Advanced Linear Continuous Control Systems
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Lecture – 20

Stability Analysis in State Space: Lyapunov Stability Analysis (Direct Method) Part-V

Ok. Now, we start with Direct Method of Lyapunov that is basically for linear system.

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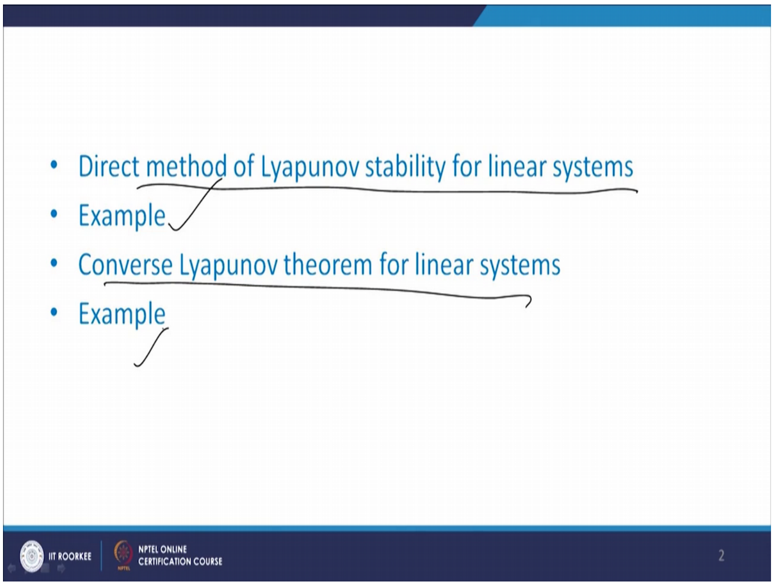

In this we will study direct method of Lyapunov stability for linear systems, the example then we will see converse Lyapunov theorem for linear system; again we see one example. Now, what a direct method of Lyapunov stability for linear system as I told last time that there are many criteria's are available for checking the stability of a linear system particularly, Routh criteria, root locus, (Refer Time: 01:06).

The main purpose of Lyapunov is for checking the stability of a non-linear system, but it is found at along with this taking the stability of a non-linear system this Lyapunov criteria is also useful for checking the stability of a linear system. Last time you seen that this Lyapunov criteria gives the sufficient condition for stability of a system for non-linear system. But when we apply Lyapunov for checking the stability of a linear system it is pound at this criteria gives necessary and sufficient condition for stability of the system. Therefore, this criteria is an alternative to Routh criteria for checking the stability of the system and in this case there is no need to calculate the eigenvalues, to just see here.

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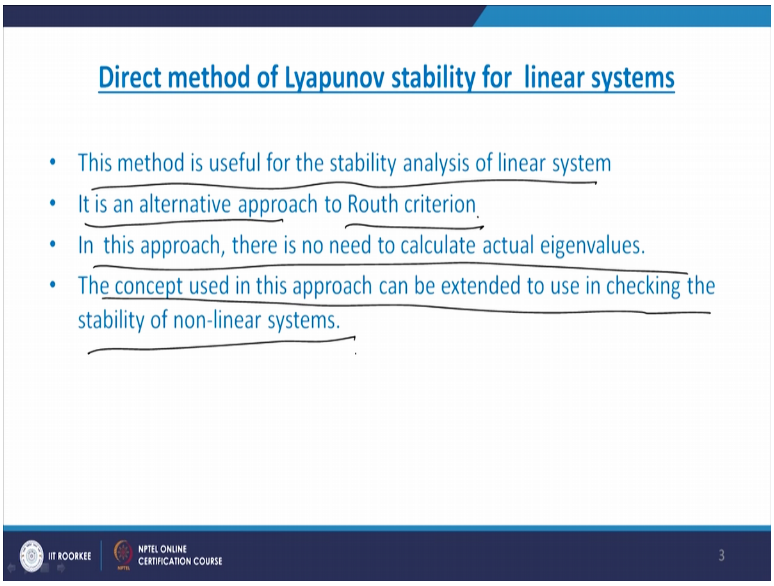

This method is useful for checking the stability analysis of linear system. It is an alternative approach to Routh criteria. In this approach, there is no need to calculate actual eigenvalues because you know eigenvalues plays important role in stability. If eigenvalue on the right hand side system a stable left hand side is a stable, but here there is no need to calculate the actual eigenvalues. The concept used in this approach can be extended to use in checking the stability of non-linear system; that is the concept of direct approach can be extended for checking the stability of non-linear system.

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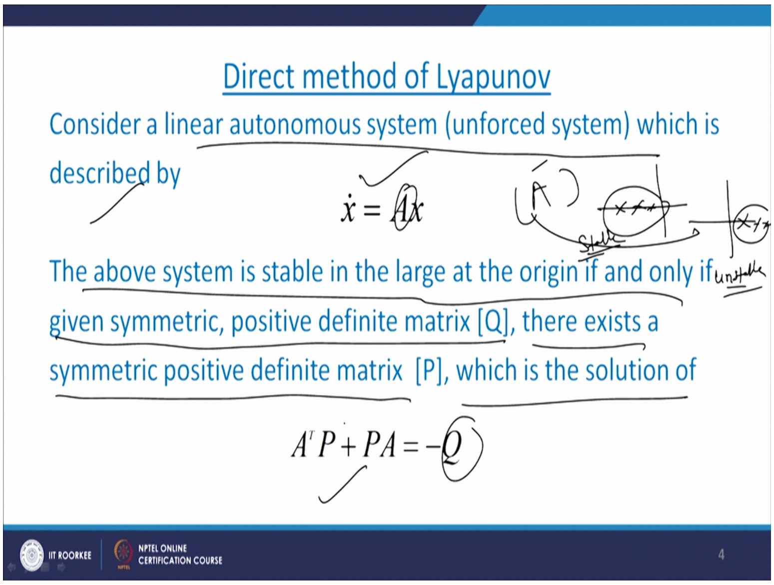

Now, what the direct method of Lyapunov. So, consider a linear autonomous system unforced system which is described by x dot equals to Ax, A is the system matrix. And how to take the stability of this? Stability of this system is governed by A matrix. If you calculate the eigenvalues of A you can create the stability of the systems, that is if the eigenvalues of A this is left side of the s plane system is stable and if the eigenvalues of A lies on the right hand side system is unstable.

Now, we are seen the alternative approach. So, it is said that a Lyapunov direct method said that the system is stable in the large at the origin if and only if given symmetric, positive definite matrix Q, there exist a symmetric positive definite matrix P, which is the unique solution of A transpose P plus PA equals to minus Q. So, it is say that if you consider Q as positive definite, you have to consider Q as positive definite matrix then determine the values of P that determine the P matrix. And it is observe that if P is coming comes to be positive definite, if P is positive definite then we can say that this A is a stable system.

And whenever we take P, P we have to consider as a real symmetric matrix that is a rows and columns are same. So, that is very important criteria for checking the stability of the system and you will find at this particular simple formula A transpose P plus PA equals to minus 2, it is used in various applications. It is useful in a checking the controllability observability of the system, it is also useful in a model order resistance. So, this approach has multipurpose applications, it is not only for stability for other control expects.

Now, this A transpose P plus P equals to Q; that means, when Q we consider as a positive in matrix then calculate the P, if P comes positive definite then we can say that this system a is stable, otherwise system is unstable. And this gives necessary and sufficient condition for the stability of the system. Now, question has come in mind how this equation has come, how this A, how we can say that A transpose P plus P equals to minus Q; now we start deriving this.

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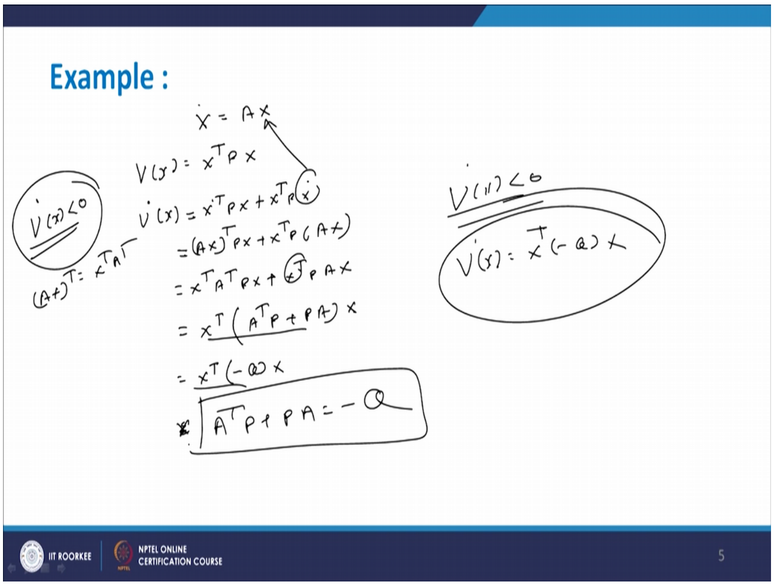

Now, we take say x dot equal to Ax this is a plant is system. Now, what we take in Lyapunov the concept is based on the function that is a positive definite function and then positive function we can express in a quadratic form. Because, we have because when the given system in a quadratic form that particular matrix in that form is always a symmetric matrix. Therefore, here we are considering V of x equal to x transpose P x, this is in quadratic form this P is real symmetric matrix.

Now, the concept of Lyapunov is based on the energy V of x function. Now, when this system become unstable sorry when the system is stable, when this system is stable when V dot x is less than 0; there is a concept of the energy. So, V of x you have taken x transpose P into x now, we want V dot x. So, what we do? We can we will determine V dot x this V dot x that is equal to x dot transpose P x plus x transpose P x dot, that is the normal methodology of differentiation we are used that is x dot transpose P x plus x transpose P x dot.

So, what we are knowing we are knowing x dot equals to x. So, when we, but here x dot transpose. So, we can replace x dot equal to x and we make it transpose. So, here we can write down as x dot equals to Ax transpose into P x plus x transpose P, what is x dot x dot equals to x we replace here. Now, here A x transpose so, A x transpose is nothing, but x transpose A transpose. So, here we can write down as x transpose A transpose P into x plus x transpose P into A into x.

Now, after this equation can be written as x transpose, here x transpose, here x transpose we are taking outside. So, we can write down as A transpose P plus PA and x is there so, you can get x outside. So, x transpose A transpose P plus PA. Now, here what is here A transpose P PA that we have got, but here it is say x transpose P plus P equals to minus Q. How this Q will come?

Now, why this Q will come because for stability of this A V dot x must be negative definite, that is V dot x must be V dot x must be less than 0. So, here in order to get V dot less than 0 what we can write x transpose, I can write this is a Q into x. So, now V dot x x transpose minus Q x, if this is the case then our A is stable therefore, we have written x transpose minus Q into x.

Now, we can equate this equation. So, after equate this equation we can we will get A transpose P plus PA equal to minus Q. So, this is the final result; that means, in order to check this stability of A we have to assume Q as positive definite matrix and then calculate the P. If P comes to be positive definite, then system is stable otherwise system is unstable. Now, we have to see some example based on this and then we see the what is the converse Lyapunov approach. So, be prior to this one we start solving one example.

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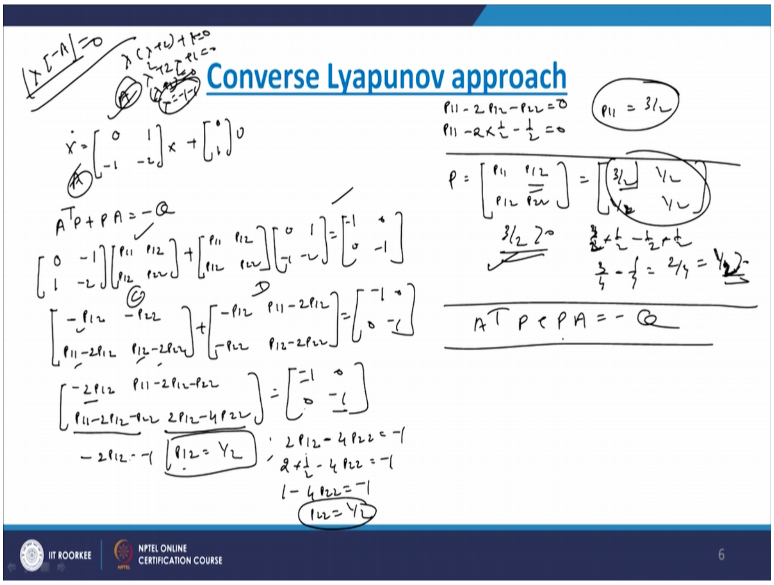

Now, here we start solving it. So, x trans x dot equal to 0 1 minus 1 minus 2 x plus 0 1 u, this is the system x dot equals to 0 1 minus 1 minus 2 x plus 0 1 into u. Now, we have to check this stability of this system and concept which we had to use that is Lyapunov direct approach ok.

Now, we are written the state space model as 0 1 minus 1 minus 2 x plus 0 into u. Now, if you want to check the stability of this system. So, how to check the stability of this system? The stability of the system you can take this A, you can take this as A A matrix and we will calculate the eigenvalues of this matrix. If you write down this eigenvalue of the matrix has lambda I minus A equal to 0, you can get the eigenvalues of the system.

So, if you solve this so, what we will get will get the eigenvalues of this system as lambda plus lambda plus 2 plus 1 equals to 0. So, we will get it as lambda square plus 2 lambda plus 1 equal to 0. So, lambda 1 plus 1 whole square equal to 0 so, lambda equals to minus 1 minus 1. So, it is simply 0 1 minus 1 minus 2 x 0 1 u. So, these A as these A matrix stability is determined by calculating the only the eigenvalues so, system is bound to be stable.

Now, what here we have to do? Here we have to use the concept of the Lyapunov that is a direct approach. So, what we can do here we can write down as A transpose P plus PA equal to minus Q. Now, A transpose P plus PA equals to minus Q. Now, here if you write down this equation as that is A transpose that is 0 minus 1 1 minus 2 and this P we can take it as P 1 1 P 1 2 here P 1 2 here P 2 2 here in P 1 1 P 1 2 P 1 2 here P 2 2. And what is A matrix? A matrix as 0 1 minus 1 minus 2 and Q as 1 0 0 1 and this is minus 1.

So, we have written this equation A transpose P plus P equals to minus Q this equation is there. So, P is the real symmetric matrix that is why we have taken P 1 2 that is equal to P 1 2 that is P 1 2 equals to P 2 1. But, here both are same that is why P 1 2 equal to P 1 2 same like this. Now, we solve this equation. So, after solving this equation we can get result as, minus P 1 2 minus P 2 2 P 1 1 minus 2 P 1 2 this P 1 2 minus 2 P 2 2 plus minus P 1 2 P 1 1 minus 2 P 1 2 here minus P 2 2 here P 1 2 minus 2 P 2 2; this is equal to minus 1 0 0 minus 1.

So, how you you have got this. So, here will find that if 0 is multiplied with P 1 1 minus 1 multiplied P 1 2. So, we will get P 1 2 then here 0 and minus 1 is multiplied with respect to P 1 2 and P 2 2. So, we got 0 multiplied by P 1 2 0 so, will got P 2 2. Similarly, here 1 is multiplied by P 1 1 minus P 1 1 then this minus 2 multiplied P 1 2, we will get this. Then similarly, here 1 multiplied by P 1 2 we will get this and this minus 2 into P 2 2 we will get this. Similarly, this side we have got these results and now we again solve it.

So, after solving we will get minus 2 P 1 2 P 1 1 minus 2 P 1 2 minus P 2 2 then here P 1 1 minus 2 P 1 2 minus P 2 2 and here we will get 2 P 1 2 minus 4 P 2 2. And here we will get again same minus 1 0 0 minus 1, that is what we are done we have arrayed the elements of this both matrices; this is let us say matrix is C and D, the C and D we are added so, we have got this results. Now, what we want? We want respective values of P 1 1 P 1 2 and P 2 2. So, what we will get? So, we can get it as minus 2 times P 1 2 equals to minus 2 so, this equal to this. So, what we will get? We will get P 1 2 equal to 1 by 2.

Now, we have got P 1 2. So, what we want? We want P 1 2 and with as P 2 2. So, here what we will get here we will get this 2 P 1 2 minus 4 P 2 2 equals to minus 1. So, if we compare this equation we can easily get P 2 2. So, here we solve as 2 times P 1 2 minus 4 times P 2 2 equals to minus 1 now, you replace the P 1 2 in these equations. So, you will get it as 2 into 1 by 2 minus 4 into P 2 2 equals to minus 1 now, here 1 minus 4 times P 2 2 equal to minus 1. So, finally, P 2 2 at 1 by 2. So, we have got P 1 2 P 2 2 now, what to be determined that is P 1 1.

So, here we take this equation where to compare with this. So, here we can write down as P 1 1 minus 2 times P 1 2 minus P 2 2 equal to 0. Now, here P 1 1 minus 2 P 1 2, what is P 1 2 P 1 2 is 1 by 2 and P 2 2 is also 1 by 2. So, we can write down like this equal to 0 and after solving these we will get as P 1 1 equal to 3 by 2. So, we got all values of P 1 1 P 1 2 and P 22.

So, now we write down in a matrix form this P equal to P 1 1 P 1 2 P 1 2 P 2 2 and here we can write down as this P 1 1 is 3 by 2. What is P 1 2? P 1 2 is 1 by 2 1 by 2. And what is P 2 2? P 2 2 is also 1 by 2. So, we got a P and what is the concept of stability, we have to check the definiteness of the P. So, we will find that in order to take the definites we have to use the Sylvester criteria and for Sylvester criteria, we have seen earlier that we have to take principle minors. So, here you find out first principle minors 3 by 2 is greater than 0 and next about other principle minors is complete determinant.

So, 3 by 2 multiplied by 1 by 2 minus 1 by 2 into 1 by 2 that is what we will get here 3 by 2 multiplied 1 by 2 1 by 2 into 1 by 2. So, 3 by 2 into 1 by 2 minus 1 by 2 into 1 by 2. So, we will get at 3 by 4 minus 1 by 4 so, we will get 2 by 4 that is equals to equals to 1 by 4 and it is greater than 0.

So, we will find that this particular matrix P is positive definite because a 3 by 2 greater than 0. Similarly, after solving this then 3 by 2 multiplied 1 by 2 minus 1 by 2 into 1 by 2 that is 3 by 2 into 3 by 4 minus 1 by 4 so, 2 by 4 1 by that is equals to 1 by 2. So, 1 by 2 greater than 0 3 by 2 greater than 0, it means that this system A which we are consider is a positive definite system, that is your given system matrix A is stable system. That is defining Q calculating the P, we have got a system a stable system this is example.

Now, here we have taken the case just like A transpose P plus PA equals to minus Q. Here we have defined Q and calculate the P, can we do the reverse that way that is, in order to check the stability of A, what we will do we? We will consider initially P as a positive definite matrix and we calculate the Q matrix and we see what result we will get. So, in order to tackle this we see the example now, we start with an example.

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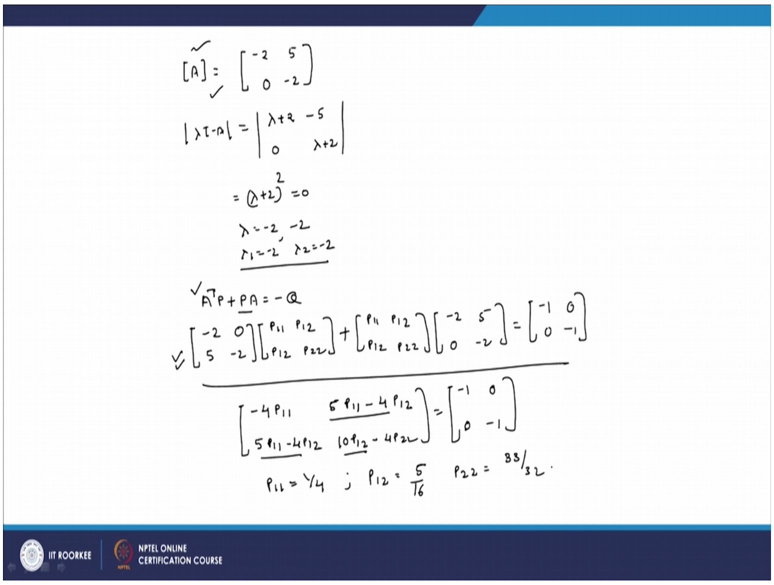

Suppose matrix A is given as minus 2 5 0 minus 2. Now, we check the stability of this system A by conventional approach, that is we will write lambda I minus A. So, it is given as lambda plus 2 minus 5 0 lambda plus 2. So, after solving this we will get lambda plus 2 whole square that is equal to 0; that means, lambda equal to minus 2 and minus 2; that means, we can say that first lambda 1 equals 2 minus 2 lambda 2 equal to minus 2.

Now, this system is stable now, we check the stability of this matrix A using Lyapunov approach, that is we write A transpose P plus PA equal to minus Q. So, here we will use the direct approach; that means, in direct approach we assume Q as positive definite matrix and then we calculate the P matrix. If P is positive definite then we can say that system A is stable. So, now, we replace A as minus 2 0 5 minus 2.

So, this is basically a transpose of A that is this A transpose and this is A. So, this particular part is nothing, but the transpose of A. Now, the P matrix P 1 1 P 1 2 P 1 2 P 2 2 now, this P we have written now what this PA. So, we write as P 1 1 P 1 2 P 1 2 P 2 2 and A matrix as minus 2 5 0 minus 2 and this is equal to minus Q means minus 1 0 0 minus 1. Now, we have to solve these equations. So, after solving this equation we will get minus 4 P 1 1 5 P 1 1 minus 4 P 1 2 5 P 1 1 minus 4 P 1 2 10 P 1 2 minus 4 P 2 2 equal to minus 1 0 0 minus 1.

Now, equate minus 4 P 1 1 equal to minus 1 these equal to 0, this part equal to 0 and these equal to minus 1. So, after solve after solving this we will get P 1 1 equal to 1 by 4 P 1 2 equal to 5 by 16 and P 2 2 equal to 33 by 32.

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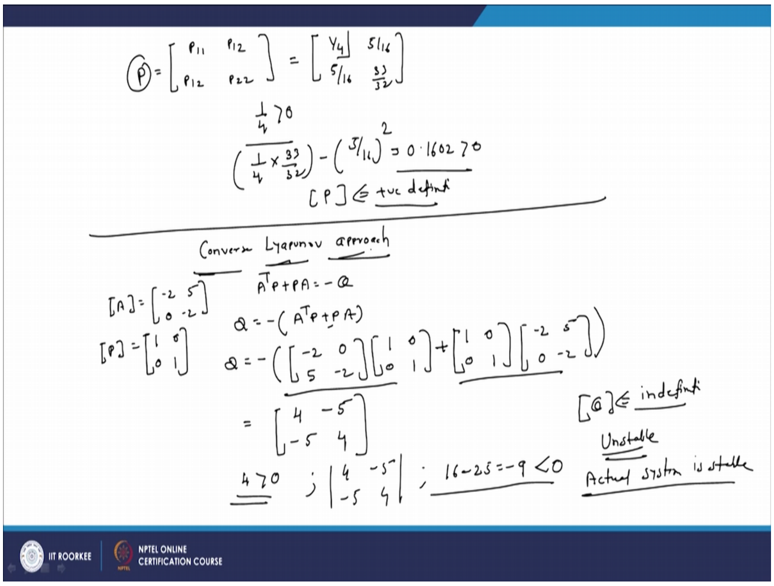

So, now this P can be written as P 1 1 P 1 2 P 1 2 and P 2 2 that is equal to 1 by 4 5 by 16 5 by 16 33 by 32. So, now, we use the concept of Sylvester's so, principle minors will take as 1 by 4. So, this 1 by 4 greater than 0 and here second principle minor that is complete determinant. If you are write we will get 1 by 4 multiplied by 33 by 32 minus 5 by 16 whole square equal to point 1 6 0 2 and this is greater than 0.

So, we will find that all the principle minors are greater than 0, it means that this P is this P is positive definite. And therefore, we can say that by means of direct method the system is stable. Now, we try for a converse Lyapunov approach, that we will try converse Lyapunov approach. So, in converse Lyapunov approach we write A transpose P plus PA equal to minus Q.

So, in the converse Lyapunov approach we assume P as positive in matrix and we calculate the Q, at the Q is positive definite the system is stable. So, as per the convention we should get Q as positive definite, when we are considering P as positive definite. Now, we see for this problem what results we will get. So, here we take same A same A matrix as minus 2 5 0 minus 2 and now, this Q equal to minus A transpose P plus

PA and in this case the P which we have to assume as positive definite. So, we write down as 1 0 0 1.

So, now we calculate Q so, Q equal to minus A transpose means, here minus 2 0 5 minus 2, P matrix as 1 0 0 1 plus this P matrix as again same thing 1 0 0 1 and here A matrix as minus 2 5 0 minus 2. Now, you multiply this, then multiply this and finally, we will get Q as 4 minus 5 minus 5 4 now, we again check the principle minors for this matrix Q. So, my first principle minors is 4 greater than 0 and about the second principle minors we will find that will 4 minus 5 minus 5 4. And here, we will get 16 minus 25 equal to minus 9 and which is less than 0; that means, this particular matrix Q belongs to indefinite.

And therefore, as per converse Lyapunov approach the system is unstable, but actual system is stable actual system is stable. Therefore, for any system when we want to check the stability we had to use direct Lyapunov approach, we cannot use converse Lyapunov approach; that is for some class of problem converse Lyapunov approach make you incorrect result.

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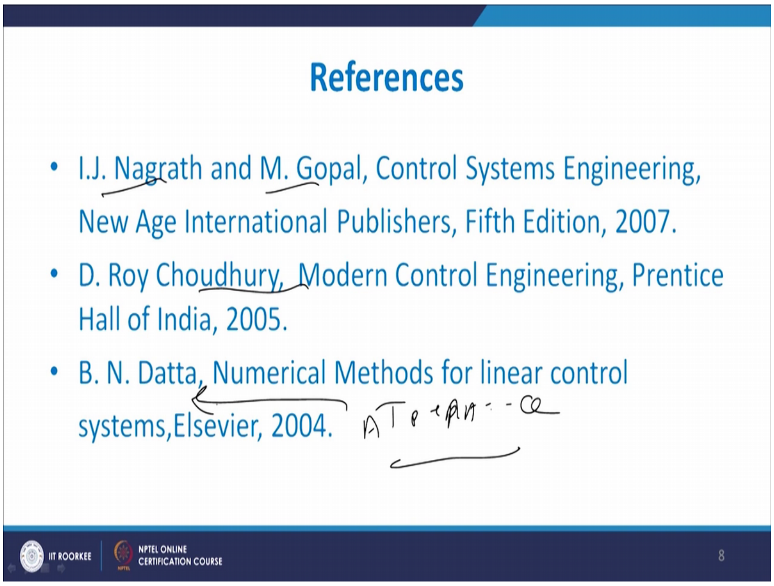

Now, these are some references J. Nagrath, Gopal, D. Roy Choudhury. And we will find at the more application of this A transpose P plus PA equals to minus Q is given in this B. N. Datta and again more application is also given in this particular book.

Thank you.