Advanced Linear Continuous Control Systems
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Lecture – 21
Concept of Diagonalization

Now, we start with the Concept of Diagonalization. In this we study introduction to diagonalizations, mathematical analysis, role of Vandermonde matrix and example.

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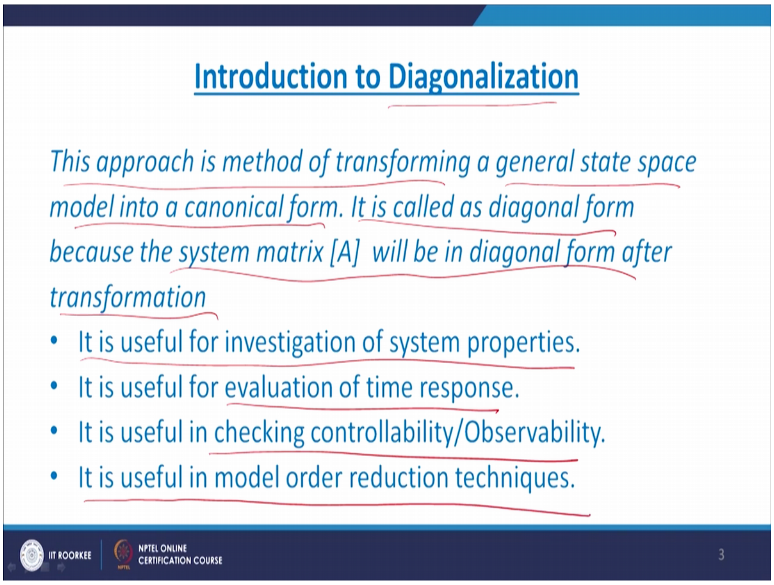

Now, about the introduction to diagonalizations; in actual practice we want to design a system in such a manner that, we should get a desired performance. So, if you having your original model in that case sometimes it is difficult to analyze that system.

Even sometimes difficult to design a controller or internal property of the system cannot be easily studied. Therefore, we need to transfer a system into different form in a such a manner that, that properties should not change. So, here we are doing the same thing a diagonalization means the same thing that, we have a one system and it has been converted into another form and that form is very much useful for control system applications.

So, here the concept of diagonalization this approach is method of transforming a general state space model into canonical form. It is also called as a diagonal form because the system matrix A will be in a diagonal form after transformation that is, a original model A is some one form. After a transformation it has been converted into diagonal form. That diagonal form is we have all the diagonal elements the remaining elements are 0s. So, how they are useful?

So, it is useful for investigation of system properties. It is useful for evaluation of time response. It is useful in checking a controllability and observability and finally, control design and, and last it is useful in model order reduction techniques. Now, you see the mathematical analysis, here is state space we have always start with the state space model.

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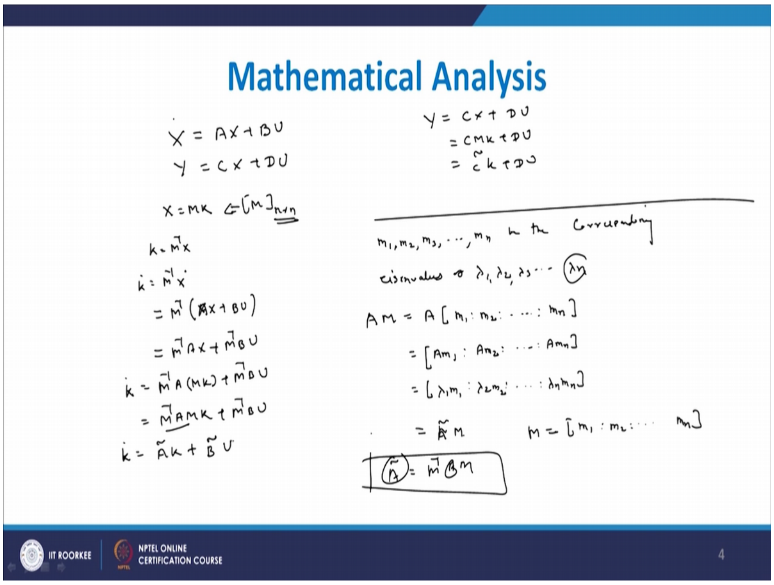

The state space model is represented by X dot equal to AX plus BU and Y equals to CX plus DU. Now here is a problem A matrix it is in general form, there are many elements are involved in a A matrix. So, our purpose is that A must be converted into diagonal form. So, X is the state vector. Now what we want here? We want to convert this state vector into a new vector. So, this is possible. So, we assume that A is non diagonal and we define a new state new state vector K in such a manner that so, we can write X equals to say M into K.

Now, what happened we have X is the original state vector this K is a additional vector new vector, we have taken and M is the some matrix model matrix. So, here M is a operated on K we got X, but now here we know X. So now, he in this case this M is nothing but n cross n matrix. So, after this we can write down this equation as K equal to M may be M inverse X. So, what we want we want state vector in terms of K. So, what we can do? We can take K dot K dot equal to M inverse X dot.

Now, after this we can replace X dot equals to AX plus BU in this equation. So, you will get M inverse X dot equal to AX plus BU. And after solving this we will get M inverse AX plus M inverse BU. So, here we have equation in terms of K here we have got M inverse AX M inverse BU. So, here K here is AX. So, I think we have to change it because we want a complete equation in terms of K because K is a new state vector. Therefore, what we can do? We can write K dot equals to M inverse A. Now what is X? X equals to MK M inverse BU and we can write down as a M inverse M K plus M inverse B into U. And now this can be written as K dot equals to A bar K plus B bar into U.

Now, we have got equation which is in terms of K dot earlier it is in terms of AX. Now here A bar equals to M M inverse AM and B bar equals to M inverse B. And so, if you determine this model matrix by determining different eigenvectors and make inverse then multiply AM. So, you will find that this M inverse AM and this A have same eigenvalues, same eigenvalue is nice means that; they have a same properties. So, though so once you got K dot now we can see about Y output Y Y equals to CX plus DU and what is the X X equals to M K plus D into U.

And here we can C bar K into DU. So, you got B bar equals to M inverse B A A bar equals to M inverse AM and C bar equals to CM. Now, point has come so, how to get the values of M. So, M we called as the modal matrix. And here the more important thing that the M inverse AM and this A must have same eigenvalues. Now we can we want to determine the values of M. So, what we can do we will assume m 1 m 2 m 3 mn be the corresponding is corresponding to eigenvalues of lambda 1 lambda 2 lambda 3 up to lambda n.

Here we are saying that lambda 1 is the eigenvalue and corresponding to these eigenvectors are m 1, corresponding to lambda 2 eigenvectors are m 2. Similarly, corresponding to lambda n eigenvectors are mn. Now we can write down the equation as A into M. So, A is a matrix operated on M model matrix. So, we can write down as A so, M involve eigenvectors m 1 m 2 up to mn. So, we can write down as m 1 m 2 this process repeated and mn.

So now we multiplied Am 1 Am 2 A M n. Now we have seen that AX equals to lambda X, now this equation we can also write down in terms of lambda in terms of eigenvalues. So, we can write down as lambda 1 into m 1 lambda 2 into m 2 equals to lambda n into mn. And this can be written in terms of A bar M and here this A bar equals to M inverse Am. And M is the model matrix and this M can be written as m 1 m 2 up to mn.

So, A and this A bar has same cached equations their eigenvalues are same therefore, there cannot be change in the properties, but here one thing is has been observed that, A bar which will come that we can get in a diagonal form. Diagonal form means only few elements are present remaining elements are 0. Now, the problem will come that how to calculate the eigenvectors, and we will find that the calculation of a eigenvector sometimes is difficult. But it is observed that if your original matrix A it is some specific form. So, we can easily determine the eigenvector of this matrix.

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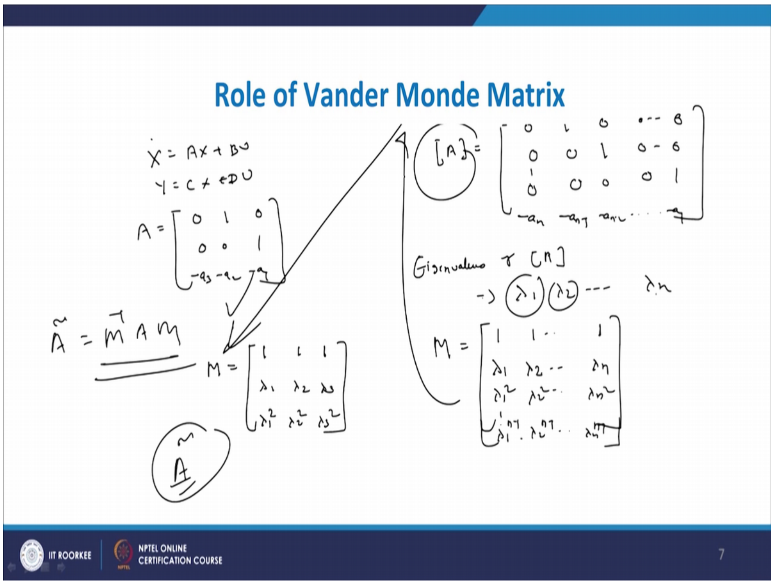

That means if you have a system which is represented by X dot equal to AX plus BU Y equal to CX plus DU and this A it has a specific form that is, 0 1 0 say 0 0 1 then here minus a 3 minus a 2 minus a 1. So, we have seen this type of form earlier this is called controllable canonical form or we can say as companion form or phase variable form. So, if you have A in this form and if you bought the model matrix M so, the how to get it because finally, we want to achieve these M inverse A into M, this we want. And this M to calculate the M that is the eigenvectors are required. So, to calculate the eigenvectors for A, we have to calculate eigenvalues for respect to all eigenvalues we have to calculate the eigenvector.

But if the given matrix A you know this particular form then we can easily write down the eigenvectors. So, how you will write down? So, in that case if the given matrix is like this. So, model matrix M we can write down as. So, let us say this third is order. So, we can write down 1 1 1. First row we write down as 1 1 1. If nth order, we can write down 1 1 up to n. Now the second circuit row so we can write down as eigenvalues are lambda 1 lambda 2 and lambda 3.

And now about third it is lambda 1 square lambda 2 square lambda 3 square this is the whole matrix. No need to calculate the actual eigenvectors. So, this M this matrix is called basically a Vandermonde matrix. So, if you know M we can calculate M inverse then multiplied by A into M, we can go we will get A bar suppose, if your matrix it is in generalized form. So, A equal to say 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 1 and lastly let us say minus an minus a of n minus 1 a n minus 2 up to say, let us say the eigenvalues of A eigenvalues of A; it is if it is written as lambda 1 lambda 2 up to lambda n eigenvalues has been written as lambda 1 lambda 2 lambda n.

In that case the model matrix M, it is written as 1 1 1. Then let us write down the eigenvalues are lambda 1 lambda 2 lambda n then lambda 1 square lambda 2 square lambda n square and we can write down as lambda 1 n minus 1 lambda 2 n minus 1 up to lambda n n minus 1. And so, corresponding to lambda 1 this is the eigenvector corresponding to lambda 2 this is the eigenvector and correspond to lambda n this is the eigenvector and as told earlier this is called as Vandermonde matrix.

Now, about the another point invariance of eigenvalues, eigenvalues under linear transformation cannot change. Already we have seen that we have given system matrix we have come into the another form now we will see also another way that the eigenvalues cannot change if you use the concept of transformations.

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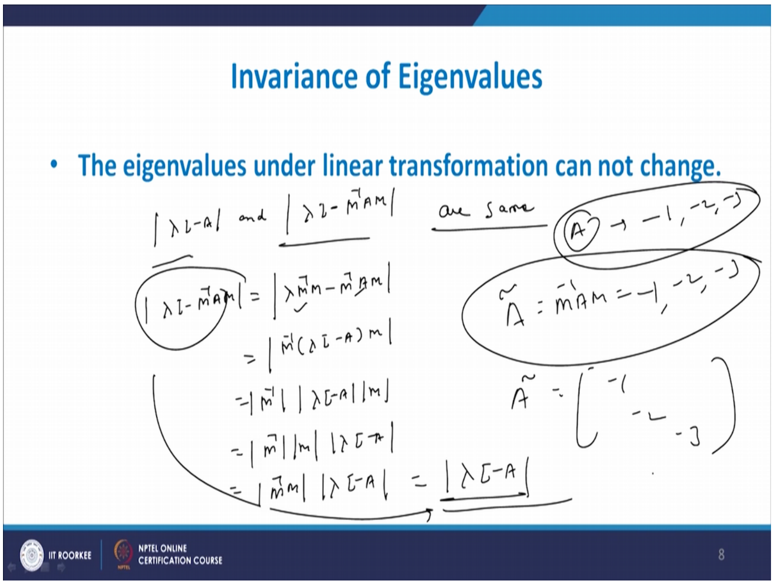

Now, our problem is we have a lambda I minus A and lambda I minus M inverse AM. And our problem is that both are same both are same. Now, how do you prove it? So, for that purpose we will write lambda I minus M inverse AM equal to now this I identity matrix we can write this I as M inverse M. So, here we can write down as a lambda M inverse n minus M inverse AM. Now, what we will do here? We have lambda is there M inverse M so, M inverse M in the both way here it is this side at this side. So, we can take it M inverse one side M on the other side. So, we can write down this equation as M inverse lambda I minus A into M.

Then this complete determinant; So, we can write down as M inverse lambda I minus A into M, and afterward we can write down this as M inverse M to lambda I minus A. And therefore, this is as M inverse M to lambda I minus A, that is equal to lambda I minus A. So, here lambda I M inverse M is same as lambda I minus A; that means, you find that eigenvalue is under transformation cannot change.

If you do it we saw like some of this example you will find that, A let us say this A matrix. Then you take this M inverse AM that is, A bar equals to M M inverse AM. We also you will get same eigenvalues lambda 1 lambda minus 2 minus 3. Now the problem is there, if older system has a eigenvalue minus 4 minus 1 minus 2 3 minus 2 minus 3 then we have come a different pump, what advantage we are getting here because same eigenvalues what is the issue here. But you will find that if you have a matrix A we have minus 1 minus 2 minus 3; that means, it is sort of type of decoupling effect you can easily observed.

So now the details of this we will see later afterward, but now here we have shown that by doing some transformations, the eigenvalue cannot change. And we have original matrix A and we can differ and another matrix say that we have got A same eigenvalues. So, only thing that we have to do some more calculations ok; now we start solving one example.

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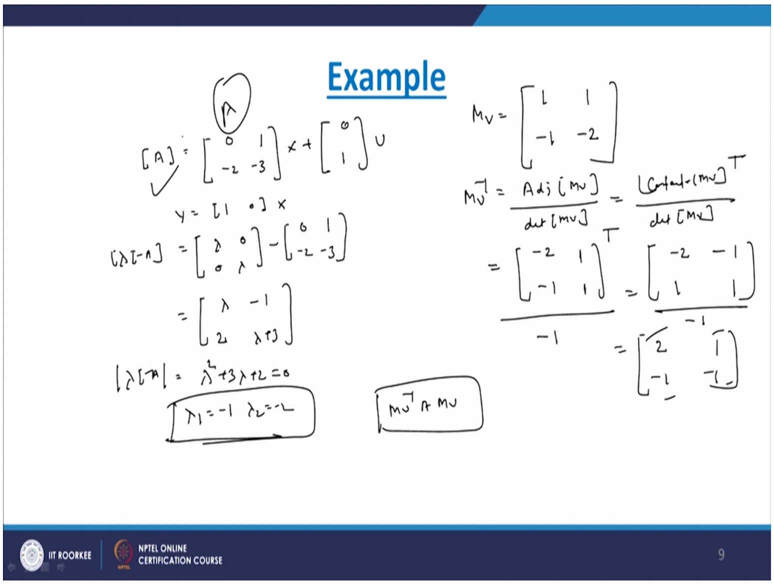

Let us say it is A equal to 0 1 minus 2 minus 3 X 0 1 U and Y equal to 1 0 X. Now here we have to use the concept of the diagonalizations. And we have redefined the model matrix as well as the diagonal matrix. We will start doing it. Now the first step we have to determine a lambda I minus A. So, that is given as lambda 0 0 lambda minus 0 1 minus 2 minus 3 and after solving we will get lambda minus 1 2 minus 1 minus 1 2 and here lambda plus 3.

Now, the determinant of this lambda minus A that is lambda square plus 3 lambda plus 2 equal to 0; So, we will get we will get the lambda 1 equals to minus 1 lambda 2 equals to minus 2. Now, we have got 2 eigenvalues. Now our purpose is to conclude the eigenvectors we have seen earlier that if you have systemizing in some different form, we can easily write down the eigenvectors. So, you will find that in this particular matrix

A this matrix is belong to a companion form or we call as controllable canonical forms let us see here 0 1 minus 2 minus 3.

So now we can to calculate eigenvectors, but we no need to calculate the eigenvector because; if this matrix in this form you can directly write down a Vandormonde matrix. So, therefore, the Vandermonde matrix is say MV model matrix. Here we start it with 1 1 and what are the eigenvalues minus 1 from minus 2 just here earlier we have seen this concept 1 1 lambda 1 minus lambda 2 minus lambda 3 lambda 1 square lambda 2 lambda 3 square, but see the order is, but in our example order is 2.

So, here 1 minus 1 1 minus 2; Afterward we have to calculate the MV inverse that is inverse of the model matrix. Now how to get the inverse of this model matrix? So, with the concept says MV divided by the determinant of MV and what is the adjoining of MV that is equals to cofactor of MV transpose divided by determinant of MV. So now, what is the cofactor of MV? Here is minus 2 1 then here we will get a 1 and here we will get minus 1 this is transpose and divided by determinant of this.

So, determinant of this we will get it minus 1. So, we can write down as these as minus 2 minus 1 1 1, but by minus 1 and a finally, we can get this result as 2 1 minus 1 minus 1. So, we have got MV inverse, now the next step what we want? We want we want MV inverse A into MV this we want.

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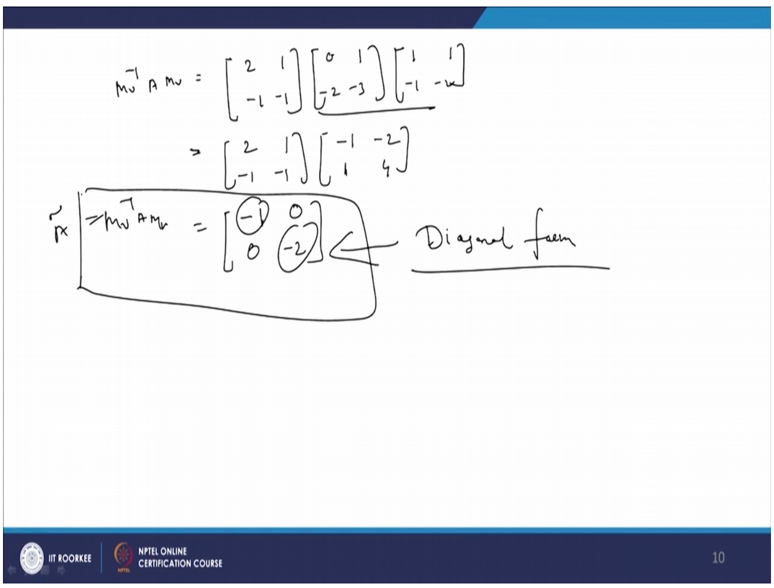

Now, this M V inverse A into MV; so, what is MV? So, MV is 2 1 minus 1 minus 1 we can check it again here ok. 2 1 minus 1 minus 1 then what is A matrix? A matrix we can check it 0 1 minus 2 minus 3. So, we get write down as 0 1 minus 2 minus 3 and what is value of MV MV is 1 1 minus 1 minus 2. So, we can write down as 1 1 minus 1 minus 2 we have to solve it. So, first of all we solve this and then multiplied by this. So, you can get 2 1 minus 1 minus 1.

So, here 0 multiplied by 1 1 multiplied by minus 1. So, we will get minus 1 then 0 multiplied 1. So, 1 multiplied by minus 2. So, we will get minus 2 then you have a minus 2 multiplied by 1 minus 3 multiplied by minus 1. So, we will get it as. So, you will get minus 2 multiplied by 1 minus 3 we will get as 1 and here minus 2 multiplied by 1 minus 2 plus 6 this we get 4.

So, 1 minus 2 to 1 4 now again we cross multiply it. So, 2 multiplied by minus 1 and one multiplied by 1. So, we will get it as minus 1 then 2 multiplied by minus 2 minus 4 1 multiplied by 4 plus 4 we will get 0. Then here minus 1 multiplied by minus 1 is 1 and this minus 4 multiplied by 1 minus 1. So, plus 1 minus 1 we will get a 0. And similarly here minus 1 multiplied by 2 2 and here minus 4 2 minus 4 we will get as minus 2 we will get MV inverse am MV.

Now, this is called a diagonal form. And we will find that the, whatever the values here they are the eigenvalues of the matrix or eigenvalues of this A bar and these are the also the eigenvalues of this A matrix.

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Now, these are some references. And as I told you that this diagonalization although we will find them in a mathematical way, but it is widely used in a applications particularly controllability, observability also a control design. So, we will see latter on how this is useful in control checking controllability, observability and also in controller design.

Thank you.