Advanced Linear Continuous Control Systems
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Lecture – 19
Stability Analysis in State Space:
Lyapunov Stability Analysis (Stability Criterion) Part-IV

Now, we start with Lyapunov Stability Analysis.

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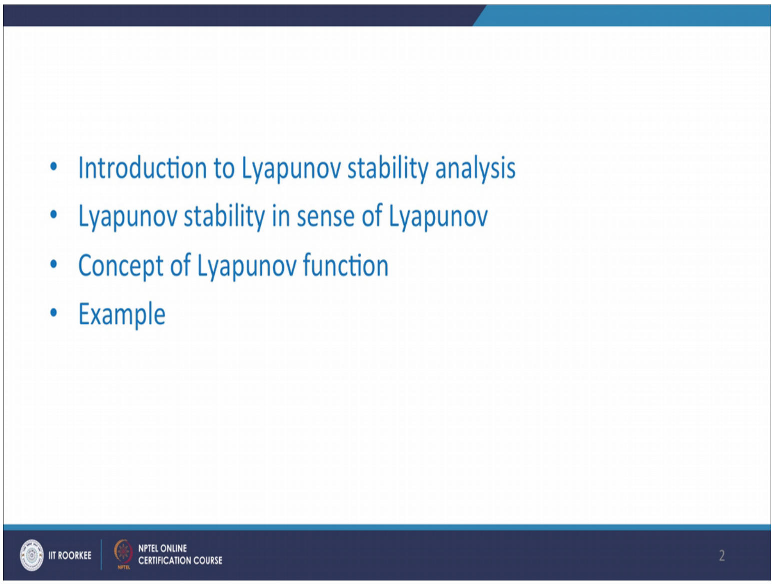

In this, we have study introduction to Lyapunov stability analysis, Lyapunov stability in sense of Lyapunov, concept of Lyapnov function and example.

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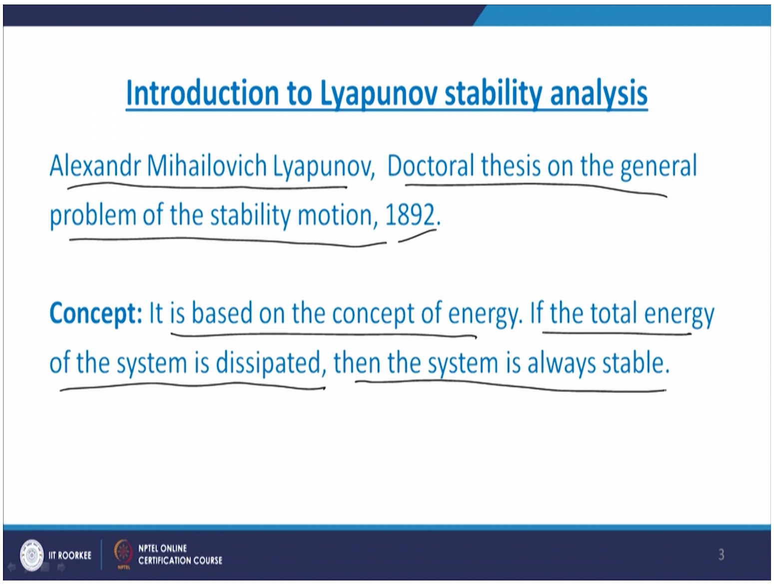

Now, about the introduction to Lyapunov stability analysis. This Lyapunov stability analysis had been develop in 1892 by a scientist name A. M. Lyapunov and because of his name the this criteria has been called as Lyapunov stability analysis and this work he has published in his doctoral thesis that is in 1892.

So, Alexander Mihailovich Lyapunov, that is A. M. Lyapunov doctoral thesis on general problem of the stability motion and in 1892. Now, what is the concept? It is based on the concept of energy if the total energy of the system is dissipated then the system is always stable,.

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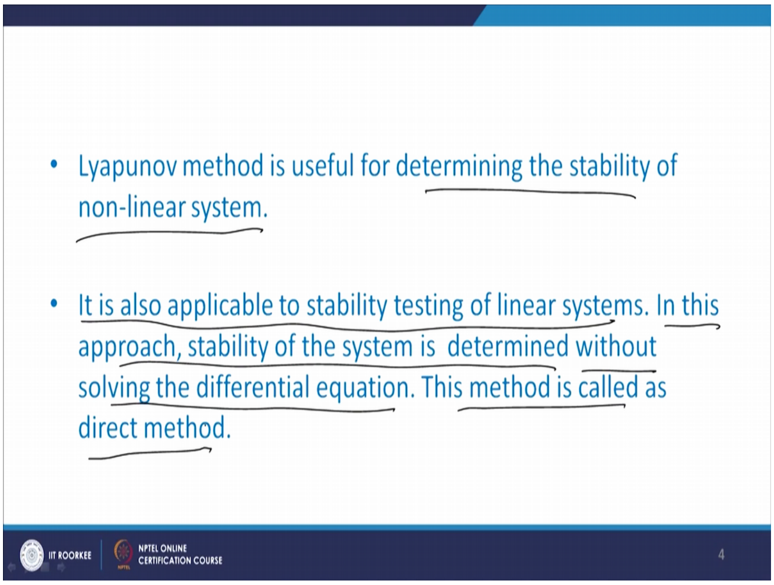

This Lyapunov method is useful for determining the stability of non-linear system. Now, another point it is also applicable to stability testing of linear systems. In this approach, stability of the system is determined without solving the differential equation. This method is called as direct method. So, we have seen some points. The main important thing is that this Lyapunov stability is based on the concept of energy and it is applicable for both linear as well as non-linear system.

Now, you know most important thing is that if you see the literature there are many criterias have been developed or are available in the literature for stability of a linear system, but as far as the non-linear system is concerned the there are very limited criterias and this Lyapunov is basically useful for checking the stability of such a non-linear system.

Now, just we see the what is the concept of this energy. Suppose I have some money or the government has given a salary in the first say first day of the month. So, so some salary I have got now what was happened, till last date of this month upon that month my salary has been increased.

So, what you say you have salary in the first day of month which is high, but in thirty as last date of that particular month salary has been increased. So, you will say that this is not possible; that means, whatever we have salary as time progresses that must decrease if it is not decreasing that please there is something problem.

Another example is we can say suppose we are driving a vehicle it may be and in that case what fuel we are using be using as a diesel or a petrol. So, we have started drive your vehicle and what is happened after certain time interval you check the petrol or diesel that is the increased, then what you say? How can it possible? So, that means, that is not a correct phenomenon. So, what Lyapunov say that whatever the energy is there as time progresses that energy must decrease.

So, if the energy is decrease then in that case the system is stable, if it is not decrease it means that system is not stable. So, that is the concept of the Lyapunov. So, this is a very important concept which is given by the Lyapunov. Then the question has become how it can be useful in checking the stability of non-linear system. See here whatever system is there, system is more of different functions out of these functions some are very important, some are not important.

So, what we are thinking if you take the specific function and, but what will function if you check the stability then if it is follow the concept of energy then you can say that your system is stable. Just say you take another example suppose a project has been given to two students and they have started doing work, but suddenly one day it is found at is to one student saying that it is not possible to adjust with that particular students then what I will do I will identify the problem what is the problem in that.

But, in that case I have found at that particularly in that part has been comfortability for a studying or developing some algorithm both have same opinion. So, what I will do I will say that ok, you be there only for that work and do not meet afterwards that means, that particular function we assume and I am checking and we have to check the stability around that function. So, what will happen around that particular function if the system behavior like just like energy is there, it decreases then we can say that your system is stable.

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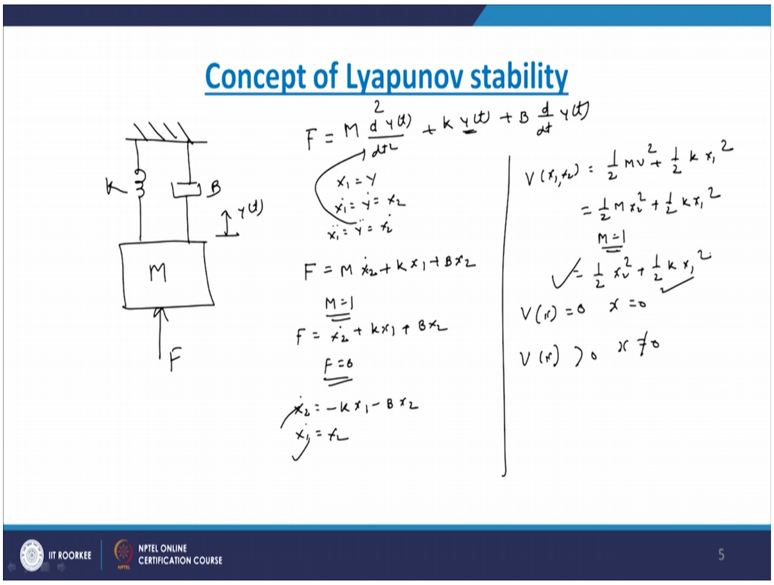

Now, we see one practical example; now, we take one spring mass damper system example we have already seen that example earlier. So, we have spring and this is a damper and here is a mass this M and if force applied this is K this is B and let us say displacement Y of t and force F is applied.

So, last time we have seen how to determine the equation of motion for such type of system. So, now, I am again writing this equation. So, F equal to M d square y of t divided by dt square plus k function of Y of t plus b d dt of Y of t. So, Y is a displacement.

Now, we have to assume the state variable. Suppose, we have taken as x 1 equal to Y now what is x 1 dot x 1 dot equal to Y dot equal to x 2, then x 1 double dot equal to Y double dot equal to x 2 dot. So, x 2 dot equal to Y dot and Y dot is like the here is shown here. Now, we can represent the equation in this way F equal to M this d square dt square of Y of t that is x 2 dot plus k into x one this is x 1 plus this is Y dot that is equal to x 2.

Now, suppose if you take M equal to 1. So, your equation become F equal to x 2 dot plus k x 1 plus B x 2. Suppose, system in autonomous. So, let us say F equals to 0 1 for system. So, now, the equation we can we can written as x 2 dot equal to minus k x 1 minus B x 2 and here original equation x 1 dot equal to x 2. Now, this equation has been formatted applied force is 0 M is 1.

Now, total energy of the system consists of the kinetic energy of move inverse and potential energy of the spring. Therefore, here we are writing down one function which is respect to kinetic energy as well as potential energy. So, how you write down? Now, we can write down as v of function of x 1, x 2 equal to 1 by 2 MV square plus 1 by 2 k x 1 square, this is M for mass, k is for spring.

Now, this equation can be written as 1 by 2 M x 2 square 1 by 2 k x 1 square. Now, here what we are doing we are assuming M is equal to 1. So, we have return M equal to 1. So, above equation can be written as 1 by 2 x 2 square plus 1 by 2 k x 1 square. Now, here V V of x equal to 0 for x equals to 0 and this V of x is greater than 0, then x is not equal to 0. Now, we will calculate or we will determine the rate of change of energy.

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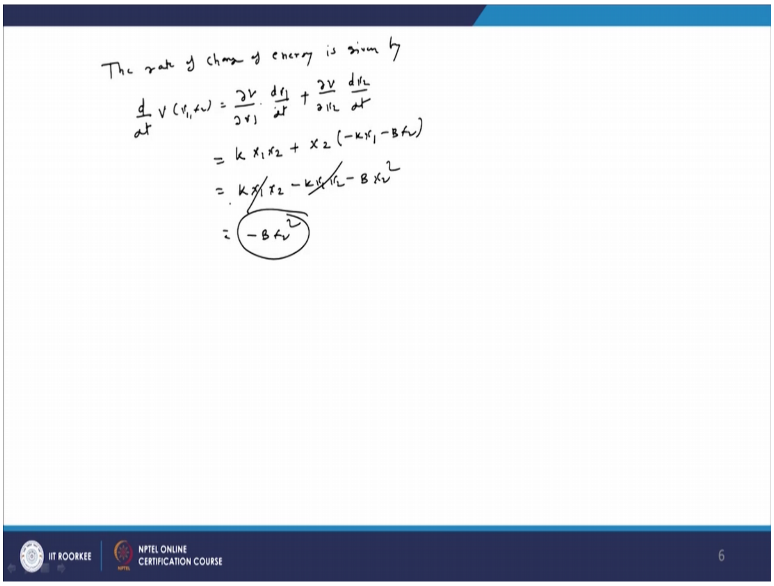

So, it can be written as, the rate of the rate of change of energy is given by it is given as differentiation of t V x 1 x 2 equal to del V by del x 1 into dx 1 of dt plus del V by dx del x 2 multiplied by dx 2 by dt.

Now, this equation we have written we have got del V by del x 1. So, V function we have already taken here and what about dx 1 by dt and dx 2 by dt. So, dx 1 by dt equal to x one dot and dx 2 2 by dt equal to x 2 dot. So, we have already values of x 1 dot x 2 dot. So, we replace the values of x 1 dot x 2 dot in this equation and this del V by del x 1 and del V by del x 2 we calculate from this we calculate from this.

So, finally, we can get equation as. So, del V by del x 1 will get with as k x 1 x 2 plus del V by del x 2 equal to x 2 minus k x 1 minus Bx 2. Now, after solving what will get k x 1 x 2 minus k x 1 x 2 minus B x 2 square. So, this will cancelled. So, finally, we got B x 2 square.

So, if you have function and if you differentiate this function if it decreases it means that your energy is decreasing what I am saying if you have function if you define and we take rate of change of it if it decreases that means, that your energy decreases. So, you will find at for this spring (Refer Time: 12:15) we have taken a function, we are differentiate it and we will find it this always negative for when x 2 is nonzero. So, here we will find at this system is stable system.

Now, we have to see the some definitions that is Lyapunov stability that in sense of Lyapunov. As we have seen the different definition of stability that is an input is bounded output is bounded then when the that is called stable then if your roots or Eigen value on the left hand side system stable, right hand side unstable. So, here these there are some definition with spectrally Lyapunov. So, we will see what are the definitions.

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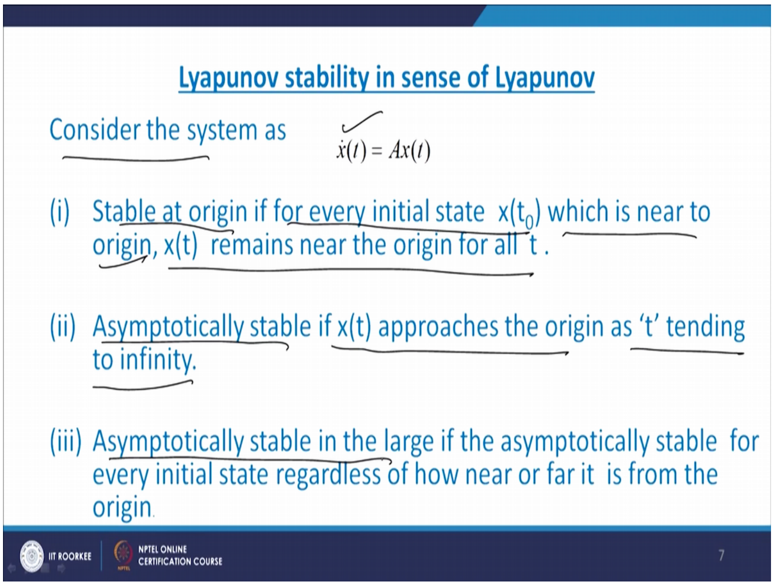

Consider the system as this x 1 dot equal to A; A of x t this is a unforce systems here this is no u input is 0. So, this system is stable at origin if for every initial state x of t 0 there are some initial state which is near to origin, origin is the flip point let us say state which is near to origin this x of t remains near the origin for all t, that is, when the system is stable.

Second system is asymptotically stable if x of t approaches the origin as t tending to infinity. What happened we have say origin now initial state. So, earlier what happen the stay started at till remain near to origin, but now at td into infinity whatever stay started it will approaches towards the origin that is asymptotically stable.

Now, another point is that asymptotically stable in the large how to define it. So, if you take different initial state for every initial state the trajectory or state will move towards the origin then we can say that it is asymptotically stable in that; that means, for whatever be the initial state for every initial state system is approaching towards the origin or trajectory approaches towards the origin, we can say that system is asymptotically stable at large.

Now, you see this definition in different aspect.

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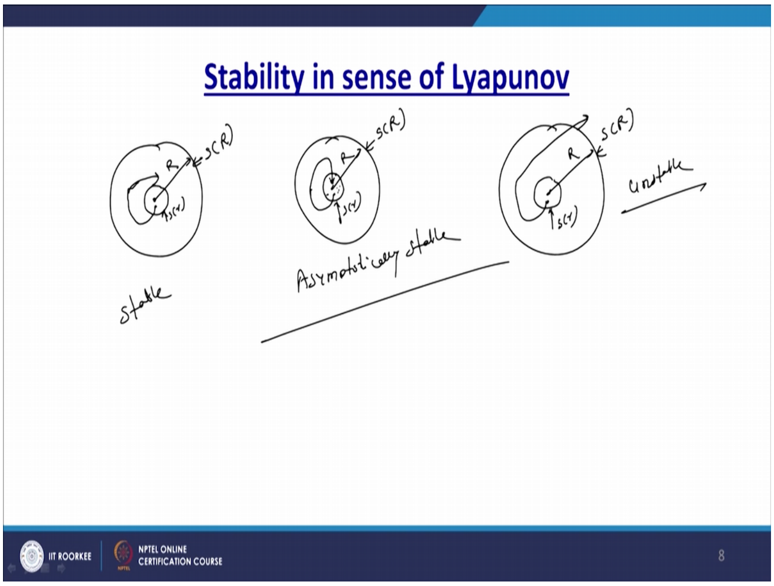

Now, we take one region this small region and we take another regions and what we want that that this particular region is the stable region; that means, whatever trajectory is should remain in that region and in small regions which is defined as S of small r and hideous here is this R, under this is origin, this is the equilibrium point.

Now, you take them on trajectory. Trajectory let us say which is near to origin which is near now it is start weighing. So, when you start weighing it will moving and it will not coming back, but it will remain here; that means, here is system is only stable system is only stable. Now, we take another region. So, the initial or in any other region outer region we have determined now here again this is R. So, region is S of R here, S of small r.

Now, now here a trajectory again which is near to origin and it is start moving and finally, at t tend to infinity it is approaches toward origins here it is approaches like this if this is the condition this is call asymptotically stable system.

Now, you take same regions, inner regions and now here is a outer regions again this is radius R this is region S of R now here is this small region S of r small r and now here is the origin. Now again the trajectory now trajectory which is near to origin starting of here it is not coming back and now it is moving outward. So, this is known as unstable, this is unstable. This is only stable and this is asymptotically stable.

Again, if you find at here if you take this initial initial condition for different initial condition here, here, here if the trajectory again going towards the origin then it is called asymptotically stable at large. So, we will see it here.

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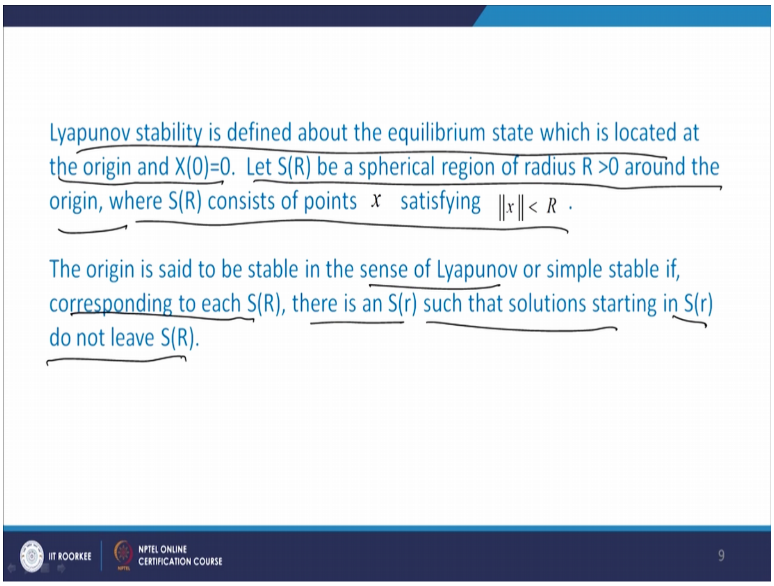

Lyapunov stability is defined about the equilibrium state which is located at the origin and x of 0 equal 0. Let S R be the spherical region of radius R greater than 0 around the origin, where S of R consist of points x satisfying norm of x less than R; so, this is the case. See here this is the case and this origin is said to be stable in the sense of Lyapunov or simply stable if corresponding to each SR, S of R there is S small r such that solution starting in S of r do not leave S of R.

So, will find at here the solutions started from here it will remain in region it is not going outside to that is why this we call as the system is a stable system.

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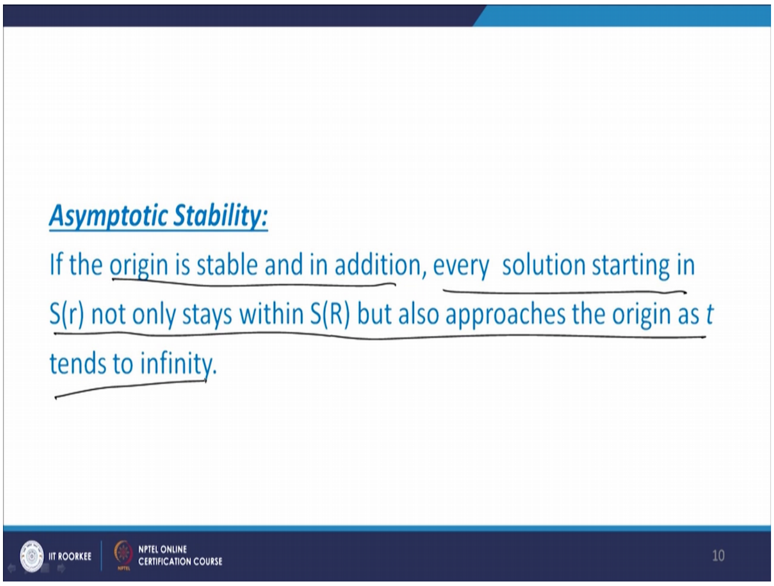

And, asymptotic stability if the origin is stable and in addition, every solution starting in S of r not only stays within S of R, but also approaches the origin as t tends to infinity. That means that what are the initial state it always approaches to the origin then we call as it is asymptotically stable system.

Now, we start solving some example based on this concept. As I we seen that this Lyapunov stability is basically applicable for both linear as well as some non-linear system. Now, first of all we see the stability testing for a non-linear system. So, here we are taking an example say.

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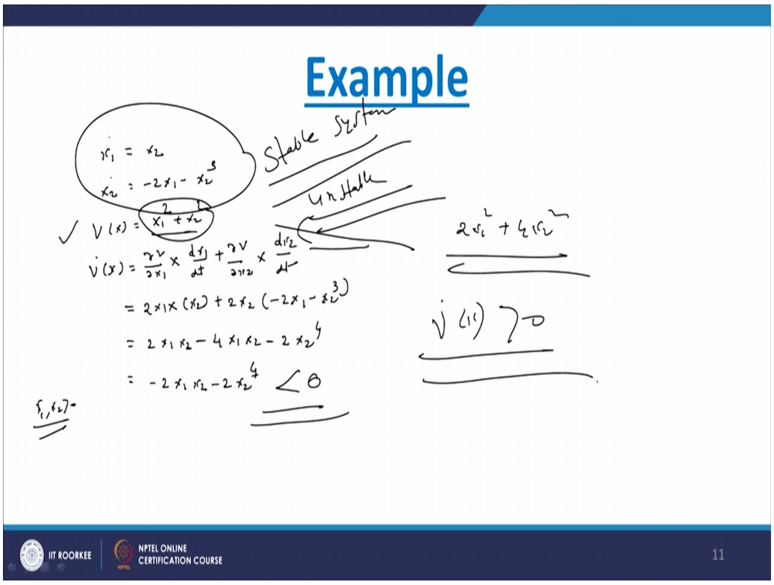

x 1 dot equal to x 2 and x 2 dot equal to minus 2x 1 minus x 2 whole cube. Now, this is the x step is followed which is non-linear form.

Now, we have to check the stability of the system, now how to check it? As I have told you that so, non-linearity means that non-linearity means for any system non-linearity just it is not like that for it will remain everywhere. So, for small parts to maybe linear, therefore, what we are doing we have to assumption function. Now, here basically function which we have to assume that must be positive definite function that is where last time we have seen the how to check the definiteness.

So, here our main purpose is to assume a positive definite function. So, you know find at earlier also this particular function this is also a positive different function that is for all x 1 and x 2 not equals to 0 this view of x 1, x 2 always greater than 0 except at different point when x 1 equal to 0, x 2 equal to 0. So, here we have to assume a Lyapunov function.

So, which is the Lyapunov function we assume. So, the best Lyapunov function we assume that is we can assume V of x equal to x 1 square plus x 2 square. So, we will find at if you x 1 and x 2 greater than 0 V of x is always greater than 0. So, that is a positive definite function which is very simple one. Therefore, when you solve the example generally we assume this particular function. So, we have taken V of x equals to x 1 square plus x 2 square.

Now, we have to check the stability of the system. So, here what will take what will do V dot x equal to del V of del x 1 multiplied by dx 1 dt plus del V by del x 2 multiplied by dx 2 by dt because we have as I we have discussed earlier we have take a function it is if take the derivative of the function if it decreases means that energy decreases that is why here we also do the same thing beyond to the V dot x and we have to see that whether that V dot x is negative or not or in other words you have to see that whether V dot x is negative definite or not.

So, here we have written in a equations now we replace the one is of del V by del x 1 del x 1 by dt del V by del x 2 also dx 2 by dt. So, here for the value of del V by del 1. So, del V by del x 1 if you see here this equation. So, what you get you get 2 x 1 multiplied dx 1 by dt that is x 1 dot and that is equal to x 2 plus del V by del x 2 del V by del x 2.

So, here which is x 2 to x 2, so, what we will get will get 2x 2 and del x 2 by dt; that means, x 2 dot you will get it as minus 2x 1 minus x 2 whole cube. Now, we solve it. So, what we will get 2 x 1 x 2 here 2 into 2, 4 minus 4 x 1 x 2 and here minus 2 x 2 raised to 4 and now, here if you solve it you will get it as 2 x 1 x 2 minus 2 x 2 raise to 4. So, we got V dot x.

Now, here you see you take any values of x 1 x 2, but which must be greater than 0; that means, if you take this x 1 x 2 x 1 x 2 greater than 0, for any of these values of x 1 x 2 which is greater than 0 you will get V dot x is always less than 0 except one condition when x 1 equal 0 or x 2 equal 0. So, if this is the case it means that V dot x is negative definite less than 0 it means that our system which we are consider this is a stable system this is stable this is stable system.

But, here these of issue because here we have taken one specific function so, it may possible that if you take a different function it may possible that this system is unstable maybe may come unstable. Suppose, what I am saying that each should have this function we have taken some different function. Let us say we have taken this function as 2x 1 square plus 4x 2 square, this also positive definite function; if you take it by chance after taking this one if you got V dot x. Let us say greater than 0 then you say that system as stable it means that here for V of x. We can get the stable for V dot x we got this particular function we have got unstable system.

It means that that this particularly Lyapunov criteria is only giving the sufficient condition for stability of this team; that means, a stability concern with the function only; that means, if the system is stable that particular function only and if the system is posed to be unstable we cannot do the system is unstable it is it is unstable only for that function only.

So, this is the one important point regarding the stability of the system stability of a non-linear system using the Lyapunov method at the most important point that the Lyapunov approach is only giving the sufficient condition of stability of the system and mainly it based on particular function for the particular function if the system is stable then we can say that the system is stable with respect to Lyapunov for that particular function only.

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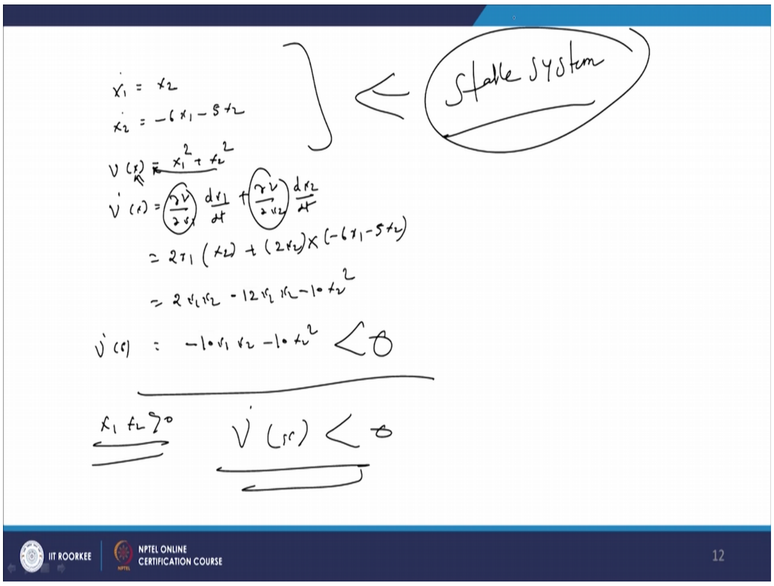

Now, we take another example let us say x 1 dot equal to x 2 x 2 dot equal to minus 6x 1 minus 5x 2 then we take Lyapunov function as a simple function we can take it as V of x equal to x 1 square plus x 2 square then where we can take V dot x V dot x equal to and then same del V by del x 1 dx 1 by dt plus del V by del x 2 dx 2 by dt,.

Now, we have V for solve it seen here del V by del x 1 that is here and del V by del x 2 is here. So, after solving it you will get 2x 1 here we get x 1 dot that is x 2 plus 2x 2 multiplied by x 2 dot that is minus 6x 1 minus 5x 2 and here after solving you will get 2x 1 x 2 minus x 1 x 2 here minus 10 x 2 square and again here you will get it as minus 10 x 1 x 2 minus 10 x 2 square.

So, again you will find that you can take any values of x 1 and x 2 or any positive values of x 1 x 2 you will find that V dot x is always less than 0, that is, if you take x 1 x 2 greater than 0. So, this V dot x is always less than 0 and we say that this is also a stable system, stable system and you will find that in this particular case there is no need to calculate the Eigen values only we have taken the function and differentiate it you can get the stability of the system and this is very much important criteria.

Even though this criteria is very old say around 130 years back it has been developed, but you find that in the literature on particularly research domain still this criteria has been widely used in application; so, this is a very important criteria.

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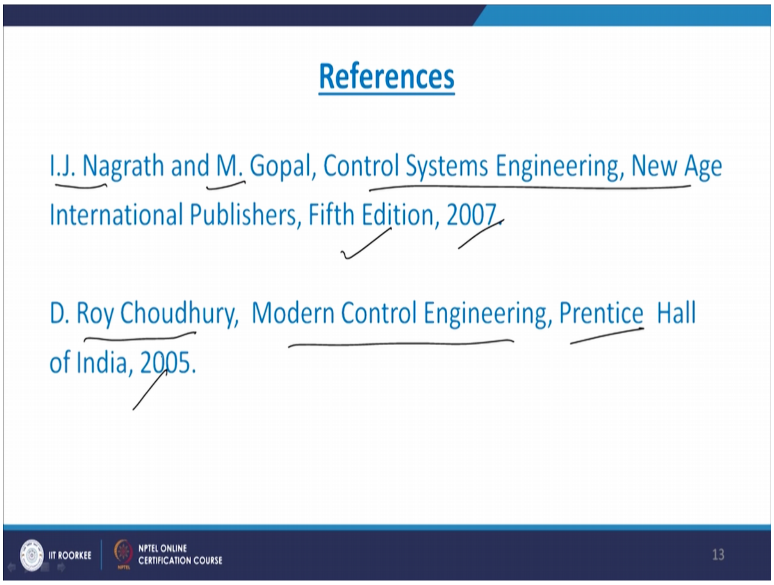

Now, there are some references I. J. Nagrath, M. Gopal, Control System Engineering, fifth edition, 2007. D. Roy Choudhury, Modern Control Engineering, Prentice of Hall India, 2005.

Thank you.