Advanced Linear Continuous Control System
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Lecture – 11
Modelling of Mechanical Systems in State Space

Ok, now we start with Modelling of Mechanical Systems in State Space, the question has come in my mind, why we are doing the modelling of mechanical system, why it is did it.

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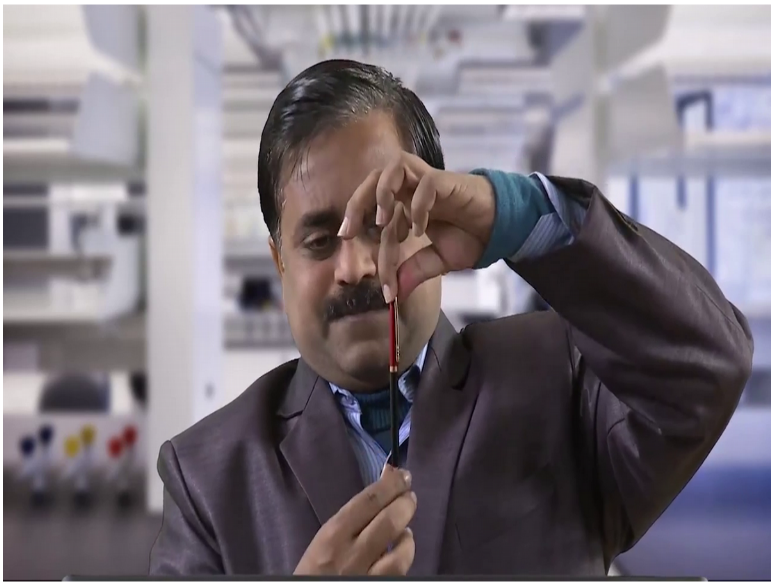

For example, I am taking let us say this is my, a pendulum, a pendulum if you make inverted what you happened, it is fall down. So, here what happen we had to develop certain controller or we have design a controller which will stabilize this pendulum, inverted pendulum. So, may how to make stabilization, for stabilization purpose we required a modelling.

See mechanical system involves pendulum, robotic arms, various type of mass damper systems, so here we required to do model. Let us say another example, robotic arm, what is the purpose of robotic arm? Used to pick the object and place and place it. For example, this is in my arm and this is an motor, this motor is here and as this motor rotate, this arm will move.

So, this is a position of a motor, initial position of a arm and it has to move here, but what happened if you move like this, but these arm making oscillations. So, what happened when arm making the sessions, the placing of the object may not be proper. Therefore, to reduce this vibrations, oscillations, we need a controller and for the controller when we design control, whenever we have model.

Therefore, the modelling of a system, particular mechanical system is very much important. Now, what we will see, how will you model a mechanical system.

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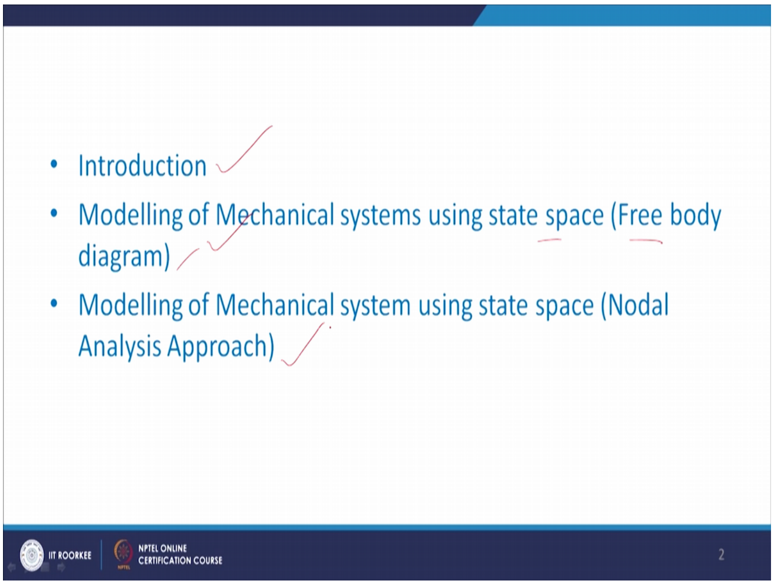

So, in this we will study introduction, modelling of mechanical systems using state space, free body diagram approach. Third, modelling of mechanical system using state space nodal analysis approach.

The mechanical system involves two types of motion, one is called translation motion and other is called a rotational motion. What is mean by translation motion? The translation motion is along the straight line or maybe along the curved path, at rotational motion be motion along the axis. Particular rotational motion let us say motor or motor which using robotic arm that also a rotational motion.

So, when the translation motion is there we apply a force, but when come the rotational it comes a torque, just like electrical system we have register, inductor and capacitor. Here also in a mechanical engineering and mechanical system the elements are represented as mass.

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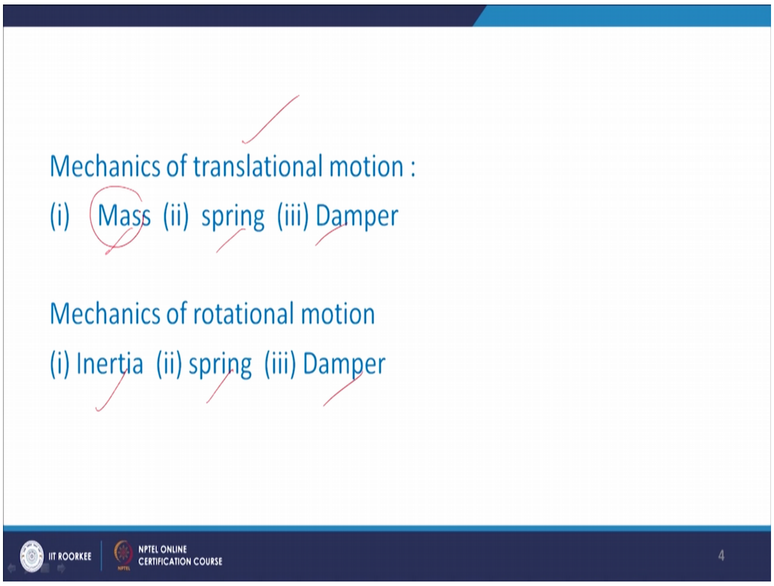

Mass, spring, damper that is in case of translation motion and in case of rotational motion it is inertia, spring and damper, only change is mass.

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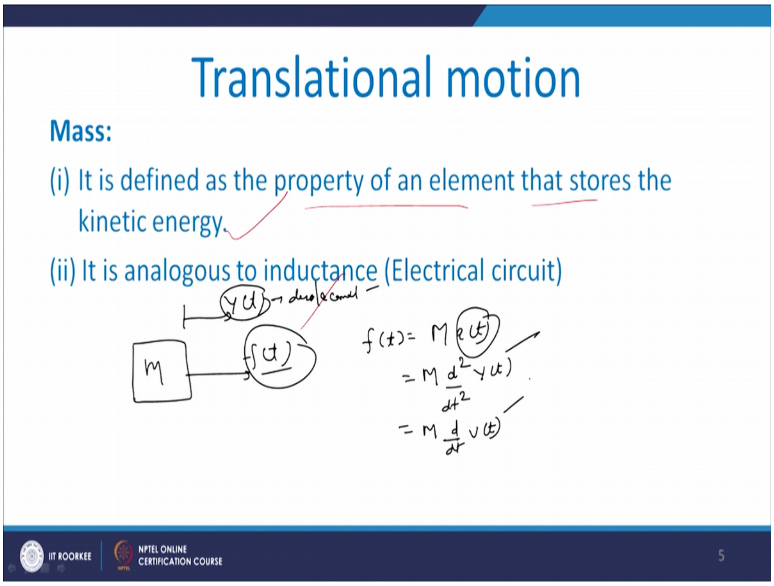

Now, how do you defined the mass, it is defined as the property of an element that stores kinetic energy therefore, it is analogous to inductors. So, how do you represent it mathematically? So, here I am showing this is mass M, force f of t and here output y of t.

So, here this Y of t is displacement, Y of t is displacement and f of t so this f of t equal to M into a of t f equals to M a, this standard expression you might have studied in physics. So, here f of t equals to M into a of t is oscillation so how do you write down the oscillation with respect to velocity. So, first of all write M equal to d square by dt square into Y of t there is equal to M d of t t into v of t.

That is a displacement and this is velocity, now coming to spring.

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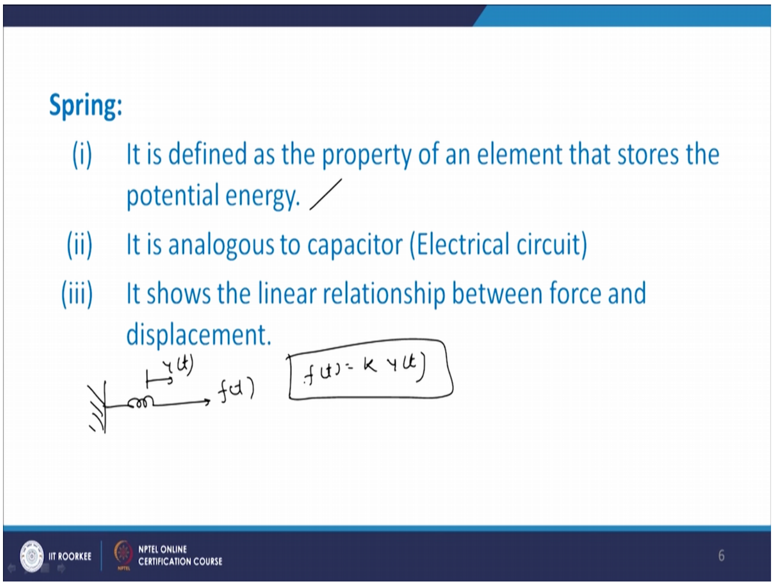

It is defined as the property of an element that stores a potential energy, it is analogous to capacitor in electrical circuit and it shows linear relationship between force and displacement. Then, how do you represent this spring in mathematical way is just see it.

So, here I am showing a spring, here force f of t and displacement y of t. So, this f of t equal to K into Y of t, this is the mathematical representation of a spring.

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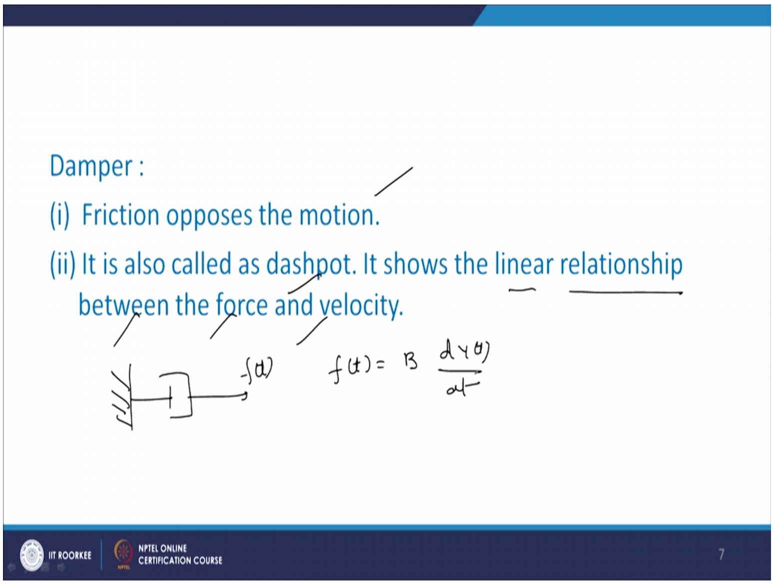

Now, come to damper, damper means it is just like as a frictions which oppose the motion. So, here friction opposes the motion it is also called as dashpot, it shows the linear relationship between the force and velocity.

So, now we represent it mathematically. So, here this goes f of t and this f of t that equals to B d y of t by dt. So, this is the representation of mass, spring and damper.

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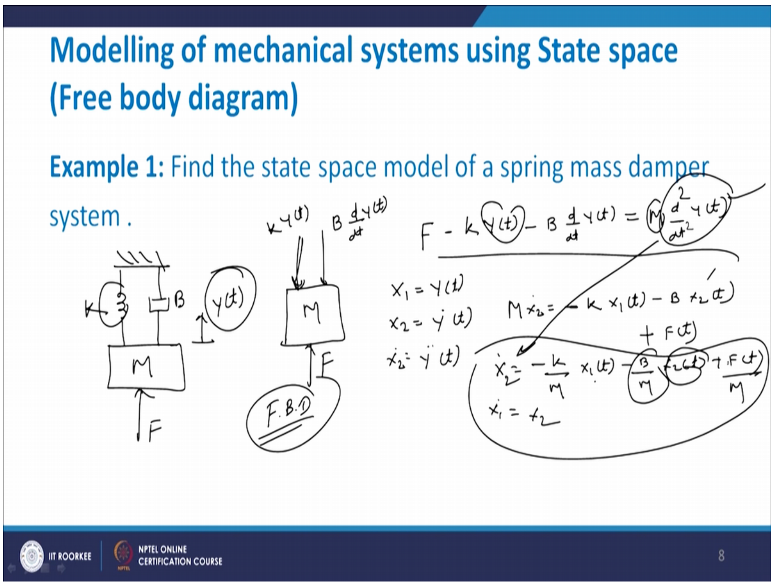

Now, we see if an example a by means of this example, we can determine state space model of a mechanical system. So, first, the first example is related simple system, these involvement only three elements spring, mass, damper. And then we will take the higher order systems, that is many masses, many springs and dampers have been involved. We have start with the first system, this is spring, this is damper, this is mass M, this is force supplied this is k, this is B and displacement is we have shown as Y of t.

So, we have mass, let us say the mass and we apply force to this mass. So, we apply force to this mass it is opposes by the spring and damper. So, here this is a damper and spring, so it is it is opposing. Therefore, we write draw the free body diagram of this one you get a free body diagram like this, this is mass M, we apply force F and it is opposes by k into Y of t.

There is spring force and here it is B d of d t of y of t. So now this is a free body diagram and now why try down the equations. So, equation is we write down as force F minus k, this is opposing. So, we write down as k Y of t minus p differentiation of t into Y of t equal to M d square by d t square into y of t

That is this equation we can write as in this way. Now, once this equation where we have got, now we have to draw, we have to determine it state space model, now we have to assume the state. So, how many states are there, states is based on the displacement see here these are only one displacement that is y t. So, we will take one state, first it as X 1.

So, we can write down as state X 1 equal to Y of t, then another state X 2 we can write as Y dot of t and this X 2 dot equal to y double dot t. So, therefore, if you see these particular equations, now we have to represent this equation in a state space form. So, this state we have to merge here so we will get. So, he will get M right we will get M X 2 dot equal to minus k into X 4 of t minus b X 2 of t plus F of t.

So, here this is M. So, you have y double dot t, so Y double dot equal to X dot minus k, k what is Yof Y of t equals X 1 of t and here is B we this is Y dot t equals to X 2 of t and this F of t. Now, we take X dot equals to minus k by M into X 1 of t minus B by M into X 2 of t plus F of t.

And what about X 1 dot we are X 1 dot that is equal to X 2. So, finally, we write downs in this equation in terms in terms of X dot equal to a X plus B u.

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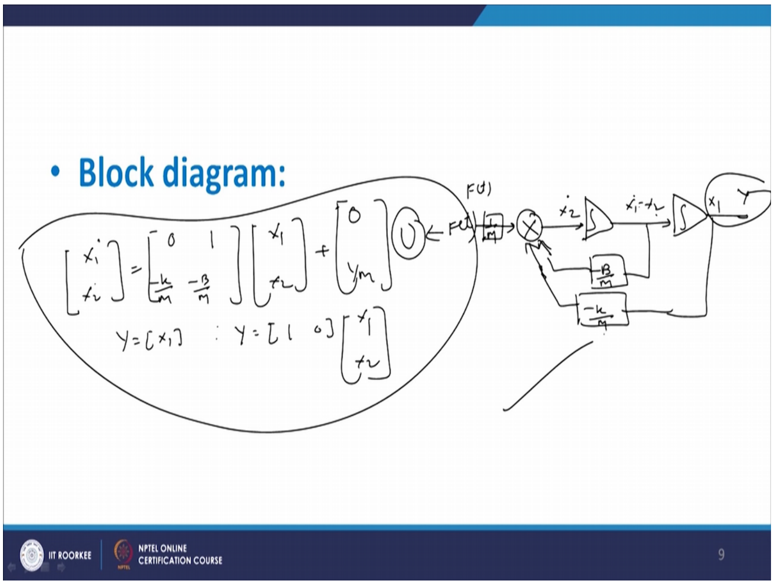

So, here we write down this equation as X 1 dot X 2 dot. So, what is X 1 dot equal to X 2 so we write down as 0, 1, X 1, X 2 and now this is about into U. So, X 1 dot equals X 2 so terms of U is 0 now about X 2 dot. So, we will see what is the X 2 dot, that is here X 2 dot equals to minus k by M into X 1 of t, B by M into X 2 of t F of t. So, here this is basically d square y of t is like this, this is X 2 dot of t.

So, we write down as minus k by M, minus p by M and here 1 by M and this is U basically input easier function F of t. So, you just see here so X 2 dot equal to k by M into X 1 t and what about the terms of X 2 X 2 of t is minus B by M. So, here we written minus B by M and terms of F of t here if you divide by this is become F of t divided by M.

So, finally, the equation is X 2 dot equal to minus k by M into X 1 of t, B by M into X 2 of t, a by M F of t, that is one by M into F of t so we should like this. Now, what is the output, output Y equal to X 1 therefore, this Y equal to 1, 0 X 1 X 2. So, this is the state space model of a mechanical system where these environment of spring, mass and damper.

Now, what you do, we have to draw its block diagram. So, how many states are there, 2 states. So, how many integrators are required, there are 2 integrators are required. So, we show 2 integrators here, integrator 1 this is integrator 2. Now, here output Y then here this integrator, this integrator here is X, here is X 1, here is X 1 dot equal to X 2.

Now, here is X 2, now here is the summing block and here is the force is coming F of t and here is basically a block of 1 by M. So, here X 2 dot equal to 1 by M into F of t and here we have to show minus k by M, k by M, X 1, then here minus B of M into X 2.

So, Y equal to X 1 and this is X 2 dot, X 2 dot equal to minus B by M into X 2 then minus k by M into X 1 plus 1 by n 2 F of t. So, this is a state space model as well as the block diagram of a mechanical systems, now we start with another example.

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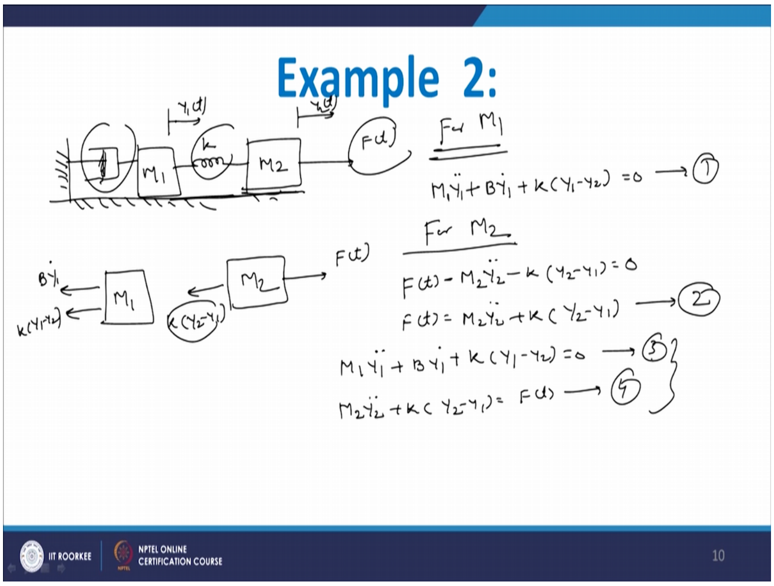

Now, here we are taking the two masses, mass M 1 and M 2 along with we have a spring and damper. So, I am drawing block the diagram of this system to the 0.

Now, here is a damper then we have a mass M 1, this is spring k or this is mass M 2 and this force F of t and this is displacement Y of t, there is a Y 2 of t and here displacement Y 1 of t. So, you can write down Y 2, Y 1 of t, Y 2 of t or simply Y 1 and Y 2 there is no issues. So, now, here we are neglecting the gravity effect and we have to represent in a state space form.

So, here force is applied, the force applied F of t is opposes by the spring as well as this damper and because of that there are two displacement, one displacement is y 1 t and another displacement is Y 2 of t. So, now, what we do, we draw a free body diagram for both mass M 1 as well as mass M 2. So, for mass M 1 if you see is there force acting know, your force acting to 0 because force acting to M 2 only.

Therefore, but this M 1 opposes, opposes this force F of t therefore, it we show in these directions, it goes to B Y 1 dot, that is Y 1 to t or simply Y 1. It goes to Y 1 dot, then here this spring. Spring is proportional displacement whereas, damper proportion velocity X 5 we have shown Y 1 dot. So, here we write k Y 1 minus Y 2 this spring involvement of two displacement Y 1 and Y 2.

So, we have written k Y 1, Y 2 now coming to M 2, this M 2. So, M 2 the force F of t has been applied and it has been opposed by only spring k Y 2 minus Y 1. Now, next step we have to write the force equation for both mass M 1 and for mass M 2, Now, for mass M 1, for M 1 for M 1.

So, for M 1, so this M 1 into Y 1 double dot, because for mass M 1 we represented in terms of a oscillations, for supply to mass because M into a is oscillation. So, displacement to 2 dot Y 1 we have shown plus B into Y 1 dot plus k Y 1 minus Y 2 equal to 0 this is equation number 1 ok. So, M 1 Y double dot plus B Y 1 dot k Y 1 minus Y 2 equal 0 then for M 2, for M 2.

So, for M 2 applied force is F of t, it has been opposed by the own mass that is M 2 Y 2 double dot minus k, this k Y 2 minus Y 1 equal to 0. And finally, this equation can be written as F of t equal to M 2 Y 2 double dot plus k Y 2 minus Y 1 this is equation number 2.

Now, this equation we can again reformat it as, that is M 1 Y 1 double dot plus B Y 1 dot plus k Y 1 minus Y 2 equal to 0, This is equation number 3 and equation number 4 is M 2 Y 2 double dot plus k Y 2 minus Y 1 equal to F of t, this is equation number 4.

So, we have got 4 equations and but we are mainly concerned with equation 3 and 4, now we have to assume the states, now how many states we can assume. So, here there are 2 displacements therefore, definitely we can take 2 states for displacement and then you will find that these are terms, this term involvement of your d square, that means, another states we have to take that is, that means, the here we will find that the number of states are more.

Now, you see how to formulate the states. So, first of all we will take a state X 1, X 1 equal to Y 1 then a displacement Y 1 is represented by X 1.

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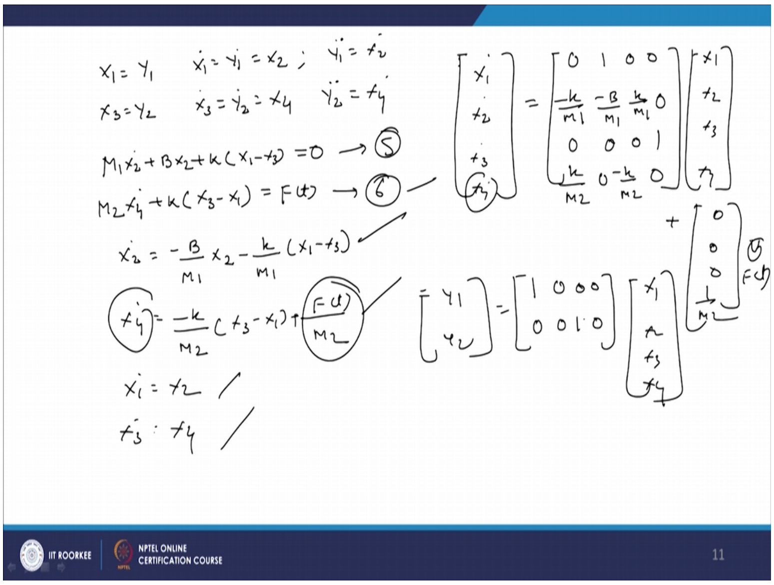

Now, we can write X 1 dot equal to Y 1 dot that is equal to state X 2 and then here we will see, required Y 1 double dot therefore, we can take Y 1 double dot equal to X 2 dot. So, here in a first equation we have taken 2 states, now coming to another equation that is this equation, equation 4.

So, for this equation also we required 2 more states. So, if we write down as X 3 equal to Y 2 and this X 3 dot equal to Y 2 dot and this X 3 dot equals to Y 2 dot. So, that can be assumed by another state X 4 and therefore, Y 2 double dot equal to X 4 dot. So, assuming all these states we have covered both the equation 3 as well as equation number 4.

So, now we write down the equations in terms of the states. So, our equation will get M 1, X 2 dot plus B into X 2 plus k X 1 minus X 3 equal to 0. Now, this is equation number 5, second equation M 2 X 4 dot plus k, X 3 minus X 1 equal to F of t. This is equation number 6.

So, we have to the equation number 5 and 6. So, now, we want the equation in terms of state space model, therefore, what we will do from this equation 5 we can write down as X 2 dot equals to minus B by M 1 into X 2, minus k by M, M 1 into X 1 minus X 3. And similarly from equation 6 we can write down as X 4 dot equal to minus k by M 2, X 3 minus X 1 plus F of t by M 2 and another state that is we have taken X 1 dot equal to X 2 and X 3 dot equal to X 4.

So, here we have got X 2 dot X 4 dot X 1 dot and X 3 dot, now this we have to represent in a state space form. So, how to write down state space form? So, we write down as X 1 dot, X 2 dot, X 3 dot and X 4 dot that is equal to; so we get elements of A matrix, the elements of X 1, X 2, X 3, X 4 plus we will get the elements of BU, that is U mean here force F of t.

Now, what is X 1 dot? So, X 1 dot here is equal to X 2. So, we write down 0, 1, 0, 0, now is it what is X 2 dot? So, we will find that X 2 dot equals to minus B by M by X 2, k by M 1 X 1 plus k by M into X 3. So, we can write down as k by X M 1 minus B by M 1, k by M 1, 0. Similarly, we write the elements of X 3 dot, what is X 3 dot? X 3 dot equals to X 4. So, we can write down as 0, 0, 0, 1 and lastly the elements of X 4 dot, you will find that elements of X 4 dot this is elements concerned with the force F of t.

So, we can write down as first of all we have 0, 0, 0 for meaning this state X 1 dot, X 2 dot, X 3 dot and particularly for X 4 dot we can write down directly 1 M 2, now about the other elements we can write down as k upon M 2 0 minus k upon M 2 into 0. Now, about the output, so there are 2 outputs, so here Y 1 and y 2. So, Y 1 equal to X 1, X 1, X 2, X 3, X 4. So, here 1, 0, 0, 0 and here Y 2 that is equal to you have X 3; that means, we can write down as 0, 0, 1, 0.

So, this is the state space model of a mechanical system. So, similarly you can try a many more examples and try to develop the model, but here we find that, if you draw the free body diagram, if you take many higher order systems or multi many masses spring and damper, it is sometimes very difficult. So, in the literature there is one method is called nodal analysis method.

So, that method can be also useful to get the state space model. So, first of all we will see the nodal analysis approach, what are the steps to be followed, the step 1.

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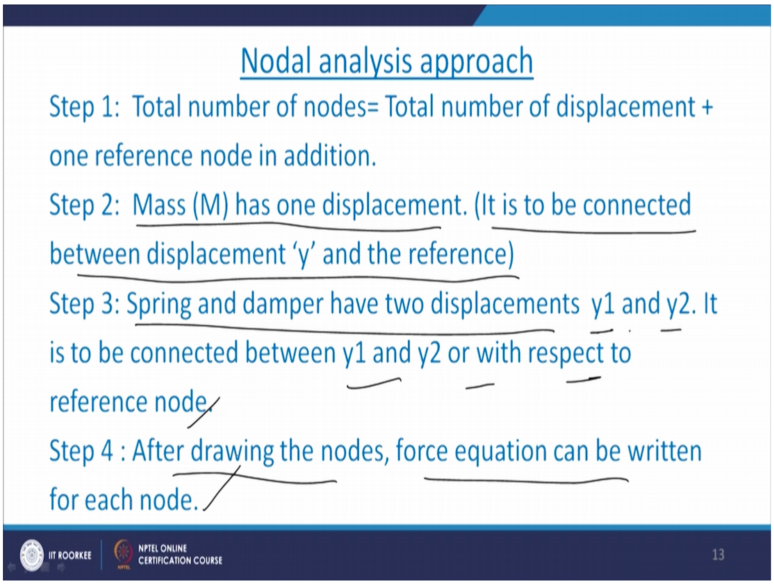

The total number of nodes equal to total number of displacement plus one reference node in addition. Then step 2, in step 2 we will see that as there are mass, spring and damper, particularly mass M is state to write. Where, to represent like this mass M has one displacement, it is to be connected between the displacement y and reference and about the spring and damper. So, spring and damper have two displacements y 1 and y 2 and it is to be connected between a y 1 and y 2 or with respect to reference node. And lastly step 4, after drawing the nodes force equation can be written for each node.

So, these are the steps, which is to be followed to get the model of an mechanical system, now we see the same example and try to find out the model.

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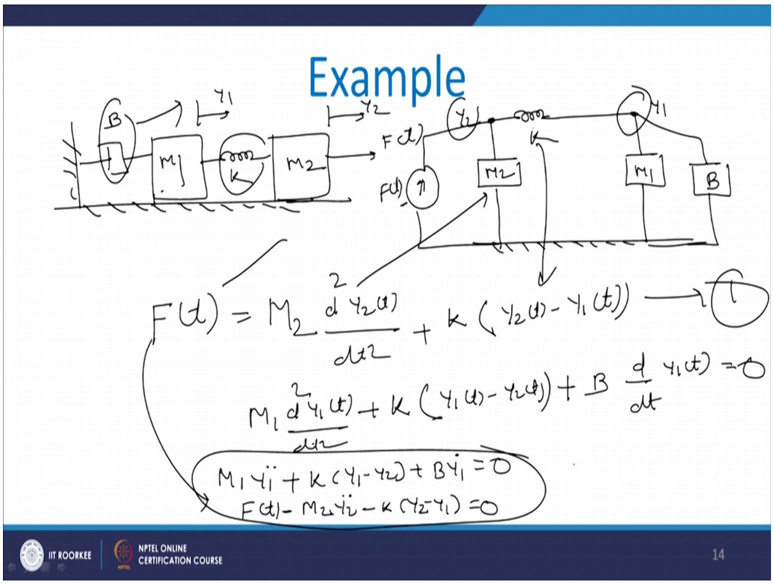

Now, we take the same system again, now here is damper, here is mass M 1 then this is a spring, then we have mass M 2 force F of t then displacement Y 2, displacement Y 1 this k and B.

So, same system we have taken and I had to reorder it, ok. Now, the step one say that total number of nodes equal to total number of displacement, along with one refers node in addition. As you see that in this problem there are two displacement Y 1 and Y 2 and then we take one reference for additions the total number of nodes in this case that is 3. So, we will show it and again the second thing about the mass M 1 and M 2, mass M 1 we have to shown with respect to node and reference, M 2 we have to shown with respect to node and reference, then this spring k it between the two nodes that is Y 1 and Y 2 and B is only concerned with the node Y 1. So, node Y 1, Y 2 displacement Y 1, Y 2 total number of nodes are 3, so we are showing.

So, first of all we are showing node Y 2 here because we have to apply the force. So, I am showing Y 2 here and Y 1 this side. So, here Y 1 is a node, so first of all what is the first step? You connect the all the masses, connected to each node. So, always mass are connected between a node and a reference, so this is a reference node. So, here I am connecting mass M 2 and here Y 1, we are connecting mass M 1. Now, where the force is applied, the force is applied to mass M 2 that is represented by a node Y 2.

So, here I am showing the force applied F of t and corresponding to this, this is spring is connected between Y 2 and Y 1 and here one damper is connected B here. So, this is a diagram using nodal analysis approach, now this is spring k, now we have to apply the equation of motion. So, force is applied, so incoming outgoing so force is applied to node Y 2 and this is outgoing. So incoming outgoing just like incoming current and outgoing current, in electrical circuit so here we have write down the equation as F of t equal to mass M to mass M 2 d square, Y 2 of t d t square plus k, this k. Y 2 minus Y 1, this is Y 2 of t minus Y 1 of t.

So, here we get this equation number 1, so all this things we are shown F of t we have shown, then this mass is shown and this spring part has been shown, now applying the force equation to node Y 1. So, for node Y 1 we write down M 1 into M 1 d square Y 1 t by d t by d t square. Here for this mass, here for this Y 1 the applied force is 0 so now, I we are always showing first of all for M 1 and B 1 and force we take as 0. So, M 1, B square Y 1 t by d t square plus k, now for k, now here we are shown like this from this side. So, your equation is Y 1 t minus Y 2 of t plus B d d t of d d t of Y 1 of t equal to 0 ok.

So, these are the equations and now what we will write, this equation in terms of M 1 here, Y 1 double dot if I add this equation plus k Y 1 minus Y 2, plus B Y 1 dot equal to 0 and about this equation F of t equal to F of t minus M 2, Y 2 double dot minus k Y 2 minus Y 1. So, these 2 equations we obtain and you find that earlier also we have got these equations, that is whenever you upload a nodal analyses. First of all we sets we have to see how many maximum displacement take one reference node addition then applied the force and see where the forces applied, where the forces applied they get that force and whenever the force is not been applied take it as 0 and then using I use the convention of the mass. Mass with respect to oscillations then damper is with respect to velocity and spring is with respect to displacement and we have to write the equation and we can get the model of the system. So, these are few references.

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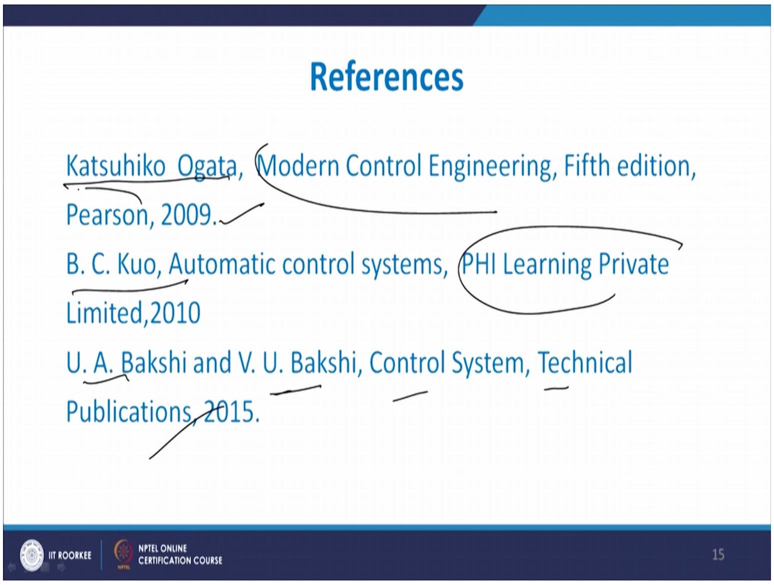

Ogata, B. C. Kuo, U. A. Bakshi and V. U. Bakshi control system technical publications. This is for PHI and this is for modern control unit, this is for Pearson.

Thank you.