1 then all will be there is a addition of these and this, these and this and we will get the X of t. So, now, this is the state equation for the unforced system, that is no force has been applied. And we will find that we have calculated this very easily.

But the practical problems involved applied force. And in that case how to get the values of X of t. And as I told you that here complete the performance is not depend on only on Y it depend upon X t just like in a medical problem whatever the result it depends on what are the internal parameter just like blood test, urine test the all the testing we are doing. Here also, these are the states of the system current flowing the inductor voltage across the capacitor so that state information or as a time changes how the X t will behave.

So, here these calculations we have done only for unforced system. So, in the next time, but we will see how to do the calculations of the state equation when there is an involvement of the force system. That is next part is concerned with X dot equal to A X plus B U Y equal to CX plus D U. If this is the plant how to calculate the X of t, because when this is the case there is a part comes to U. So, how the U will act in this case? So, we will see these things in the next part.

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Now, these are some references: Nagrath and M. Gopal, D. Roy Choudhury and B. C. Kuo. So, you can use these references for detailed studies. Here are the examples which I saw only for second order system. Now you can solve many more examples.

And you can solve this example by both the approach: you can take determined e raised to A t by I is at A square t square factorial to like that or Laplace transform. And again you see that which method is simple or if we take an example do the calculation, see that where we will get the less computation type, and moreover we require to determine the inverse; So, for a second order system, take second order we you can go for adjoint and determinant, but if you go for higher order the later algorithm seen earlier that can be also useful to use.

Thank you.