Advanced Linear Continuous Control Systems
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Lecture - 16
Stability Analysis in State Space:
Concept of Eigenvalues and Eigenvectors (Part-I)

Now, we start with Stability Analysis in State Space. In the first part is Concept of Eigenvalues and Eigenvectors.

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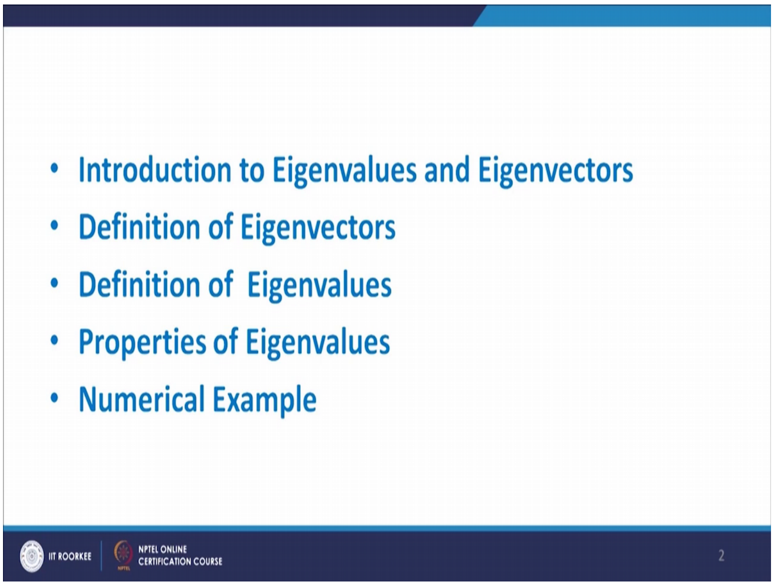

In this Eigenvalues and Eigenvectors, we study introduction to Eigenvalues and Eigenvectors, Definition of Eigenvectors, Definition of Eigenvalues, Properties of Eigenvalues and numerical examples. Now, introduction to Eigenvalues and Eigenvectors, the most important thing is that this Eigenvalues and Eigenvectors are useful in control system.

In control system, we have discussed earlier the control engineer has to perform modeling, then he has to check stability, then he has to design controller. For all these issues, Eigenvalues plays a very very much important role. If there is no Eigenvalue, no control system or we can in other words we can say Eigenvalues and Eigenvectors is the heart of the control engineering. And now, what is control engineering? Control engineering is applicable to all type of system. It may be electrical mechanical, civil, computer. Therefore, as Eigen values plays important role in control system.

So, this Eigenvalues are also place important role in all type of systems. For example, if you take a rigid body and if you move of, so, it will rotate. So, it has some axis of rotations. So, this axis of rotations is called Eigenvector and it is movement of inertia is called Eigenvalue. If you take, if we can consider another example that is if you consider the vibration of a mechanical systems, in that case the natural frequency of vibrations that is call Eigenvalue and it is and it is particularly, we can say the more of this vibrations there is call Eigenvector.

So, this Eigenvalues and Eigenvectors are usefully everywhere where is the system.

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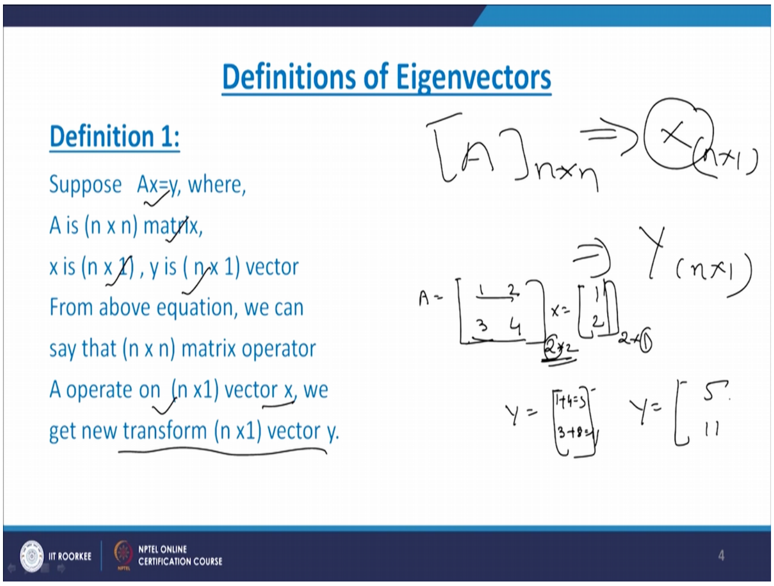

Now, how to define this Eigenvalues and Eigenvectors? Now, we will see definition of Eigenvectors. Suppose, A x is equal y where A is n cross n matrix, x is n cross 1 matrix, y is n cross 1 vector. From above equation, we can say that the n cross n matrix operator A operate on n cross vector x we get newed transform n cross 1 vector y. That is, we have a matrix which is of order n cross n when it is operate on x which is of order n cross 1, we can get y which is of order n cross 1. That is A operated on x, we can y for example, if I will take the example.

Let us say A equal to 1 2 3 4. What is order of a matrix order of A matrix is 2 cross 2. And now, this has to operated on x, let us say x is vector that is n cross 1, that is say here 1 2. Now, this x, so, order is 2 cross 1, two rows, one column. Now, what output will get output y is nothing but the rows what is row rows, these two rows, 1 and 2 operate on column this column.

So, what you get here 1 2 plus 2 4 will get 5 and here this row, operate on this will get 3 plus 8 element. That is y will get s 5 element.

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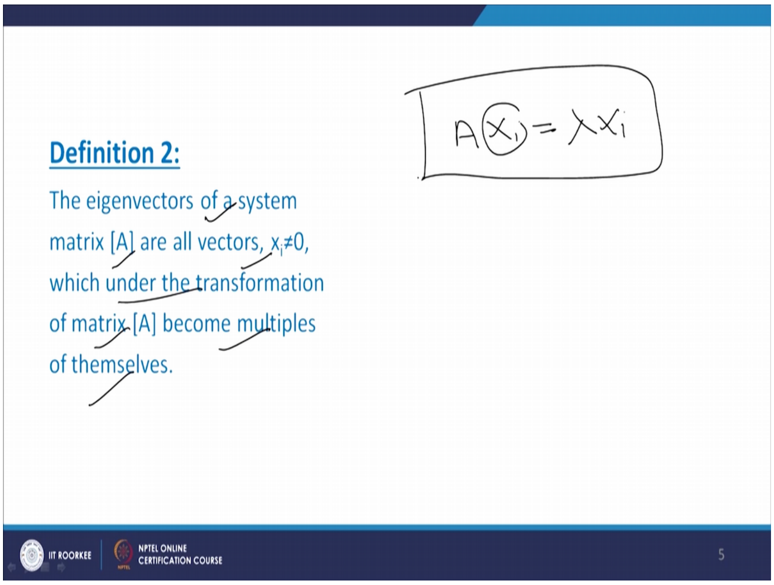

Now, another definition, Eigenvectors of system matrix A are all vectors x i not equal to 0 which under the transformation of matrix A becomes multiples of multiples of themselves; that means, here Eigenvectors are non-zero. They are never be 0; that means, here it is mentioned that a x I, there is a operate on this vector which is nonzero become multiples of this themselves; that means, we can write down as lambda in to x i. A x i equal to lambda x i; that means, here this what is happening here a an matrix operate on x and we get the new vector.

So, we will find that in that case, you to see that what is the directions. So, it is longer same directions.

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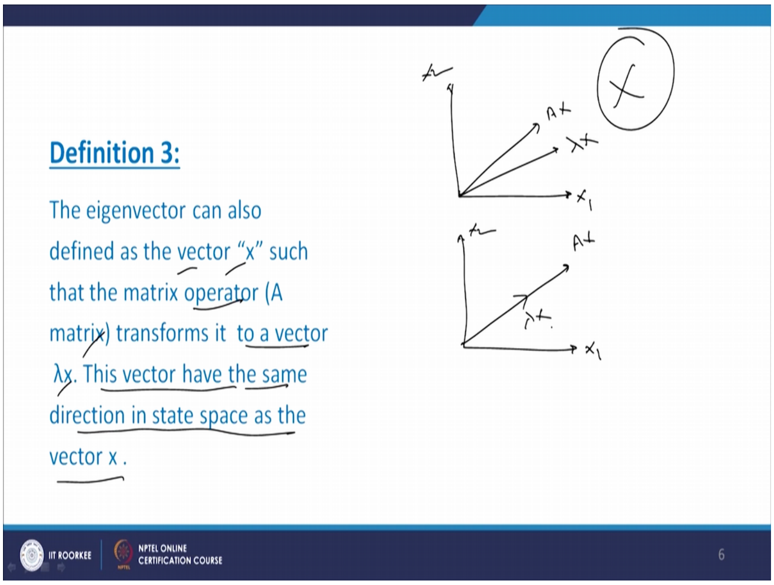

So, here it is shown Eigenvector, that is third definition Eigenvector can also defined as vector x such that matrix operator, that is A matrix transforms it to vector lambda x where to earlier. That is A equals to lambda s. So, here it is mentioned same thing to vector lambda x this vector has the same direction is state space as the vector x. If you take a any variables, it has a magnitude and directions ok.

So, after transformation, will direction remains same, then that is call Eigenvector. If the direction is not same then, we cannot call as a Eigenvector for Eigenvector. This vector is should be always nonzero it is not be 0; that means, here I am showing, so, here x 1, x 2. Now, here this A x and now, let us say lambda x. So, this is not valid that is this a x and lambda x must be in the same directions. So, here I am showing it again, so here, x 1 x 2.

Now, here I am showing A x and I can say it is lambda x, then this is called Eigenvector, this is correct.

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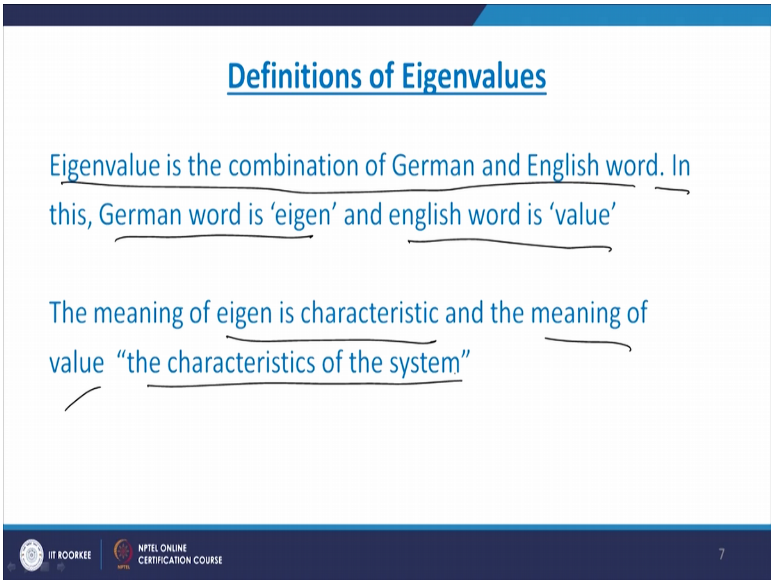

Now, what are the definitions of Eigenvalues? Actually, Eigenvalues is nothing but the combination of German and English. So, because here there are two words, Eigen and values, therefore, here Eigenvalues is the combination of German and English word. In this, German word is Eigen and English word is value, English word is value. The meaning Eigen is characteristics and meaning of value the characteristics of the system, value is characteristics of the system and where as Eigen is nothing but the characterstics.

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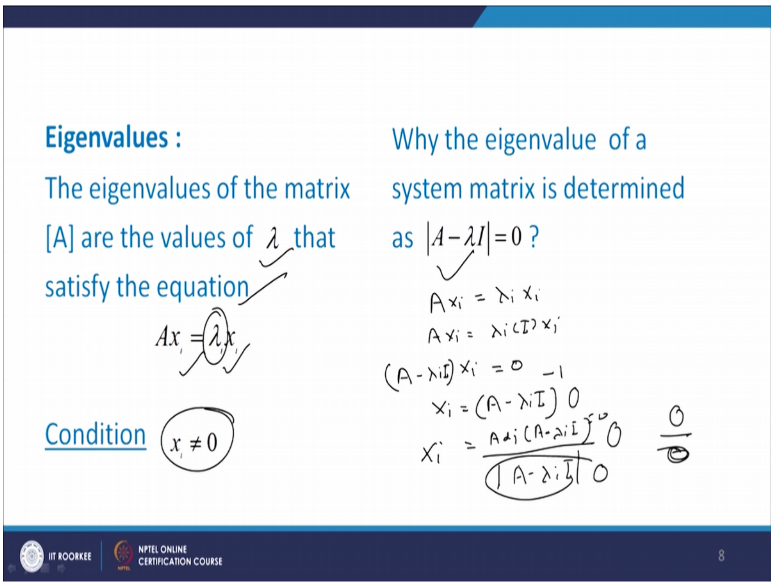

Now, how to define Eigenvalues? Eigenvalues of a matrix A are the values of lambda that satisfy the equations A x i into lambda x i and as I told, A x i should not be equal to 0.

So, this is the definition of Eigenvalues and here lambda I is nothing but the Eigenvalue; that means, a A operate on x that is sketching of the vector and be could the lambda; that means, sketching vectors is been sketching, lay that been increased that is basically an Eigenvalue. So, the factor which helps in sketching that is called Eigenvalue and the x is the Eigenvector. Now, why the Eigenvalues of the system matrix as defined as A minus lambda I equals to 0 means when you are be calculate the Eigenvalues.

Why it is says that A equals to lambda I. So, further purpose, what we do? We write this A x i equal to lambda I into x i. Now, this equation, assuming an identity matrix A x i into lambda i into I into x i, then write this equation A minus lambda I into x i, lambda into I equals to 0; that is a minus lambda into x i equals to 0. We have replacing all the elements on the one side and it is says that A i minus lambda equals to 0, that is the that is the way to calculate the Eigenvalues, but we should know, why it is so.

So, for that purpose we not taken the same equation and then I have solve it. I have compare. Now, how to do this? How to move further? So, now, what I have done this x i equal to a minus lambda I into a inverse to 0. Now, here we have got 0 and this is the inverse. So, x i become zeros, but the condition of Eigenvalues that x I should be never 0 even earlier definition we have seen, Eigenvector should never be 0, it should be always some value.

But if you see the literature, it is says in order to calculate Eigenvalues, we have to make a minus lambda I equal to 0, but here we have got this equations. Therefore, what we can do here if you write this equation a minus lambda inverse. So, it can be written as a joint of a minus lambda I divide by in determinant of a minus lambda I by I into 0 there is x i. So, now, here this in order to have x i to be non-zero, what is the possibility because this cannot be 0, if this is 0, this total percent 0; that means, if this factor is 0, then 0 by 0 cannot be define or determine.

And therefore, in order to check the stability of the system or the Eigen particularly, we can say Eigenvalues, we have to check A minus lambda I equals to 0.

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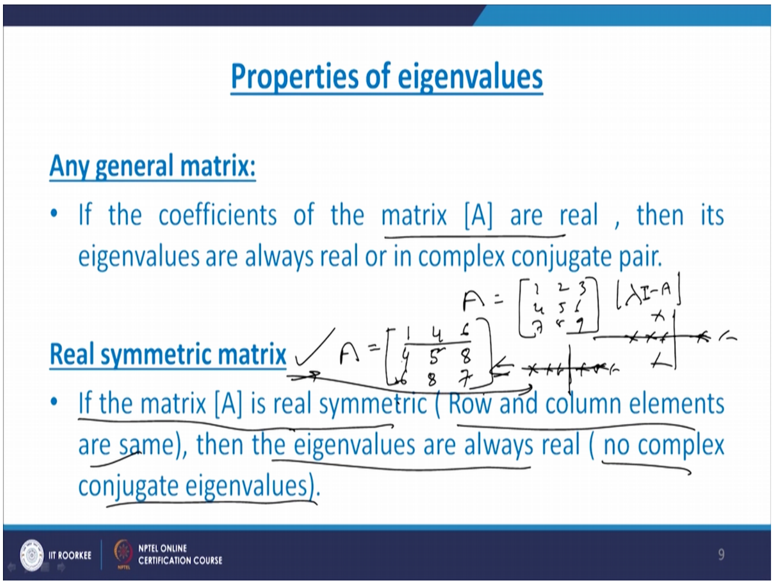

So, now, we have to see some properties of Eigenvalues. First, for any general matrix if the coefficients of the matrix a are real, then it is Eigenvalues are always real or complex conjugate pair; that means, here if I am saying A is matrix, it has some elements 1 2 3 4 5 6 7 8 9 these are real elements.

So, Eigenvalues of A, if you calculated like this lambda I minus a. So, that Eigenvalue can lie on the real axis or it may be complex conjugate. So, any possibility will be available when, when you are given matrix is real and it is general form. Now, we coming to real symmetric matrix. If matrix A is real symmetric, what is real symmetric the row and column elements are same; that is if you say take any matrix?

If the row elements and the column elements are same, that in that case that matrix is called really symmetric matrix. And when the matrix real symmetric, their all Eigenvalues a lying on the real axis, there will be no complex conjugate Eigenvalues here it is written. A matrix a is real, symmetric row and column elements are same, then the Eigenvalues are always real no complex conjugate Eigenvalues. For example, if I write this for this part here write 1 4 6 this is row. Now, it taking elements here 4, 6, then, I can write 5 it is 6, 8 you can take 8 here and 7; see here 1 4 6, 1 4 6, 4 5 8, 4 5 8, 6 8 7, 6 8 7.

So, this is real symmetric and in that case, the Eigenvalues of this are lying on the real axis only. That is respond, Eigenvalues of a matrix A and it is transpose.

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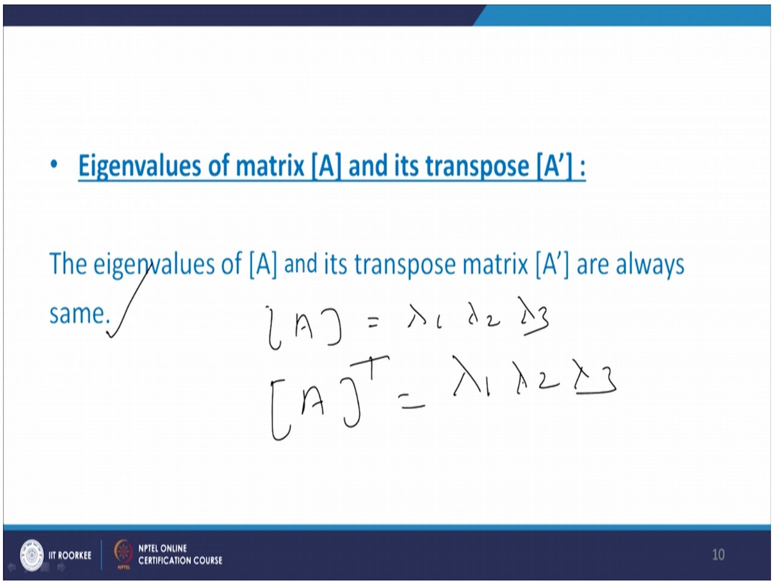

The Eigenvalues of A and it is transfer matrix are always the same; that means, if you take the A, let us say A matrix has Eigenvalues lambda 1, lambda 2, lambda 3 and if you make transpose of this, so, will we will always get the Eigenvalues lambda 1, lambda 2, lambda 3. Eigenvalues of matrix say and it is inverse that is a inverse.

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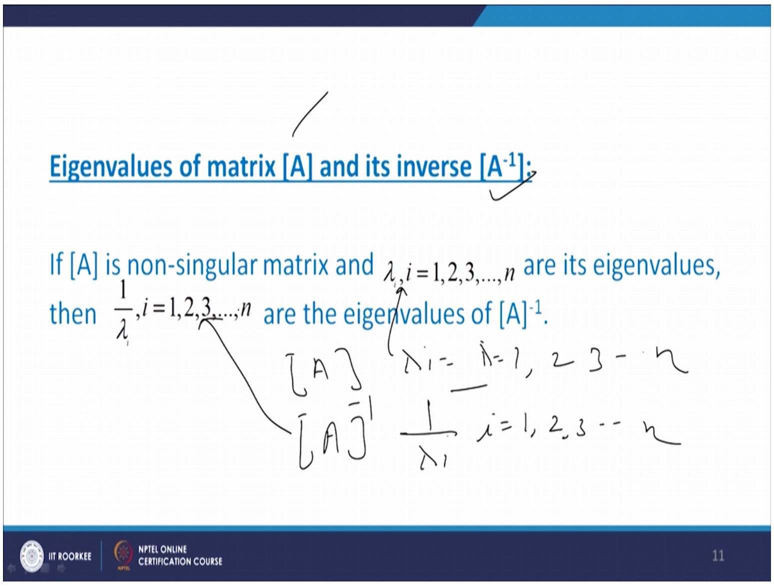

Suppose if you matrix A is Eigenvalues lambda 1, lambda i equals to 1 2 3, up to n just like this is and if you make it A inverse, so, A inverse is also Eigenvalues. So, in that case the Eigenvalues are nothing but one upon lambda i where i equal to 1 2 3, up to n that is this part Eigenvalues.

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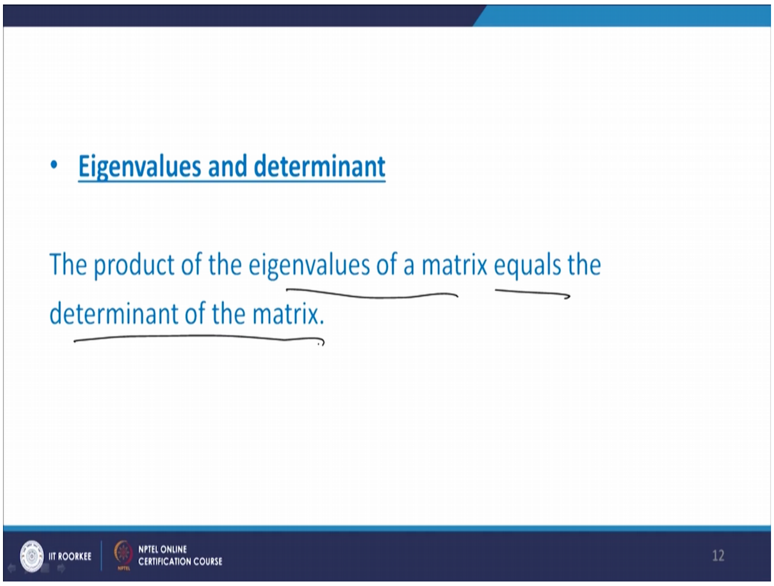

And determinants the product of the Eigenvalues of a matrix equals the determinants of the matrix. See here, if what are the Eigenvalues are there if you multiplies that is nothing but the determinant of a matrix. Then trace Eigenvalues and diagonal elements. In previous part, we were seen that using the trace of the matrix we can determine the transfer function model of the system.

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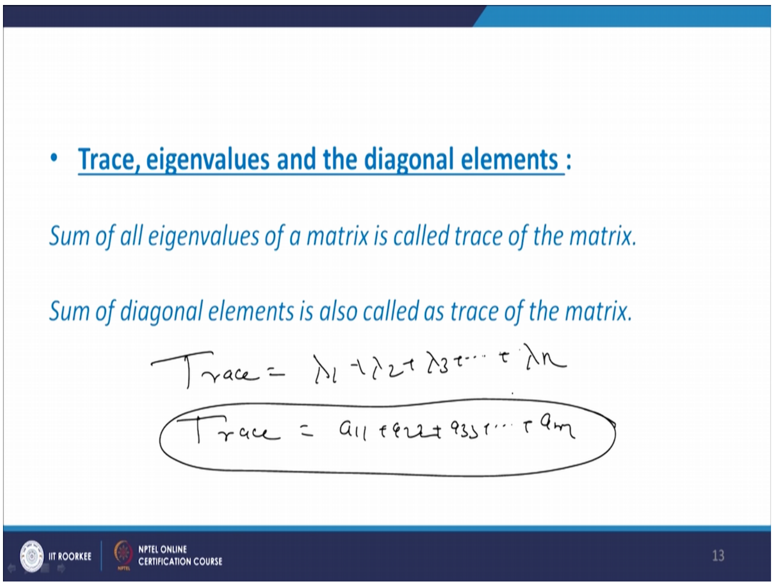

So, here the same trace the trace is nothing but equal to some of Eigenvalues that is what let us say lambda 1, lambda 2, lambda 3, up to say lambda n. Similarly, some of diagonal elements is also equal to the trace of the matrix that is if you take trace the diagonal elements, let us say a 11, a 22, a 33, a nn. So, this concept, the trace equal to sum of diagonal elements we have already use in determining the transfer function from a given state space model.

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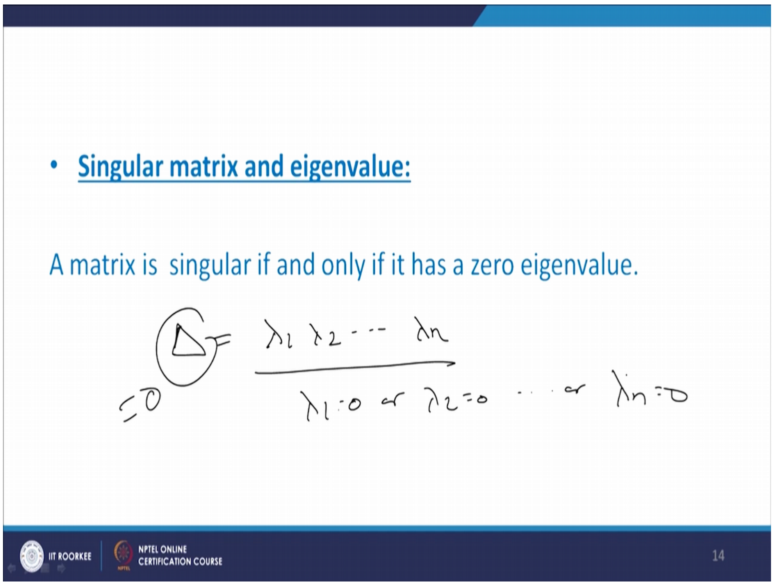

Then Singular matrix and Eigenvalues; matrix is singular, if and only it has 0 Eigenvalues matrix is singular. Singular means, the determinants is 0. Let us say this determinant delta and let us say these are Eigenvalues, lambda 1, lambda 2 up to lambda n; what is say it has 0 Eigenvalue. So, 0 Eigenvalues why it because there singular is there, open delta equals to 0, the delta means determinants equals to 0 and this is only possible when one of the Eigenvalue is equals to 0, lambda 2 equal to 0 or lambda n equals to 0. Therefore, it is shown that matrix is singular, if and only it has 0 Eigenvalue.

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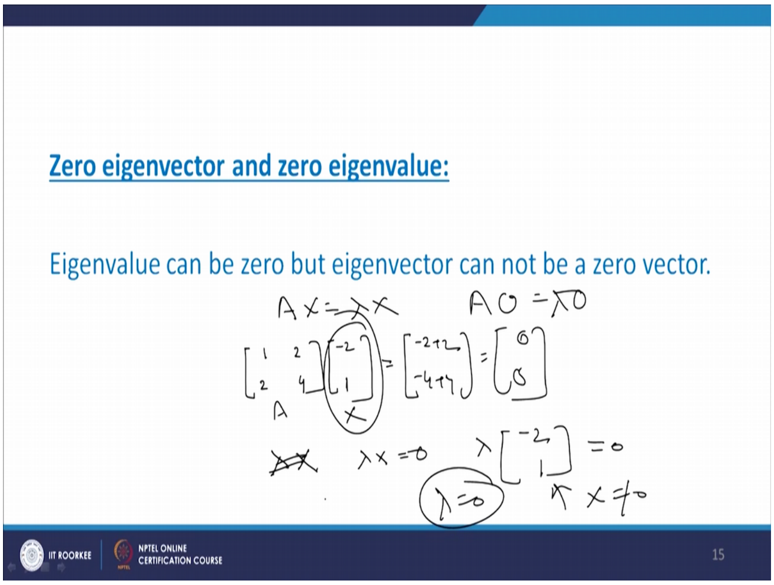

Now, 0 Eigenvector and 0 Eigenvalue, the Eigenvalue can be 0, but Eigenvector cannot be a 0 vector; that means, we are in A X equal to lambda X.

So, what is said Eigenvalue can be 0, but Eigenvector never be 0; that means, theoretically, it is possible that A 0 into lambda 0, but what you what about the actual things because actually Eigenvalue can be 0, but Eigenvector never be; for further purpose let us say if I have taken matrix as 1 2 and here you can take 2 4. So, we can write down this as here minus 2 1 equal to. So, what will get, you will get it as this is X, this is A. So, here A, X is be multiplied.

So, it is minus 2 plus 2 and here we will get minus 4 plus 4. Now, here it is 0, 0. So, now, x we have taken here. Now, what we can write we can write lambda X. So, we can get lambda X equals to 0. So, now, we can write lambda and what is X, X is minus 2, 1; that is equals to 0, but this is not 0, this x is non zero; that means, possibilities lambda equals to 0; that means, Eigenvalue can be 0, but this is not true for Eigenvector. Same Eigenvector cannot be associated with different Eigenvalues.

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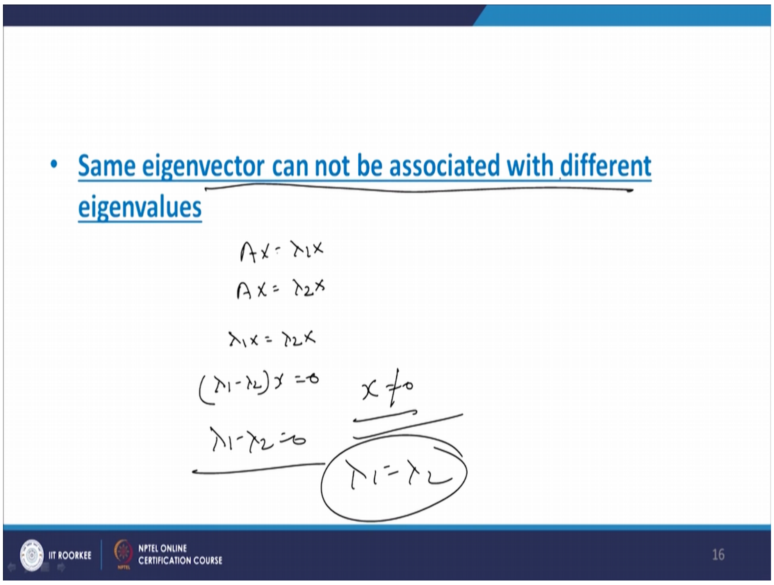

Suppose if you taken A X equal to say lambda 1 X and then we can also take A X equals to lambda 2 X, assume that X is the Eigenvector for the lambda 1, X is a Eigenvector for lambda 2. Now, if you equate it, what will get lambda 1 X equal to lambda 2 X, then if you solve it, will get lambda 1 minus lambda 2 into x equal to 0 and what about these X? X is Eigenvector, it can never be 0 Eigenvector cannot be 0.

So, in possibility is that then lambda 1 minus lambda 2 equal to 0; that means lambda 1 equal to lambda 2. But we have not taken different Eigenvalues; that means, this is not possible therefore, we have written same as Eigenvector cannot be associate with different Eigenvalues. Now, we will solves example and from this example, we will see how to determine the Eigenvalue and Eigenvector.

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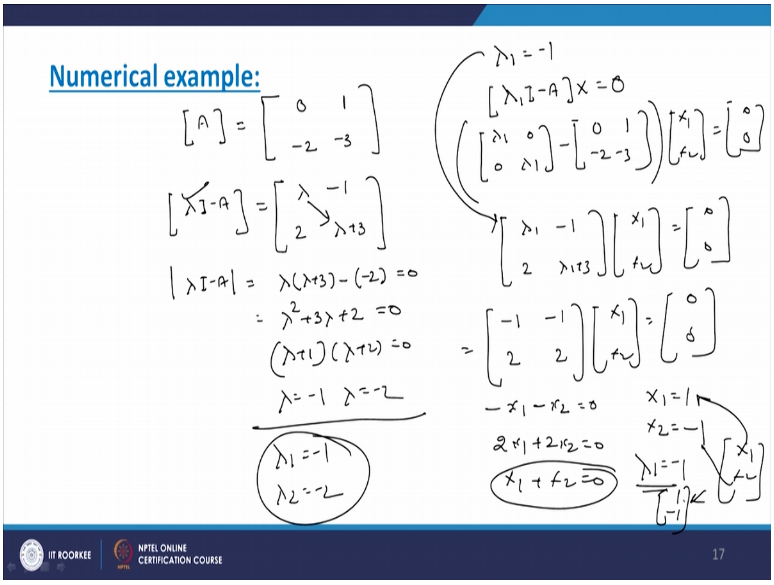

Now, we have taken example, a matrix A equal to 0, 1, minus 2, minus 3. This is matrix and our problem is, we had to calculate the Eigenvalues of this matrix.

So, how we determine this? So, first of all we write this A matrix as lambda I minus A n, lambda a minus here, I is identity matrix. So, here after replacing identity matrix and lambda we get lambda minus 1 2 lambda plus 3. Now, what is the next step? Now, we have take the determinant of this matrix. So, determinant of lambda minus A, that is equal to you solve it lambda into lambda plus 3 at this multiplication minus this is minus 2 equal to 0. So, we will get lambda square plus 3 lambda plus 2 equal to 0. So, if you solve it, we get lambda plus 1 lambda plus 2 equal to 0.

So, lambda equals to minus 1 lambda equals to minus 2; that means, we have two Eigenvalues; one is a lambda 1 equals to minus 1 and lambda 2 equals to minus 2. These are two Eigenvalues. Now, our main purpose is to determine the Eigenvectors. Now, how we determine the Eigenvector? Now, first of all we determine the Eigenvector for lambda 1 equal to minus 1. So, with equation available to us, we have same equations that is lambda 1 I minus A into X equal to 0 lambda 1 because earlier we have taken this lambda, but now, we have got two Eigenvalues lambda 1 and lambda 2.

Therefore, I have change lambda 2, lambda 1. So, lambda 1 I minus a multiply by X equals to 0. So now, we write equations as lambda 1, 0, 0, lambda 1 minus A, what is A matrix? A matrix is 0, 1, minus 2, minus 3 complete. Now, how many steps; X 1, X 2 equal to 0, 0. Now, we have to solve these equations. So, after solving, what will get will get equation as lambda 1, minus 1 2 lambda 1 plus 3, X 1, X 2 equal to 0, 0. Now, in this equation we replace lambda 1 equals to minus 1.

So, this lambda 1 equal to minus 1 we have to replace. So, what we will get? We will get it as minus 1 minus 1 2 2 and X 1, X 2 equal to 0, 0. Now, if you solve it, what you get minus X 1, minus X 2 equal to 0, second 2 X 1 plus 2 X 2 equal to 0; both are the same equations. That means, if you take X 1 equal to 1 and what is the X 2. In that case, X 2 equals to minus 1.

Because here X 1 X plus X 2 equal to 0, therefore, for lambda 1 equals to minus 1. Our Eigenvectors are X 1, X 2 that is equal to 1, minus 1, that is X 1 is 1 X 2 equals to minus 1. So, Eigenvectors are 1 minus 1. Now, next what you want? We want Eigenvectors when lambda 1 equal to minus 2.

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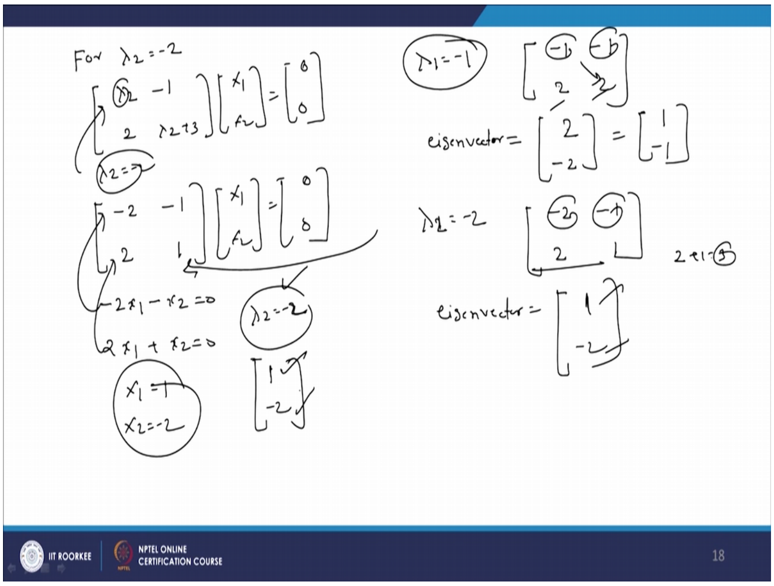

Now, for lambda 2 equal to minus 2, so, we write the equation as lambda 2, minus 1 2 lambda 2 plus 3 X 1 X 2 0 0 and now, replace lambda 2 equal to minus 2 in this equation. So, what will get? We will get minus 2, minus 1 2 1, X 1 X 2 equal to 0 0. Now, we solve this equation. So, after solving we will get as a 2 X 1 minus X 2 equal to 0 and second equation 2 X 1 plus X 2 equals to 0; that means, these equation for this.

And this is for this and we will find that again. They are the similar equations. That means 2 X 1 plus X 2 equal to 0. Suppose, if you take X 1 equal to 1, so, what is the X 2? X 2 equal to minus 2. Therefore, when lambda 2 equal to minus 2 your Eigenvector is 1 minus 2. Therefore, for lambda 1 equal to minus 1, we have got Eigenvector as 1, minus 1 and when Eigenvalue lambda equal to minus 2, we have got Eigenvector as 1 minus 2.

So, this is an second order system. We have solved easily, but if you, if you increase the order of the system, this process it quite comparison. Therefore, we want method we can give easily the Eigenvectors. So, the literature this one method which is called cofactor method which you have already seen in the linear algorithm that that mythology can be useful to get the, we can say the Eigenvector of system or Eigenvectors can easily determined by the concept of the co factors.

So, how to determine it? So, further purpose, I am taking in the same example. See here, for lambda 1 equals to minus 1, we have got this equation and after that we solve. So, what we are doing; we are not wind these steps. So, what I am what we are doing here when lambda 1 equals to minus 1, we have got a matrix as minus 1 minus 1 2 2, just you can check it; so minus 1 minus 1 2 2. Just you can see here, minus 1 minus 1 2 2.

Now, we want the Eigenvector for this one. So, what we can do, the Eigenvector. Now, cofactor means cofactor means where to the minor s. So, for minor of this minor 1 is 2 whereas, the minor of this is 2, but because here it is in the second row first column, so, it is odd. So, will get is as minus 2; that means, we divide by 2 both, this 2 and 2 will get 1, minus 1. So, you find that you get the same 1, minus 1 say, without doing any calculations, we have directly get the Eigenvector. Similarly, when lambda 1, say lambda 2 equals to minus 2. So, what matrix we have got we have got matrix is that is we have got minus 2, minus 1 2 1. Now, we want Eigenvector for this.

So, Eigenvector, Eigenvector, so, what we do we have do the principle minors of all this, minus 2, it is 1 and for this 2, but because here 2 plus 1, 3 this odd number that is 2 rows first column therefore, it become minus 2; 1 minus 2 1 minus 2 1 minus 1 here, 1 minus 1; that means, we have easily determine the Eigenvectors of a given system matrix.

Now, here we observed that here Eigenvalues are distinct, but it may possible that sometimes the Eigenvalues are repeated. They in that case, how to calculate the Eigenvector? Now, we start solving the example on generalized Eigenvector.

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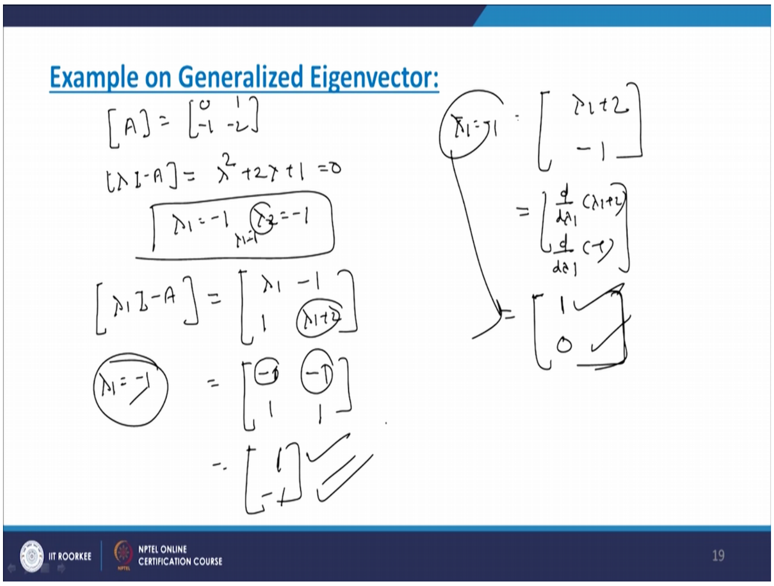

So, I have written a matrix A as a equal to 0, 1, minus 1 minus 2. So, if you solve it. So, what will get lambda I minus A, if you solve it, will get it as lambda square plus 2 lambda plus 1 equal to 0. So, we will get it as lambda 1 equals to minus 1 and lambda 2 equals to minus 1.

So, these are two Eigenvalues. We have got for this matrix 0, 1, minus 1 2 1, minus 1 minus 2. Now, but Eigenvalues are repeated. Now, how to proceed further to get the Eigenvector for such type of problems? So, for that purpose, what I am doing here will take. So, here lambda 2 we have taken we can say, lambda 1 equals to lambda 2. Now, how will proceed further, so, what I will do here? I will write equation as lambda 1 I a equal to lambda 1 minus 1 1 lambda 1 plus 2. Next, we write this as for lambda 1 equals to 2 say minus 1. So, when what will do; when lambda 1 equals to minus 1, we can write in write down this as minus 1 minus 1 1 1. Now, in order to get the Eigenvector, what will take? We will take this part as a cofactor part.

So, we can write down as 1 and for this, we can write down minus 1 that is; that means, we have got Eigenvector for lambda 1 equals to minus 1. Now, problem is that, we want to determine the Eigenvectors for another one. So, what we can do here, we can take what we do we can, we can for lambda 1 equals to minus 1, this cofactor for this is lambda 1 plus 2.

And cofactor for this minus 1 is minus 1; that means, for lambda 1 equals to by conventional method, we have got it. But to get the Eigenvector for another lambda 1 which is equal to minus 1, what we have done? From the physical variable itself, I have to lambda 1 plus 2 and for this minus 1 I have to minus 1. Now, what we do? I will differentiate this equation that is what we will do differentiation with respect to lambda 1 that is lambda 1 plus 2 and differentiation of lambda 1 with respect to this minus 1.

So, if we differentiate it, so, what we will get. Here you get 1 and this is be 0 1 0. This is Eigenvectors for lambda 1 equals to minus 1 that is second Eigenvalue and original Eigenvalues lambda 1 equals to minus 1. We have got Eigenvector as 1 minus 1; that means, whenever we need to determine the Eigenvectors, what you do? We have to write the equation in terms of generalized form, then express this write the minus and that minus.

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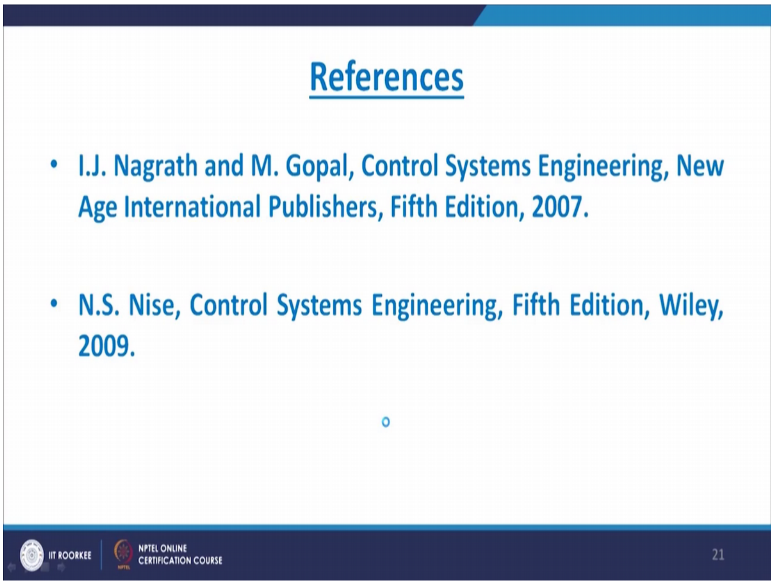

We have to differentiate and then replace the values you can get the Eigenvectors for a repeated Eigenvalues. Now, there are some references, I.J. Nagrath and Gopal and N.S. Nise.

Thank you.