Digital System Design
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K-Maps

Hello everyone. In our previous lectures, we have seen how to use Karnaugh map for minimization for Boolean minimization and optimization.

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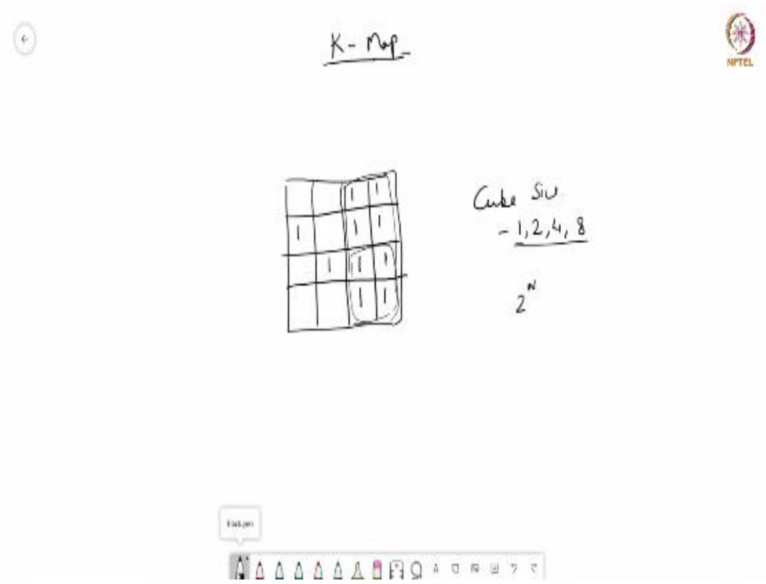

So, we have seen that if we use K maps or Karnaugh maps then any canonical form of Boolean expression like either sum of product or product of sum can be minimized into minimum sum of product form or product of sum form. So, let us summarize how do we see K maps or how do we minimize? So in K maps when we go ahead and minimize. So, first of all in a K maps we fill all the minterms or maxterms.

And then any two minterms which are adjacent to each other. So, for example, if two minterms are adjacent to each other, adjacent means only horizontally or vertically then we can combine them and whatever term we combine it is called a cube. So, cube could only be of size. So, cube size could be only 1 which means the original minterm or it can be two minterms can be combined to form a cube of 2 or there could be 4 minterms which can combine to form a cube of 4.

Similarly, if there are 8 minterms which are close to each other they can also form a cube of 8. Similarly, the size of cube can generically be only 2 is to the power N. Whenever we are trying to combine minterm. So first of all they also have to be cube size would be the power of 2 is to the power N. The other thing is that it is only adjacent. So let us say one term is here one term is here they cannot be combined because they are adjacent, but they are not adjacent.

Adjacent means they are not either horizontally adjacent they are not vertically adjacent. So, this is how we go ahead and create cubes and each cube corresponds to 1 if it is 1 then, so if it is sum of products then each cube will convert to 1 product term. So, let us see couple of more cases here.

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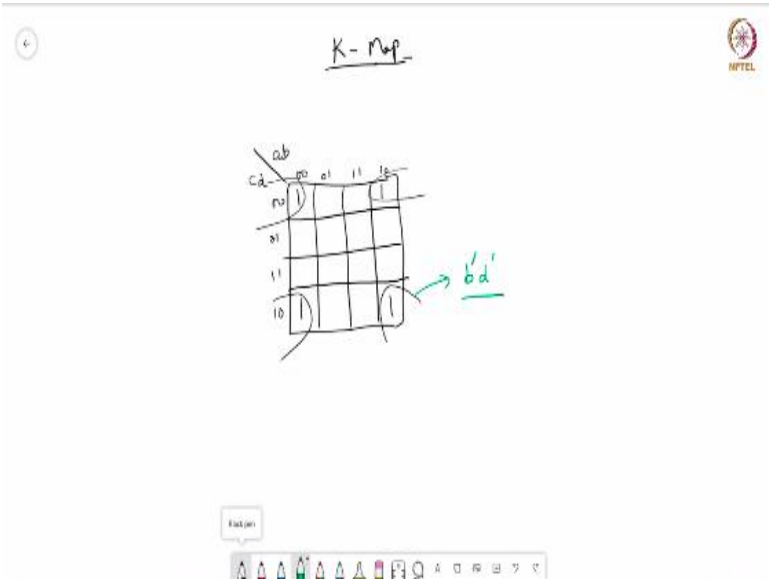

So, if there are, so if there is a one term which is present here, there is a one term which is present here then also they can be combined because they are adjacent because each term which is present in the corner they are also adjacent to each other because of the rotatory nature of K maps. Similarly, if there is a minterm here, there is a minterm here then all of these four can also be combined to form one cube.

So then the next question can also be that what the expression that would be? So, let us say these are the four 1s so what would be expression corresponding to them? Expression corresponding to them would be we will see what are the variant terms, what are the invariant terms? So, here if we see what is the variant part a is the variant and b dash is the invariant. So b dash would be here and from the side we can see that c is variant and d dash is invariant.

So these four corner 1s will form a cube and the mean term, the product term corresponding to this cube would be b dash d dash. So, similarly if there is let us try to see some more examples.

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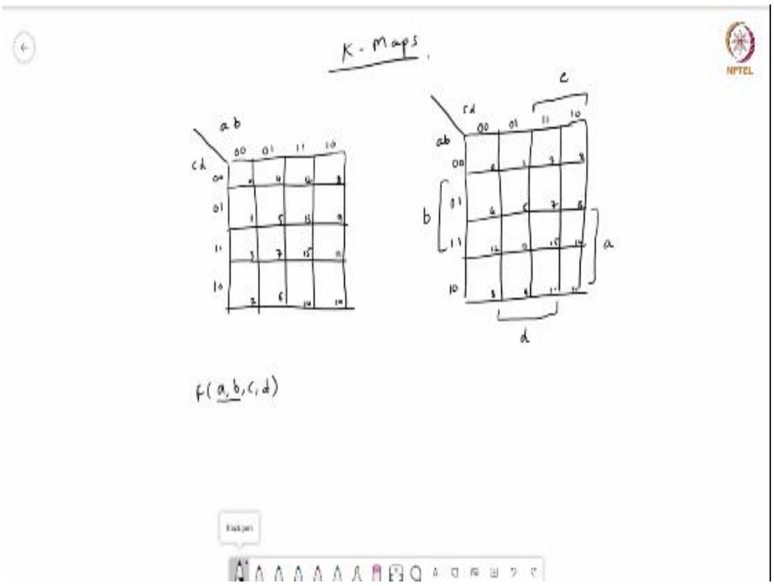

Let us also summarize how to write minterms in K maps. So, let us say this K map is for a function which has four variables a, b, c, d then we take let us say top this a is the more significant variable and d is the least significant variables so let us keep a and b which are more significant on the top and c and d which are lesser significant on the side. Then on the top we will write in terms of a gray code, 0, 1, 3, 2 and on side also we will write 0, 1, 3, 2.

And in corner of each cell we can write the minterm which corresponding to that cell so we can write 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15. Sometime this K map can also be drawn as like we can change the sides, we can take the transpose of this K map. We can say that a and b which are more significant are lying on the side and c and d which are less significant on the top.

So, if we transpose this way or if we are changing the sides we are keeping our more significant variables on the sides then the minterm the way we would write minterm would be 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. So, there is no other difference all the techniques whatever we use in this K map will also be applicable in this K map. So, there is no difference the only thing is that as a part of convenience or convention we can either follow this approach or this approach.

But as a end result there would not be any kind of difference because we can see which variables we can combine the minterms to form cubes and after forming the cubes then we can see what variable what would be the product term or sum term and that product term and sum term will come because of whatever is on this side or this side. And further sometimes to make things easy we also say that this means c and this two columns means they are corresponding to d.

And you can see these two are corresponding to a and these two are corresponding to b. This is also another representation which can help us in finding out what are the variables or what are the product terms corresponding to each cube which we create. So, this was the summary of K maps. Now let us try to understand one more question that given let us say a particular expression is given as a sum of minterms or product of maxterms. Then what is the systematic procedure to find out which all what is the minimum expression corresponding to that function.

You can see that this question has been answered partly that we can form the cubes whatever are the maximum cubes we can use them whatever is the largest cube that is the minimized form, but sometime confusion will arise because the cubes that would be created could be so many. If there are so many cubes, then which cube should be selected is always a question. So how do we select the cubes which are necessary which are important. So, to make that procedure more systematic let us define couple of new terms.

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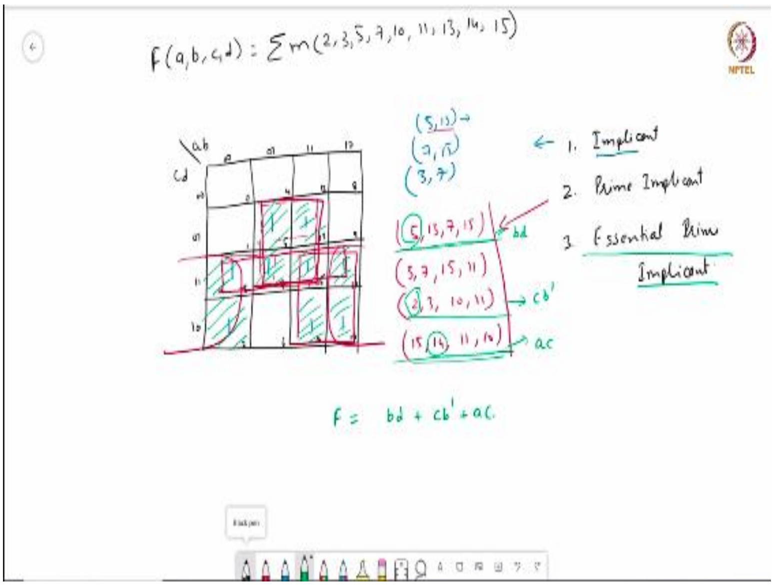

So these terms we will define three terms one is called implicant, second prime implicant and third essential prime implicant. So, let us try to understand these three terms with help of an example. Let us say the function F which is of four variables a, b, c and d is given as sum of minterms 2, 3, 5, 7, 10, 11, 13, 14 and 15. So, let us draw a K map corresponding to this expression so that to make our understanding little easier.

So this is the K map corresponding to this expression. Now let us try to define the term implicant. Implicant is all of these minterms are implicant of this function which essentially means that let us say when this minterm 2 is there then this function is true reverse is not true, but if minterm 2 is there then the function has to be true. So that means in other words any implicant is a subset of this sum of product expression.

So all these minterms form the implicant and similarly let us say we group some of them so let us say we group these two 5 and 13 can be grouped of course they can be grouped. So they also form implicant. Similarly, any of these two terms can also be combined they also become the implicant and let us say these four terms can be combined yes they also form implicant.

So, if I try to enumerate the implicant then there could be many implicants so these are of course all of these minterms are implicant, but group of 5 and 13 is an implicant, 7 and 15 is also an implicant, 3 and 7 is also an implicant. So, there could be various or numerous possibilities of these implicants. Each group and further like corresponding to each group there would be one product term.

So all of these product terms are implicant of this particular function. Now to make our minimization systematic is in on top of implicant we also define a term which is called prime implicant. So, the difference between implicant and a prime implicant is, a prime implicant is one implicant that cannot be further optimized or basically you cannot create a larger cube out of that particular implicant.

So, let us take an example let us a 5 and 13 as implicant. So, is it a prime implicant no because 5 and 13 can further be combined with 7 and 15 to form a bigger cube because a bigger cube can be created out of 5, 13, 7 and 15 so 5 and 13 cannot be an implicant, but then next question is this let us say 5, 13, 7 and 15 is it a prime implicant. Yes, it is a prime implicant because it cannot be further increased.

You cannot create even a large cube which contains all of these terms. So that is why this is called prime implicant. So given this expression how many prime implicants we can find out? So we can see such one cube is this and similarly one cube we can see here that is 3, 7, 15 and 11. We can also see one cube as this and this so that means 2, 3, 10 and 11 and we can also see one cube like this 15, 14, 11 and 10.

So, these are all prime implicants. Prime implicants are the largest implicant which can be formed. So there could be few redundant so that is why we now define a new term which is called essential prime implicants. So essential prime implicants are those implicants which has to be form of our, which is to be part of our final solution. How do we find out that which cubes should be part of our final solution? How do we find that?

So to find out which cubes are part of our final solution what we see is we try to look at our minterms and we see some minterms which are covered by only one prime implicant. So, let us see this minterm 2 so this minterm 2 is covered only by 1. So, this become essential prime implicant. Similarly, if I see so this is because of this two it become a essential prime implicant.

So, similarly if I see this term that means 5 this 5 appears only in this particular prime implicant. So that means this is also an essential prime implicant. Now, let us see this one 14 because of 14 this particular term, this particular prime implicant become essential prime implicant. So the process is first we will try to find out all the prime implicants then we will find out what are the essential prime implicants?

After finding essential prime implicant we will see here are we able to cover all the minterms? If we are able to cover all the minterms then we are done. If we are not able to cover some of the minterms then for the remaining minterms we will see that what would be the optimal solution. So, let us see in this case because of this 5 13 is covered, 7 is covered, 15 is covered because of 2, 3, 10, 11 2 is covered, 3 is covered 10 and 11 are covered.

So because of this 14 15, 14, 10, 11 these are covered. So, essentially in this particular example if I take all these three essential prime implicants they are sufficient to cover all the minterms, so they form our final solution. So, how do we write our final solutions? Let us quickly see so for this 5, 13, 7, 15 the expression so minimum the product expression would be 5, 7, 13, 15.

So it would be b is in the invariant form and d is the invariant from so this is b and d 2, 3, 10, 11, so here invariant is c and here invariant is b dash so it is cb dash. For 14, 15, 10, 11 invariant part is a and c. So, the final function F I can write as bd plus cb dash plus ac this is the minimum expression, minimum sum of product expression for this function. So, let us quickly take one more example that help us in understanding this whole procedure of finding essential prime implicant and finding the minimum or most optimized expression.

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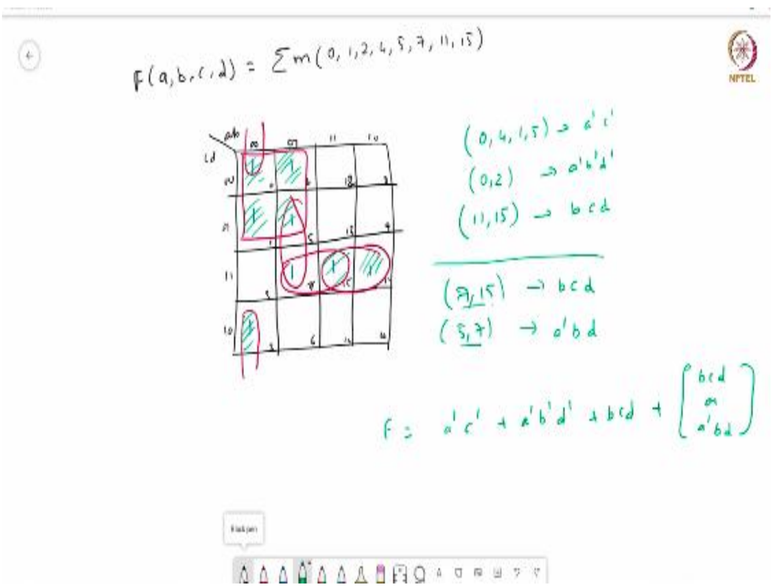

So let us say the expression is F sum of minterm 0, 1, 2, 4, 5, 7, 11, 15. Now this is the K map let us see how to write this expression into K maps. So, this would be 0, 1, 2, 4, 5, 7, 11 and 15. So, let us see which all essential prime implicants so what are the prime implicants? So first find out the prime implicants? So this is the one big cube which we can see, what are the other one?

So we can see this is also another cube which we can create and this is the another one and this is the another one and this is the another one. So these are the prime implicants we have seen. Now what are the essential prime implicants? Visually, we will see that which are the minterms which are covered by only one prime implicants. So we see that this is because of this 0, 4, 1 and 5 will become essential prime implicant.

And this is also covered by only one prime implicant. So 0 and 2 is also another essential prime implicant and this is also covered by only 1 prime implicant. So 11 and 15 form the another essential prime implicants. So these are the three essential prime implicants. Now if we see the terms which are covered by these essential prime implicants these are this, this all these four terms are covered, these two terms are covered and this and this is covered.

So, what remain uncovered is one is minterm 7. So, minterm 7 is remaining minterm. So what means we have to choose now there are two prime implicants. One is 7 and 15 and the other one is 5 and 7. So, these are the two prime implicants which are covering this minterm 7 we have to choose one of it to make our expression complete. So, here the size of both of these expressions, the size of both of these minterms let us see what is the product term corresponding to 7 and 15?

The product term corresponding to 7 and 15 is b, c and d and product term corresponding to 5 and 7 is a dash b and d. The size, the total number of literals which are present in this minterm, this product term and this product term is same. So, it does not matter that whether we take this or whether we take this. Both of them would essentially lead us to similar cost of our final expression.

So what would be our final or minimized expression? So this F can be written as so let us see first of all what are the product terms corresponding to these essential prime implicants? For 0, 4, 1, 5 it is going to be a dash, c dash and for 0 and 2 it is a dash, b dash and d dash corresponding to 11 and 15 it is going to be b, c and d. So, I can write this as a dash, c dash plus a dash, b dash, d dash plus bcd plus if I take any of them either I take bcd or I take a dash b or d both of them are equivalent.

So, from this example what I can conclude is that there is a possibility that there are multiple minimum expressions from there could be multiple minimum expressions or multiple optimized expression which can have similar cost or similar, yeah which could have similar cost. So then it is equally probable, so basically we can choose any of them both of them would result in the correct solution. So, in summary this is how we make use of Karnaugh map to minimize a expression which is given in terms of sum of minterms or product maxterms.

And what is the best part we have seen in this K-maps that because of their visual nature we can quickly find out what are the cubes or what are the maximum group of 1s which we can combine. And so because they are visually appealing, they are quite fast and we can quickly come up with a solution. So, because of all these reasons this K maps are also used we can also use these K maps to find whether two expressions are equivalent.

And what sometime, so there is one expression given now we have to find out what are the minterms corresponding to that expression that also we can quickly do it with K maps and then finally we can optimize it by again converting those sum of minterms to minimum expression using K maps. So, now let us try to answer another question that what if total number of variable in the K map is more than 4. Let us say total number of variables are 5. Can we do that also using K maps?

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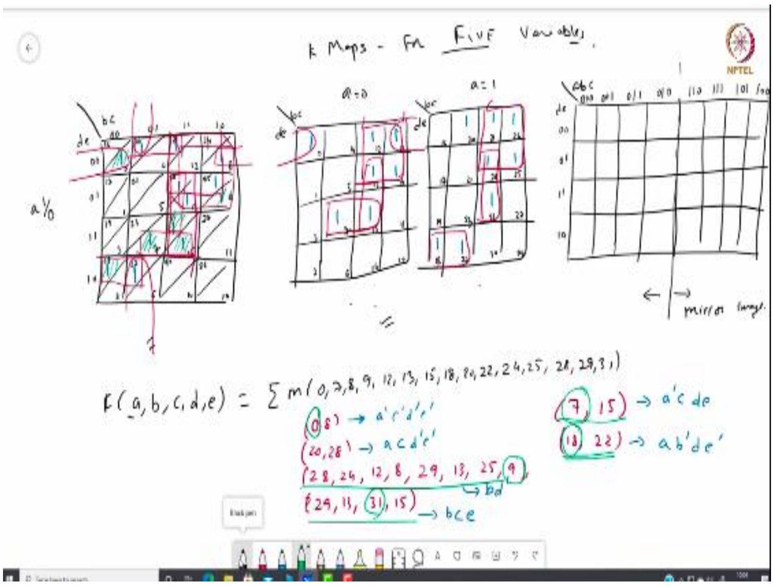

So, if we have 5 variables how will we draw a K map? That is big question because in our 4 variable K map we have already this, we have already drawn in two axes so how can we do it in for K map for 5 variables. So, there are multiple ways because it become sometime tricky that how to draw so I will now give you a multiple methods where we can try to use this 5 K maps with a 5 variables.

Let us say 5 variables are F a, b, c, d and e. So these are the 5 variables how can we draw with these 5 variables. So one particular option could be that we make this a is the more significant, b, c, d are the lower significant ones. So, we take b and c on the top and d and e on the sides and then after that we draw it like a 4 variable K map and then each cell we divide into lower triangle and upper triangle.

And a we write 1 on the upper triangle 0 on the lower triangle. So, this way we are going to divide each of the cell into two parts. The lower triangle is for cell 0 so the order the way minterm would be? It would like 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15. Now a is 1 in upper triangle so these are representing 00, 01, 11, 10, 00, 01, 11, 10. So, the upper triangle would be for 16, 17, 19, 18, 20, 21, 23, 22, 24, 25, 27, 26, 28, 29, 31 and 30.

So, this is how we can divide all the cells and the same 4 variable K map would be used to design a K map for 5 variables. The other alternatives could be that we can also design 2, 4 variable K maps one is a equal to 0 another a equal to 1 and then we can say bc, de on this side, bc, de on this side and then again divide them into 16 cells. And there is also one more possibility that we can put abc on this way and d and e on this side.

So, I am enumerating all of them just to make a sense that what could turn out to be a good solution for us and you will also see in couple of minute that why all these three different variations, why all these three different versions are there what are the benefit of them. So, these are the various ways we can draw K maps of 5 variables. So, I would suggest that either we use this approach or we use this approach.

Although, whatever you find easy you can use any of them. The basic principle still remain same that if we have given some particular function of 5 variables we will draw all these and then we will create combines we will try to create cubes and then after whatever is the bigger cube that we take as a prime implicant, essential prime implicants and out of the prime implicant then we will find out what are the essential prime implicants.

So, let us quickly see with one example how can we? So we can fill this up like this. So, after filling let us try to create cubes. So, one thing which have to be understood here the meaning adjacency means that anything let us say if these are the two terms which are in upper triangle and they are horizontally near so that means that can be created, they can form off cube, but let us see this also adjacent, but one is in upper triangle another is in lower triangle.

So they cannot be combined to create a group although 20 and 28 can be combined. So, similarly 7 and 15 can be combined we can also combine 29 and this so what about these? So we can combine these four in the lower triangle and we can also combine these four in the upper triangle. So they form a bigger cube which can be combined there all the 8 terms could be combined.

So, if I would write what could be combined with this thing so 0 can be combined with 8. So, I can write the amount of or like the number of prime implicants which I have found is 0 and 8 I have also seen 20 and 28, from here I will combine all of them 28, 24, 12, 8 then 29, 13, 25 and 9. So, similarly these four can also be combined so I can say 29, 13, 31 and 15. The other one which is remaining from here is I will write it here 7 and 15, 18 and 22.

So, these are the minterms which can form a cube or which can be combined together. If I want to do the same thing here I have to see that let us say these are the four which can be combined and these are the four which can be combined because they are overlapping. So we can create a bigger cube out of them and so we can say this can be created. Similarly, we can combine this, we can combine this and we can combine this.

Now when we are combining this, so we have, we can also see that these two cubes 13 and 15, 19 and 31 they are also overlapping in these two K map so they can further be combined. So, whenever we are looking we are combining or we are using this particular K map method then we have to observe this K map as a two different planes and if some particular group is there in the upper plane as well as in the lower plane then they can be combined.

However, in this method we see them as a upper triangle and lower triangle and if anything which is there in upper triangle as well as lower triangle they can be combined. If one like this example if a lower triangle is in a different cell upper triangle is in a different cell, then they cannot be combined. So, you can use whatever feel convenient to you, you can use any of these two methods.

The third method of representing which is also given in some of the textbooks where you represent it using like you create in one direction you create 8 cells in other directions you create 4 cells. Here, whatever is written here is the mirror image on the other side. So, there instead of overlapping on the sides you have to see whether the mirror image is correct or not. So, I myself has find these two method as convenient, so whatever is convenient to you, you can use that particular method for your optimization.

So now given these variables now how do we do that? How do we find product terms? So, let us say the product term it is the same method which can be extended here, for the product term what we can do is for example 0 and 8. So 0 and 8 they are both in the lower triangle so it will become a dash, a dash is the invariant part and here we see c dash is the invariant part and d dash and e dash.

Similarly, for 20 and 28 this is 20, this is 28, so we can see because both of them are in upper triangle so a and as is the invariant part and then here c is the invariant part, d dash e dash.

For this one, we see upper triangle as well as lower triangle is there. So a become variant part not the invariant. So, invariant part is b and d dash. For 29, 13, 31 and 15 so that means this particular cube so for them also a is a variant part now this is b, c and e.

For 7 and 15 this is a dash c, d, e for 18 and 22 this is a, b dash, d and e dash. Now, if I have to find the minimum expression I can again use the same method. So these are all the prime implicants I have found, now I need to see what are essential prime implicants. The essential prime implicants let me see that what are the minterms which are covered by only one of the minterm, one of the prime implicant.

So this one is covered by only one prime implicant that is 0 is covered only by this prime implicant 18 and 22 so that means this is only covered by this prime implicant, so this also become essential. 22 could be covered by 2 actually, so 22 could be either this or it could be this. So, now we see here 9, 9 is covered only by 1 so I can say that this is also essential prime implicant. 29, 31, sorry, so here if we see 7 is also covered by only 1 of the implicant this is also essential prime implicant.

And if we see this, this is also covered, 15 is covered by only, 15 is covered by two prime implicant actually, but 31 is covered only by 1 implicant. So that makes this also as essential prime implicant. So in this way we can see that what are the essential prime implicants and after essential prime implicant you see only 20 would be remaining and then 20 either you can combine it with 20 and 28 or you can combine it with 20 or 22 both of them would be equivalent.

So, this is like our concluding remarks here that this if we are creating, if we are minimizing K maps for the 5 variables that makes our life little complicated, but still because K maps are visually powerful we can still find out which all minterms can be combined, which all minterms can be put together to form a bigger cubes. So, there are still some effective mechanism for minimizing Boolean expressions.

What about if number of variables are more than 5 then our life will be more complicated. So, for examples, these two K maps will become 4 K maps then we have to visually see that whether a particular cube is overlapping in two layers or in some other two layers. So, all of those thing will make it little more complicated. So, we have to find out some other mechanism which can be used for more effective treatment for minimization if the number of variables are more than 5. So, with this I would like to close today and we will see that more systematic method which can be more generic in our next lecture. Thank you very much.