Advanced Linear Continuous Control Systems
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Lecture - 18
Stability Analysis in State Space:
Lyapunov Stability Analysis (Sylvester's Criterion) Part-III

Now we start with Stability Analysis in State Space. Today, we will see the third part Lyapunov Stability Analysis, Sylvester's criteria.

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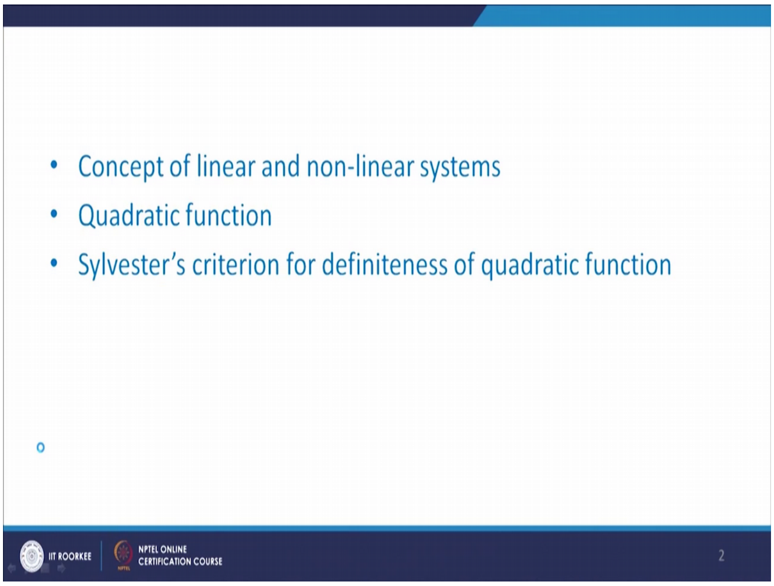

In this, we will study concept of linear and non-linear systems, quadratic function and third one Sylvester criteria for definiteness of quadratic function as are we have to study a Lyapunov stability criteria. But the main important thing in lyapunov stability criteria is, that this criteria is useful for both linear as well as non-linear systems. Therefore, now we have to study what is the difference between the linear and non-linear system?

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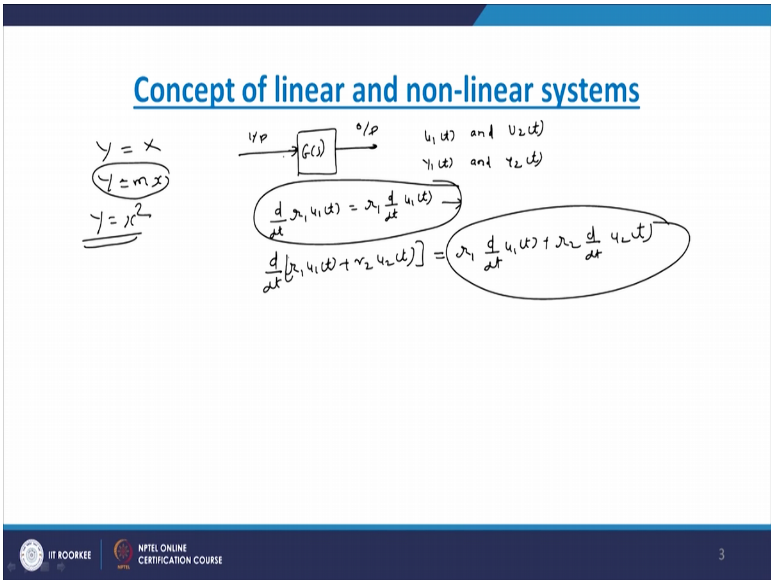

Sometimes we say that y equal to x or Y equal to mx. So, this particular system is called as a linear system, but when we are saying Y equal to X square this is a non-linear system. So, why we are saying this is linear or we or non-linear? Suppose, let us say you have a fine a very good friend in the morning meet you. So, what will happen? He is smiling.

So, morning he is smiling, but when you will meet in the evening we will find that, he is not smiling. He just cried he is not in good mood. So, and you are imagine why this is happened morning. He is smiling and evening he is crying. So, behavior is not good. So, that behavior is called non-linear system. If the behavior of a person is same every time then, this is called a linear system.

Now, what is a difference between this basic difference between the linear and a non-linear system? The first important point is that, in linear and non-linear system that is homogeneity principle, and a super position principle is not applicable in case of non-linear system, whereas, it is applicable in case of linear system. Just see it suppose, I have taken a plant g of s. Now here, this is input and here is the output.

Let us say, the input for this plant is given as u 1 of t and u 2 of t and the output is y 1 of t and y2 of t. So, 2 input 2 output now, I am writing down the equation. Here, like this differentiation of dt r 1 into u 1 of t. So, r 1 is a any arbitrary constants.

So, this I can write down as r 1 differentiation of differentiation with respect to t that is, u 1 of t. And now, if you write down the equation as d of dt r 1 into u 1 of t plus r 2 into u of t u 2 of t.

Similar, to r 1 here, r 1 and r 2 are the arbitrary constants. So now, we can write down this as, r 1 into differentiation of t that is u 1 of t plus r 2 differentiation of t into u 2 u of t. So, this nothing but the principle of superposition and this part is nothing but the principle of homogeneity. So, whenever system is linear both these super position which both means principle of super position and homogeneity are applicable, whereas in non-linear system these things are not applicable. Then, you find that a linear system we are using the concept of Laplace transform; that means a system in discrete form. Then we have to use z transform. So, all these tools are applicable in case of linear system whereas, they are not applicable particularly Laplace transform and z transform are not applicable in case of a non-linear system.

Then, other important point is that, if you see the linear system they have only one equilibrium point. Let us say, I have taken a pendulum here, this is a pendulum, this is point, this is the equilibrium point. So but, if you take the non-linear system, we will find that there are different equilibrium points. Therefore, the analysis of such systems are quite difficult other point is that suppose, if you apply input to the linear system.

Let us say, this is the same system. Now here, some input is applied, input means, let us say step input applied now. If I am applying the step input, definitely we get some output. Now what happened afterward if you change the magnitude of this input?

So, what happened the system which is stable for one particular input? Step input where magnitude have some value. So, the slight changes of the magnitude system become unstable. That means, the stability of the system it depends upon the input as well as the initial states.

Whereas, if you consider a linear system the stability is depend upon the poles or we can say the Eigen value of the systems. And now, the most important point which our focus is on Lyapunov stability criteria. The most important thing is that, Lyapunov criteria is applicable for linear and non-linear, but as far as what criteria you have seen studied earlier? Particularly, a Routh criteria root locus liquids pond, they are difficult to apply for non-linear system. It is difficult it is not possible you can say; whereas Lyapunov is applicable for a non-linear system.

So, in order to understand the Lyapunovs theorem, we should know understand, what is meant by a quadratic function?

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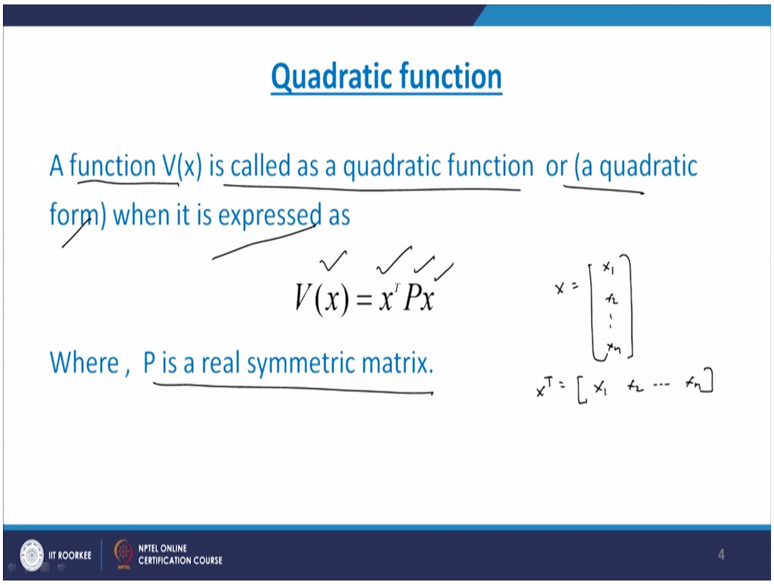

So now, quadratic function what is meant by a quadratic function? Quadratic function V of x is called as quadratic function or a quadratic form. When it is expressed as V of x equal to x transpose Px where P is a real symmetric matrix. P is a real symmetric matrix means, the rows and columns are same.

So, if you take a P as a matrix which is say a real symmetric and X transpose X like this. So, what will happen we will get a quadratic functions. So, this X can be written as X 1 X 2 up to X n. And if you make it X transpose, it is X 1 X 2 X n. Now we will try to study how to formulate the quadratic equation. And how we are saying that this P is real symmetric and then, how this is useful in our Lyapunov stability analysis. So for that purpose, here I am taking a third order of plan that is the order of the P, I am taking as a third order.

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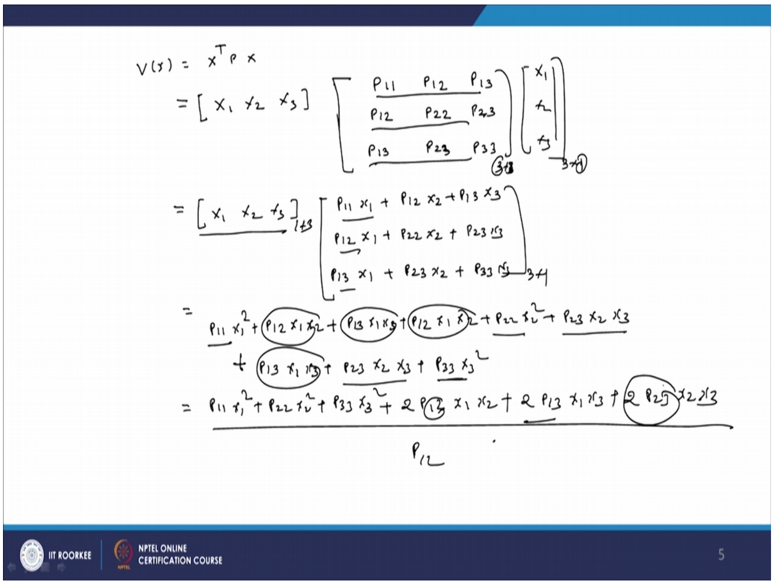

So here, I am writing down V of x is a quadratic function. We can write down as X transpose Px. So, as I am considering a 3 state and other words the order of the P matrix is 3. So, I am writing down this equation as X transpose can be written as X1 X 2 X 3 and this P matrix. We can write down this P matrix as P 1 P 12 P 13 first row.

And now, you just see in the second row I am writing down the elements as P 12 P 22 P 23. This is second row P 12 P 22 and P 23 and you will find that the element. This P 12 and P 12 is same, because, I am considering a P as a real symmetric matrix. Now third row P 13 P 23 P 33 X1 X 2 X 3. So, X 4 X 2 X 3 this is for X transpose, this X is represented by like this. And here, we will find that, this P 13 and P 13 is same P 12 and P 12 is same P 23 and P 23 is same.

Now, we solve this equation. So, we can write down as and here, we can down write down as, now this complete row we have to multiplied with this column. So, how we will get P 11 into X 1 P 12 into X 2 P 13 into X 3? So, we can write down as P 11 into X 1 P 12 into X 2 plus P 13 into X 3 a row column multiplications and afterward it has it has been added.

Now, about the second row second row is P 12 into x 1 plus P 22 into X 2 plus P 23 into X 3 this is second row this column and now the third row with this column. So, we can write down as P 13 into x 1 plus P 23 into X 2 plus P 33 into X 3 here this matrix is 3 by 3 by 3 matrix and what about this, this is 3 cross one.

So, your final matrix it is in terms of rows and column. So, we had 3 rows with one column see here these 3 rows one 2 3 and this one column. So, it is 3 cross one and what about this X 1 X 2 X 3 how many rows one row and 3 columns. So, we can write down as one cross 3 now we again multiply this. So, after a multiplication let us see what we will get; now, you to multiply this with this.

So, we can get it as P 11 into X 1 square plus P 12 P 12 X 1 into X 2 plus P 13 X 1 into X 3 then P 12 X 1 into X 2 then P 22 X 2 square plus P 23 X 2 into X 3 plus P 13 X 1 into X 3 P 23 X 2 into X 3. And lastly P 33 into X 3 square see here this we have multiplied by this we have got this results.

And now we can add the elements and after the adding all this element what will get P 11 into X 1 square this one then this P 22 into X 2 square and lastly this P 33 into X 3 square P 11 P 22 and P 33 plus. So, we will find that this is a P 12 X 1 X 2 this is P 12 X 1 X 2. So, we will get 2 times P 12 X 1 X 2 plus now this here P 13 this is P 13 and here also P 13.

So, we can write them as 2 times P 13 X 1 into X 3 and lastly P 23 X 2 X 3 and this P 23 X 2 X 3. So, we can write it as 2 times P 23 X 2 into X 3. So, this total equation we have solved and we have come across these particular results. And now this we are like this from reverse thing is also possible if this equation you can write down in this form.

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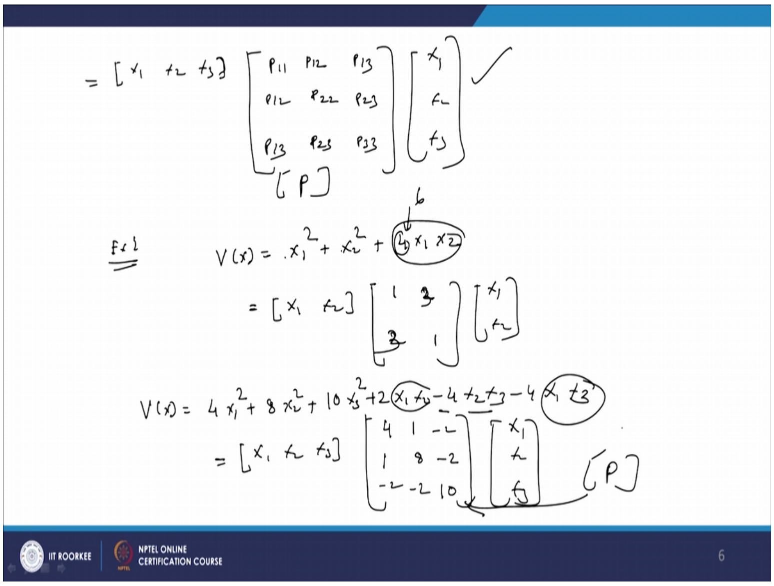

That means here we can write down as same equation as X 1 X 2 X 3 and a about the p elements see here P 11 P 22 P 33. So, this element we have write down in diagonal form.

So, we can write down as P 11 P 22 P 33 now here we have P 12 that is a element P 12 2 times. So, what we can write and P 12 P 12 similarly you will find 2 times P 13. So, how you write down here P 13 and here, P 13 and then P 23 that is, here you find P 23 this part. So, P 23 can be written as, here P 23 and here P 23. And here, X 1 X 2 X 3; that means, the any quadratic function is given that, can be written in this form and particularly this P matrix and this P matrix is a real symmetric matrix.

And why this is important in Lyapunov trans. Lyapunov stability because, if he having a quadratic function. If a P matrix then, it is very easy to check the stability of the system. If the general matrix is there, it is quite difficult that is why for Lyapunov stabilities. This type of quadratic function is required now, we solve 1 or 2 example and we will see how to get this function, suppose example 1, suppose we have written a quadratic function as a V of x equal to X 1 square plus X 2 square plus 4 times X 1 into X 2.

So here, from these equations quadratic functions there are 2 states. Now, what we want we want to write this in terms of matrix form we want to write in terms of a real symmetric matrix? So here, we can write down this as X 1 X 2 and now write the elements of this. So, you see the elements P 11 and P 22, here the elements are 11 and about this X 1 X 2. The elements we can write them as 22. How 22 this is 4 division of this is 22 just like here this is 2 t 1 2? So, we added we have written P 12 and P 12.

So, then what are the elements are there we have to divide by 2 and then, we have to write down. Here suppose here a instead of 4 let us say the element of 6.

Then, what would have we written we have written this is s 3 and, this is s 3 and again here, X 1 X 2 now we will take another example suppose this V of x equal to 4 X 1 square plus 8 X 2 square plus 10 X 3 square plus 2 X 1 X 2 minus 4 X 2 X 3 minus 4 X 1 X 3.

Now, this is also a quadratic function and we have to express this in terms of real symmetric matrix that is in terms of X transpose Px. So, how to write down the P? So, P can be written as here that is here X 1 X 2 X 3. Now write down P matrix, here X 1 X 2 X 3. So first of all, we have to write down the diagonal elements. So, which are diagonal elements, diagonal elements means you just take the square elements, square let us say X 1 square X 2 square X 3 square.

So, we write down the elements 4 8 8 and diagonal form that is, we can write down as 4 8 and 10. Now other elements X 1 X 2, X 2 X 3, X 1 X 3. So, whatever value is there we have to divide by 2 and we have to write down here. That is, particularly X 1 X 2 element is 2 divided by 2 2 by 2 means 1. So, we can write down as 1 1 then about this 4 X 2 X 3. So, divided by 2 we will get minus 2 and minus 2.

So, we can write down here minus 2 minus 2 and about this X 1 X 3 we can write down, write down as again minus 4 minus 2 and minus 2. So here, minus 2 and minus 2 and now this is P matrix and this is complete a quadratic function. Now here, one problem is there this P has come. Now, from this P can we get some idea about the Eigen values of these P matrix. In which side, they are lying whether the left side right side.

And here, I am not talking about the imaginary axis, because, real symmetric matrix involves the elements on the real axis. That is, we will get the elements on this axis, real axis no complex conjugant elements. Then this quadratic function can it used for a stability of the system. Now first of all, we will see how to handle this is that is how to get the information about the Eigen values of this matrix P.

So, in order to get some information about the Eigen values not likely stability. Because, Sylvester the criteria which are developing without knowing whether, it is useful for stability or not. Therefore, here only we are focusing on whether to get means in which direction we are getting the roots whether on the left side or the right side.

So, for that purpose in literature one criteria is there, well famous criteria, that is called Sylvester criteria and that criteria is useful in order to determine the definiteness means, whether the roots or a Eigen values on the left side right side or at the particularly origin.

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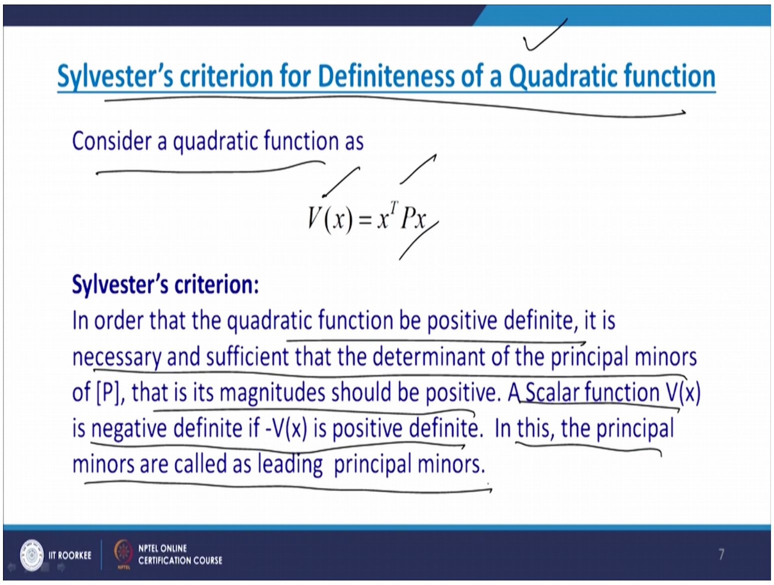

To proceed Sylvester criteria for definiteness of a quadratic function; now, consider a quadratic function as V of x equal to X transpose Px. So, what a Sylvester criteria say in ordered that the quadratic function be positive definite. It is necessary and sufficient that the determinant of the principal minors of P that is the magnitude should be positive a scalar function V of x is negative definite.

If minus V of x is positive definite in this the principle minors are calculated as a leading principle minors that is, this criteria is basically based on principle minors that is for example, this is a P matrix n-th order matrix. Now, we have to determine the leading principle minors. So, which are the leading principle minors in this case? So, here let us say a 11 this is one.

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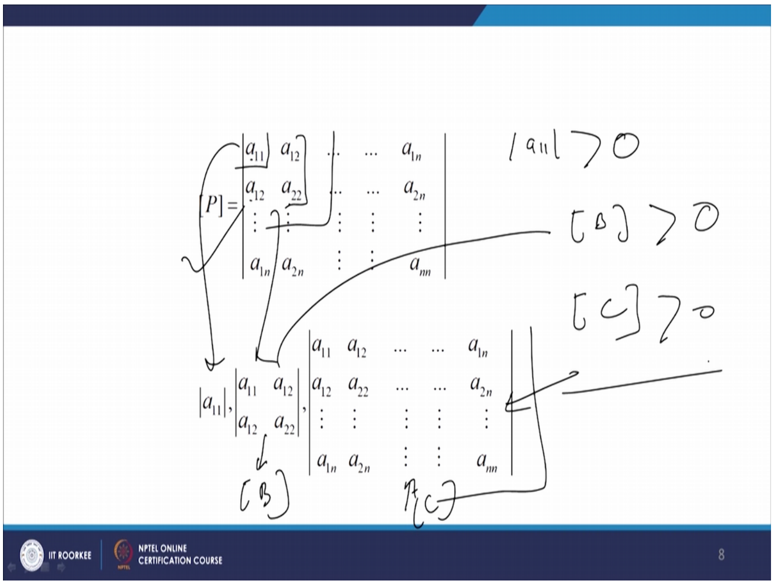

This a 11 a 12, a 12 a 22 this is another see and this is a 11 here showing and this is like this. And similarly, we can go for a another third principle minors 4th like this and finally, it is complete determinant. So, it has been said that, if all these principle minors or the determinant of these principal minors are greater than 0. We can say that, this given system matrix P is positive definite; that means, this magnitude of a 11 should be greater than 0.

Similarly, this magnitude let us say we can write down as a B. This is B, must be greater than 0 and total matrix the determinant of complete matrix. Let us say C we define a, let us say defined as C. C must be greater than 0. So, if this is the case then we call as a positive definite matrix.

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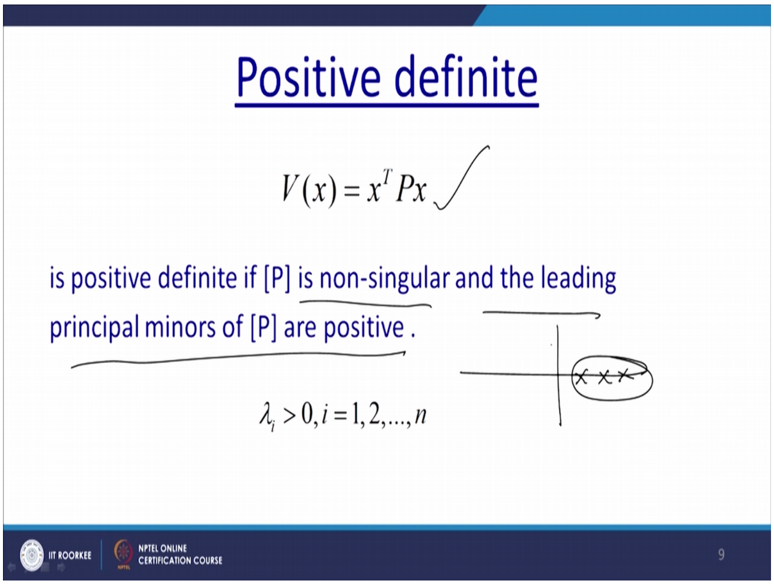

So, your V of x equals to X transpose Px is positive definite. If T is nonsingular and the leading principle minors for P are positive in other words. It can say that, here all the Eigen values of P are lying on this right side of this plane this side then it is called positive definite matrix.

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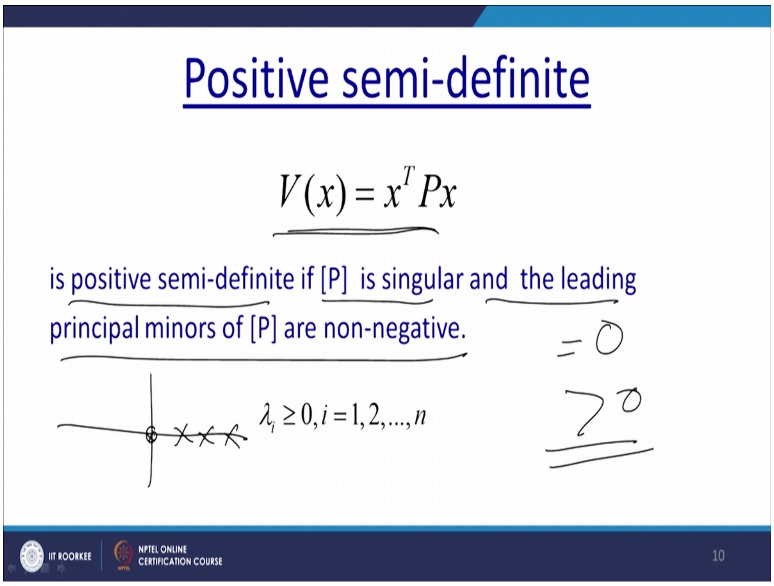

Now, this is a another word positive semi definite same function how to define a positive semi definite matrix. So, in that case whatever principle minors we calculate if there is a at least, one principle minors equal to 0 and other principle minors are greater than 0. All other principle miners then, we call as positive semi definite matrix. That is V of x equals to X transpose Px is positive semi definite. If P is singular and the leading principle minors of P are a non-negative.

In other words, the Eigen values of P they are lying on the right side and at least a one Eigen value and this original. If this is the case it is called a positive semi definite.

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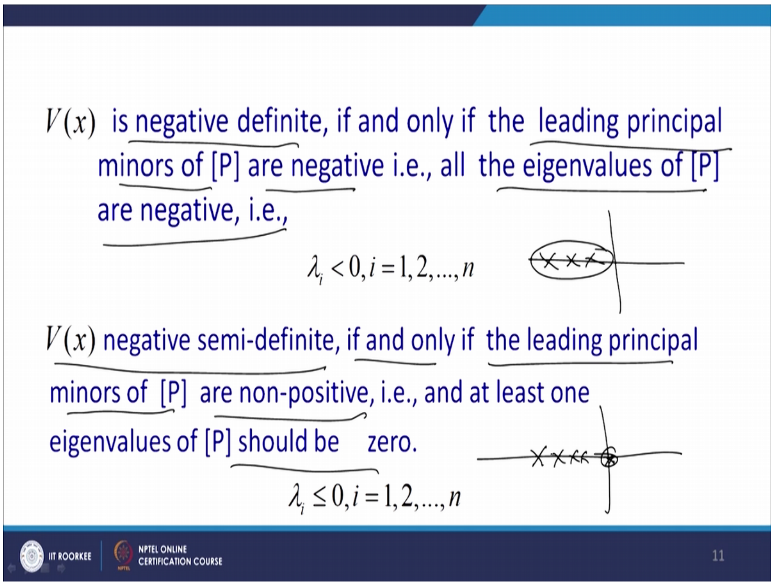

Now, the reverse case is possible in case of negative definite. So, V of x is negative definite. If and only if, the leading principle minors of P are negative. That is, all Eigen values of P are negative; that means, V of x is negative definite. If the Eigen values of P are lying on the left side of this plane, then that particular matrix P or real semi matrix P is called negative definite.

And similarly V of x is negative semi definite if and only if, the leading principle minors of P are non-positive. That is at least one Eigen values of P should be 0. It means that this particular P matrix is negative semi definite. If there are some Eigen values on the left side of the X plane and at least one Eigen value at the origin then this V of x is called negative semi definite.

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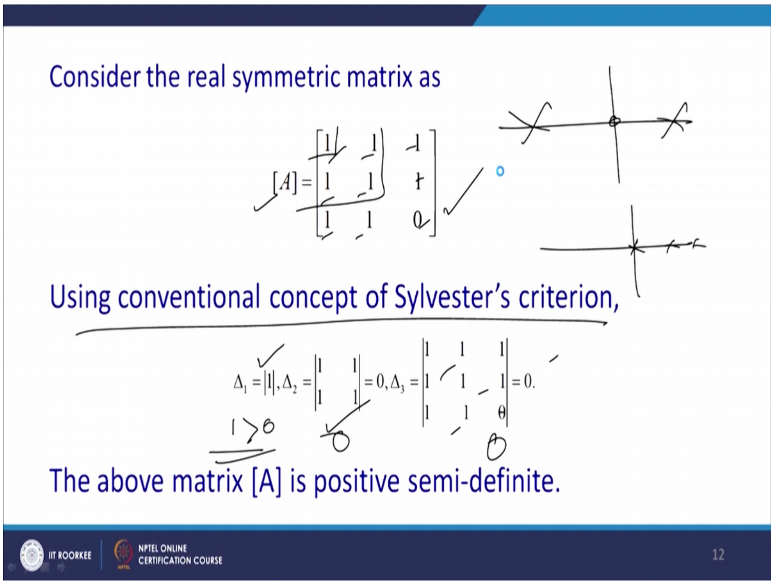

Now we see one example, consider the real symmetric matrix A as 1 1 1 1 1 1 1 1 0. Now our aim is to check the definiteness of this matrix. Whether, it is positive definite, positive semi definite, negative definite, negative semi definite or in other words that is one definition it is called indefinite. Indefinite means, that if there are a Eigen values on the right side, left side at the origin, then it is called indefinite matrix. There is no definite definiteness only independent.

So, we will see this is example. So now, you use the concept of the Sylvester criteria and you will find out this type of, this criteria is given or what I have explained in most of the textbook? So, it says that in order to check the definiteness of A, we have to determine the all leading principle minors. So, for this matrix A we start developing the leading principle minors. So, first minor is 1 see at this 1 now here written 1, this 1 is greater than 0.

Now, we go for the second principle minors for second principle minors, we like this 1 1 1. So, it is 0 that is 0 and third this 1 1 1 1 1 1 1 1 0 see this. So, this is determinant of this is 0; so 0 0 and greater than 0. So, this is the condition of positive semi definite. So by means of this, we have got the matrix A as positive semi definite matrix as the definiteness. It is concern with the Eigen values we have seen that. If this matrix is positive definite; that means, their all Eigen values must lie on the right side of this plane; that means, it must have Eigen values either here, here, here.

That means one Eigen value should be at origin and remaining Eigen values at the right side of this planes then, and then this condition is satisfied now. What we do that, we just calculate the Eigen values of this matrix A. Now the Eigen values of this matrix A.

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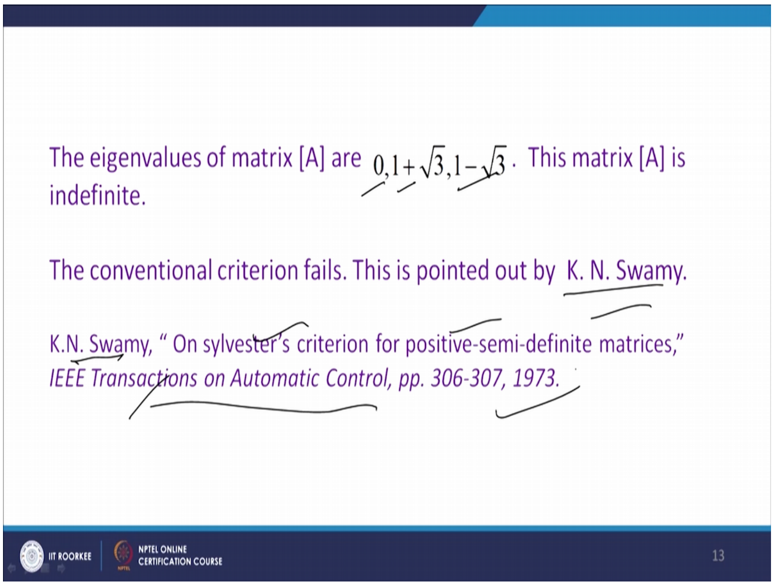

The Eigen values of matrix A are 0 1 plus minus plus root 3 and 1 minus root 3 this matrix is indefinite. See here, by calculating the Eigen values this matrix comes indefinite, but actually this given matrix is positive definite. Why this is happened by means of this approach conventional approach? We are coming a positive semi definite matrix.

But actual Eigen values are like this. One at a origin is here So, this indicate indefinite. So, this is the issue has been pointed out by K.N. Swamy he published a paper in a I triple E transactions on automatic control. Very important journal see here, K.N. Swamy on Sylvester criteria for positive definite matrix I triple E transactions on automatic control 1973. And in this case, he said that in order to check the positive semi definiteness of a matrix, we have to check not a Eading principle minors leading all principle minors need to be check.

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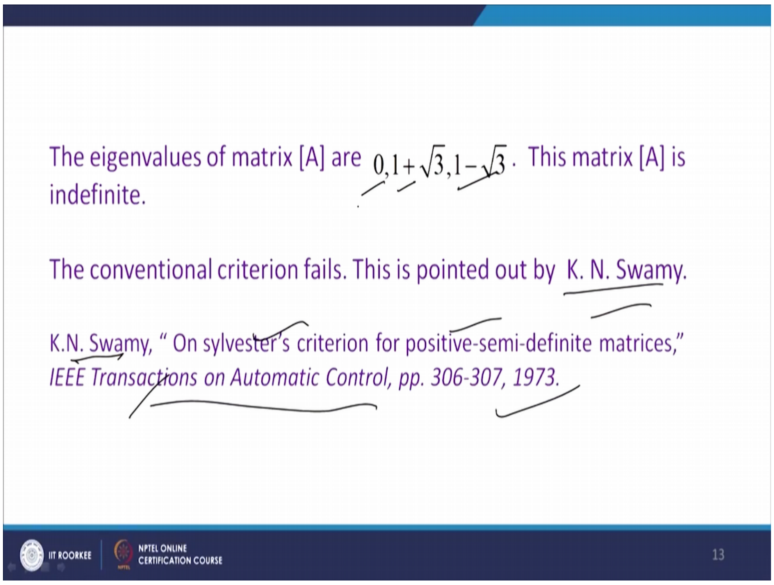

That is according to K.N. Swamy a quadratic function is positive semi definite. If and only if, all the principle minors of the system matrix are non-negative; that means, if the matrix of a order n has n of p principle minors of a order P, where n of p is represented by factorial P n minus P.

That means, here in a given problem we have determined 3 principle minors. So, they are not sufficient for checking the positive semi definiteness of a matrix. In that case, we have to check all principle minors let us see after checking the all principle minors, we are getting the result corresponding to Eigen values or not.

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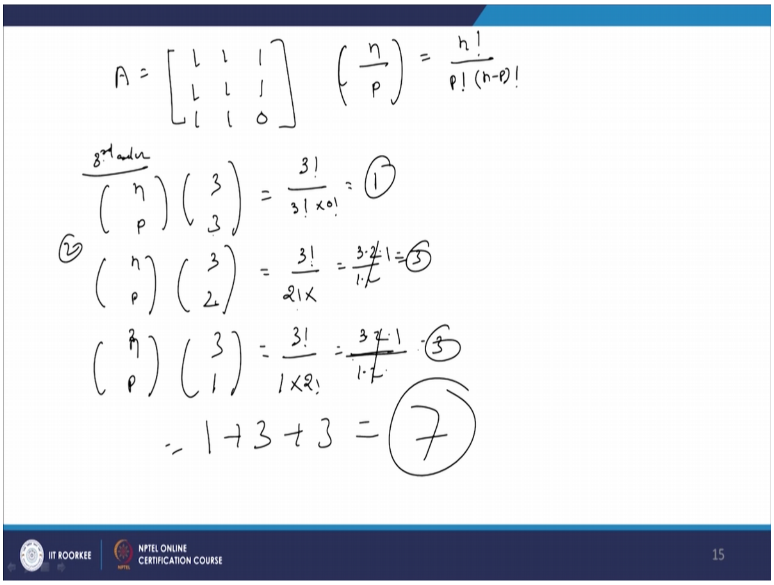

Now, I have writing down the matrix A same 1 1 1 1 1 1 1 1 0, this is the matrix A. As I shown earlier same matrix, now we have to see how many principal minors are possible here. So, according to K.N. Swamy for a third order that is n of p where p is 3. That is, 3 of 3 how many principal minors are possible that is third order a principle minor that is factorial 3. Then here, 3 multiplied by 2 0 that is 1 that is, how I determine here? This n of p equal to factorial n divided by Pth order principle minors n minus P.

So, your third order third principle minor n of p here is 3 of 3; so factorial 3 3 here factorial 3 a 0. So, we will get one that is, we will get one principle minors. Now you see the second order principle minors. So, here n of p So here 3 this, is 2 ok. Earlier we have taken 3 this is for second. So, how you get it So, n is your factorial 3. Then here, pi is factual 2 I multiplied by here 3 minus 2 1 factorial 1 that is 1.

So, what we will get 3 2 one 2? So, we will get 3 principle minors, and then number of principle minors for order one that is here 3, 3 n P again 3 1. So, factorial 3 1 that is, P is 1 and n minus P 3 minus 1 2 this factorial 2. Again 3 2 1 2 that is 3 2 1 3 2 1 this; so you will get 2 cancel 3; that means, for third order.

So, total principle minors which need to be checks r 1 plus 3 plus 3 7. Whereas, the conventional approach say that you have to check the 3 principle minors. Now we will actually find out the principle minors and then, you will check the determined and we will see the result see here.

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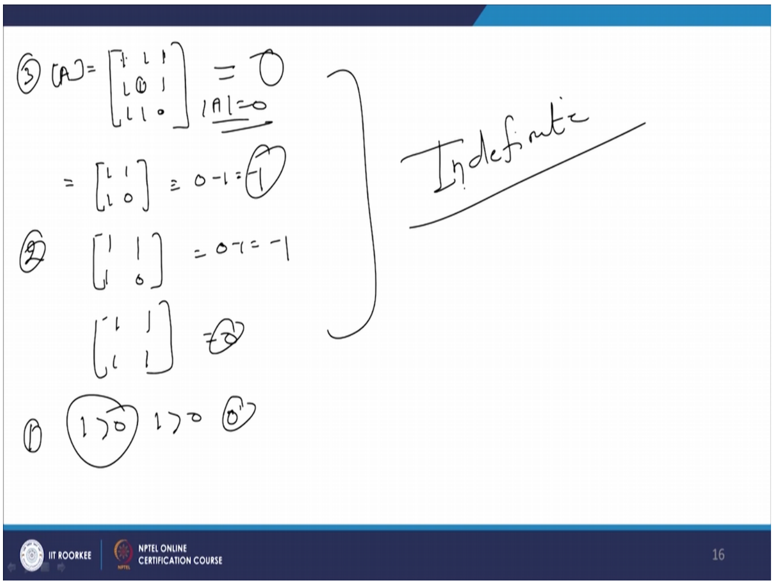

The first principle minors of order 3 that is here, 1 1 1 1 1 1 1 1 0: so this is 0 for order 3 first principle minors and we will find that, the determinant is 0. That is, let us say determinant of say A determinant of A equals to 0, now about the principle minors for second order.

So here, we can get it as for second we will get it at principle minors as for with respect to 1. Here 1 1 1 0 then, second we will get it as with respect to this we will get 1 1 1 0 and for with respect to this we write down at 1 1 1 1. So, if you solve it we will get 0 minus 1 equals to minus 1. Here, 0 minus 1 equals to minus 1 here, equals to 0. That is for principle minors of a order 2 and now, the principle minors of a order 1 for a order 1 is here, this is a 1 this is 1 greater than 0. Then this is 1 is also greater than 0 and here, is 0 that equals to 0; that means, here we will find that there are 0 then there are some minus 1 and here is 1 greater than 0.

So, we will get principle minors they are 1 at origin other left hand side right hand side; that means, the actual value of principle minors or greater than 0 less than 0 at the origin; that means, that this matrix is indefinite. Indefinite whereas, the existing criteria say this given matrix is positive semi definite. So, this issues has been identified by K.N. Swamy. Therefore, whenever we require to check positive semi definite inverse of a matrix we have to check all the principle minors not the leading principle minors.

Now, whatever I had taught particularly this Lyapunov this quadratic function. So, this function will be useful in checking the stability of the system. The Lyapunov who, Lyapunov is basically Lyapunov; Lyapunov is a scientist who developed this theorem: he has used this concept in a checking the stability of both linear and non-linear system.

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These are some references I.J. Nagrath Gopal, D. Roy Choudhury and this K.N. Swamy the people which we are refer too.

Thank you.