Introduction to Time – Varying Electrical Networks
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Lecture 6.5
Applications of Inter-reciprocity: The Switched RC Network

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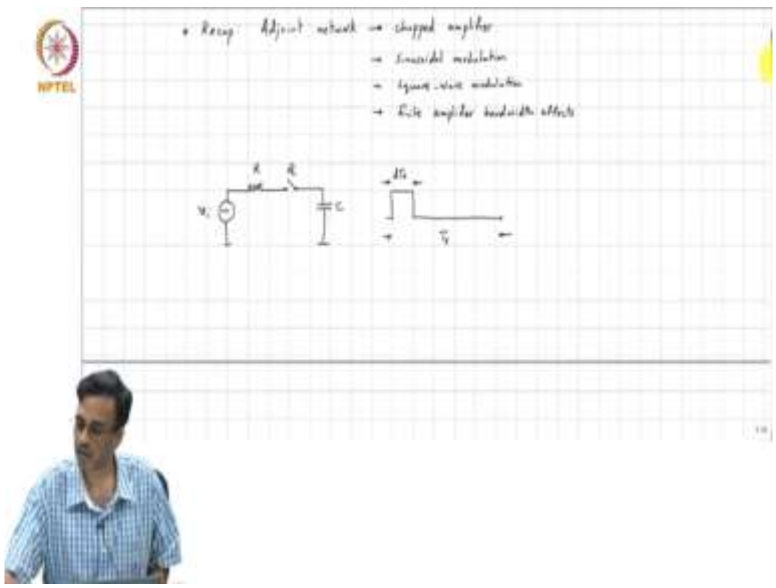

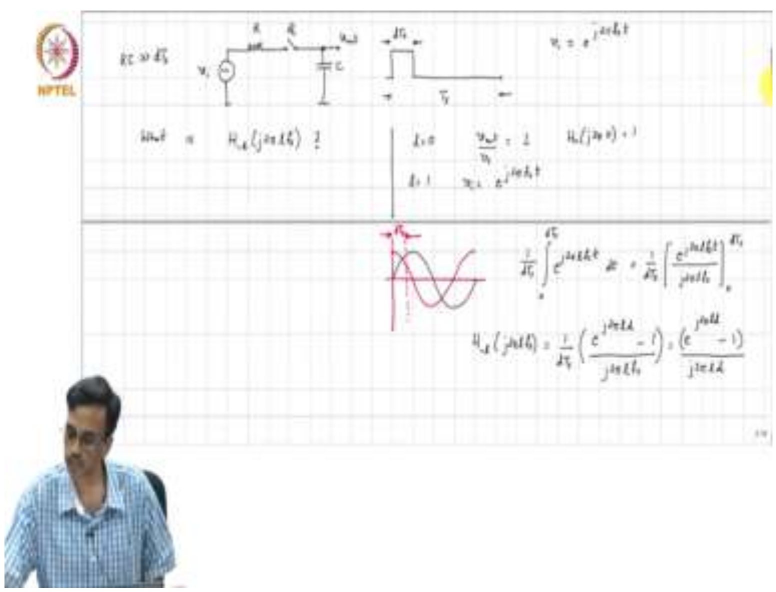

So, in the last class, we saw how adjoint network techniques were useful in determining the aliasing functions in a chopped amplifier. We considered not only sinusoidal modulation for the chopping, we also saw square wave modulation and the motivation for this was that in practice it is much easier to implement square wave chopping and cmos, especially because you can simply flip the direction of switches.

Then we also saw finite bandwidth effects. And so, this is the overview of what we did yesterday. Let us take a final example on the use of adjoints, now, this time at a more circuit level. We have already seen the switched RC network, this is R, this is a periodically operated switch and this is the capacitance and this is v i and phi 1. Let us assume that it has a duty cycle d. So, this is d times t s and the whole period is T s.

And let us say we are interested in finding, as you can see, this is depending on the value of R and C. If R is 0 of course, this is nothing but a sample and hold. If R is very large, it turns out that this can, it is used as a mixer. And where the input is typically close to f s t or integer multiples of f s, so the input is a sinusoidal tone or a narrowband tone, signal which resides at around multiples of the frequency at which the switch is open and closed and the output, therefore the dominant component of the output will be at DC because you are sampling a sine wave at f s at roughly the same rate.

So, the question therefore is, what is H sub minus l of j 2 pi l f s. So, this is what we need to find let us say. So, of course we can do this the hard way which is basically say, well let us do it the hard way, you apply for l 0, well, it is DC so what comment can we make about v out by v i? Well, if you put DC, the output voltage across the capacitor is 1 so v out by v i is 1 and therefore H 0 of j 2 pi 0 is 1. l equal to 1, so that therefore, the v i that you put in is cos, i mean e to the j 2 pi f s times t.

And we are interested in the output at DC and typically if you assume that, let us assume for argument sake, that RC is much larger than t s by t s times d. So, basically it means that the capacitor is not discharging very much or it is slowly discharging and so how would we do this? Well. if you put e to the j 2 pi f s times t, what the capacitor is going to do is simply average that part of the cycle which is between 0 and...

So, basically let us, if we think of the complex exponential as being, there is a cos, the sin and there is a cos, so and this is d times t s and what the capacitor is doing is, essentially averaging that part of the waveform. Because it keeps seeing the same overall cycles and therefore the only way the DC current to the capacitor can be 0 is if the voltage across the capacitor is the average of that part of the waveform. Now, what is the average of that part of the waveform?

Well, you can do sin and cos separately or you can do it together if you keep complex notation. So, this is nothing but e to the j 2 pi f s times t, d t, this d t is the calculus d t and not to be confused with the duty cycle d and this is integral 0 to d capital T s. And this has to be averaged over that period only, not the entire period. So, this over d times T s.

So, this is the, and if the input frequency is not f s but l times f s, you just integrate e to the j 2 pi l f s times t d t from 0 to d T s and this is nothing but one over d t s times e to the j 2 pi l f s times t divided by j 2 pi l f s and this is evaluated from 0 to d T s. So, H sub minus l of j 2 pi l f s is given by 1 over d t s times e to the j 2 pi l, f s times t s is 1, so that is simply l times d minus 1 divided by j 2 pi l times f s, which as you can see f s times t s is 1 so this is nothing but e to the j 2 pi l d minus 1 divided by j 2 pi l d.

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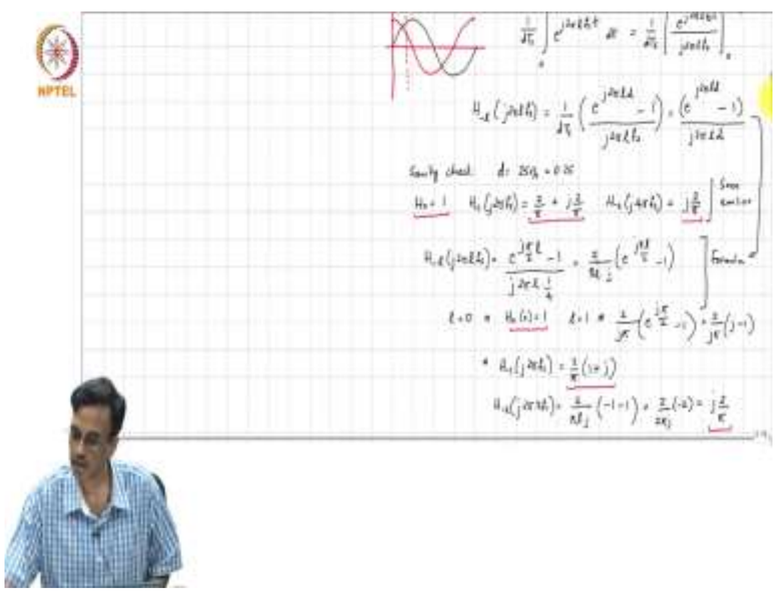

And sanity check, we have already done this for t s by 4, let us say d is 25 percent, which is point 25 then, H 0 of course is 1, we knew that already. H minus 1 of j 2 pi f s is 2 by pi plus j 2 by pi from our, we are averaging this over one quarter cycle. And H minus 2 of j 2 pi f s was nothing but we are averaging over half a cycle, so if you average the sin over half a cycle, we are, sorry, H minus 2 of j 2 point 2 f s this is j 4 pi f s. And what will we get there the sine will continue to be 2 by pi the cos will be 0.

So, that will be simply j 2 by pi. And let us see if this makes sense, so what we have calculated from our, this we have seen earlier, now let us see if we calculate this. So, H minus l of j 2 pi l f s is nothing but e to the j, d becomes one fourth, so pi by 2 l minus 1 divided by j 2 pi l times one fourth, which is nothing but 2 by pi l j, that is correct, times e to the j pi l by 2 minus 1.

Fortunately, clearly as l tends to 0, the formula also gives 1, l equal to 1, what do we see? This is nothing but 2 by j pi times e to the j, e to the j 5 by 2 minus 1. This is nothing but minus, so this is 2 by j pi into j minus 1 and that is nothing but 1 plus, minus 1 is nothing but j square... So, this is H minus 1 of j 2 pi f s is nothing but 2 by pi, times 1 plus j, which is consistent with our previous thing.

Similarly, H minus 2 of j 2 pi f s or j 2 pi times 2 f s is nothing but 2 by pi l times j, times l becomes 2 e to the j pi becomes. it is j pi, so that becomes minus 1 minus 1, hold on, I made a mistake here. Here we made a mistake, yes, we are fine, so this basically is, this is 2 by 2 pi j times minus 2, minus 2 is nothing but minus 2 j square. 2, 2 goes away, so times minus 2 which is nothing but j 2 by... So, this is consistent, these two are consistent, similarly, these two are consistent and of course these two are consistent.

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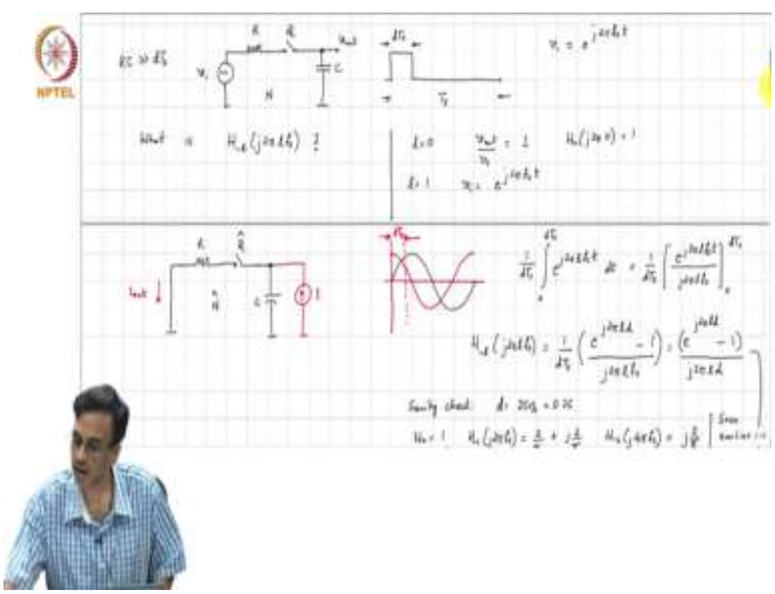

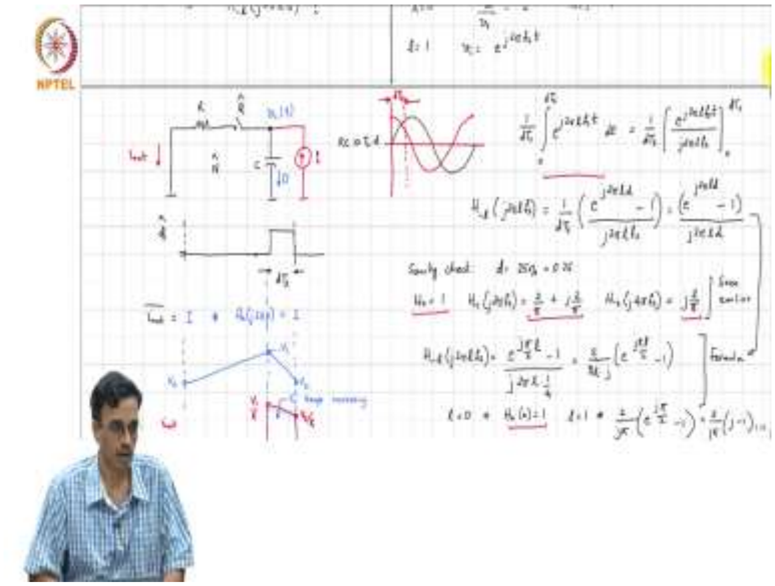

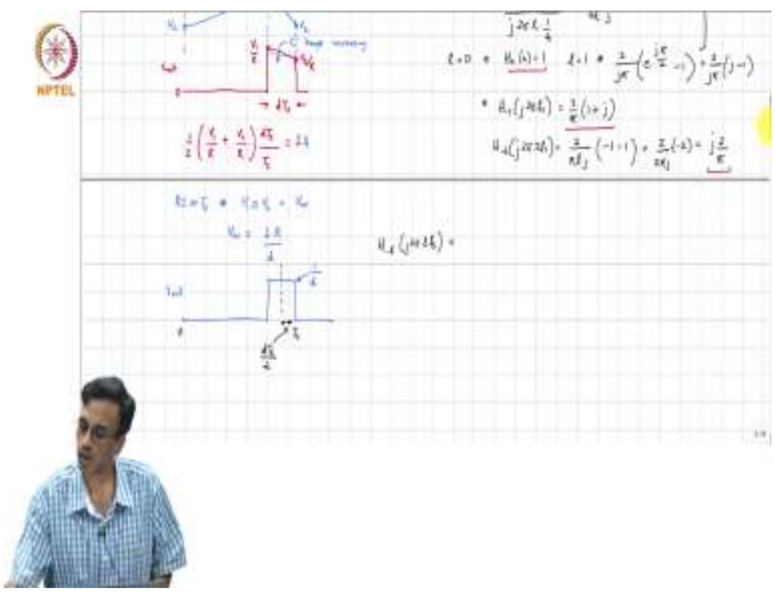

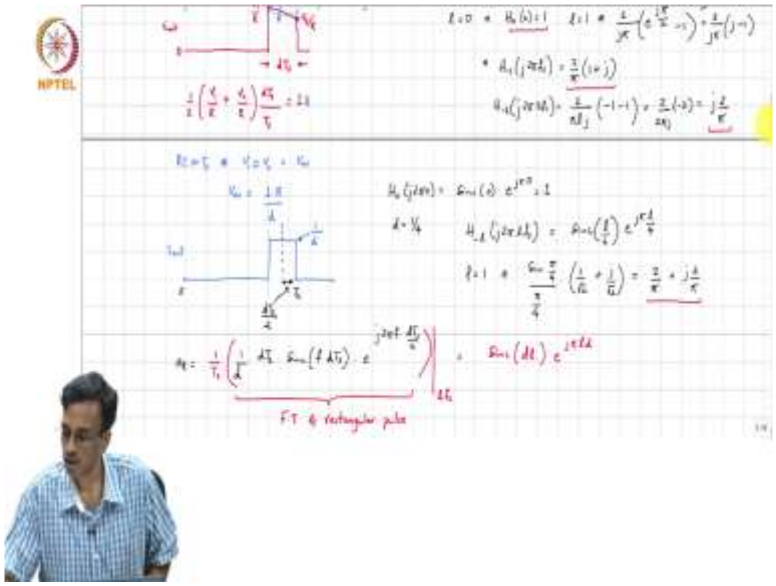

So, now let us see how we apply the adjoint technique. So, what do we do? What do we need to do? We need to first, this is N, so we need to draw the adjoint network and what is that joint network? Well, the voltage source is gone and the resistors remains as it is. We have to flip the timing of the switch and then you have a capacitor here and we are interested in the output frequency which is DC and we are interested in a voltage transfer function.

So, what should we do in the adjoint? So, this is N hat, what should we do as far as adjoint is concerned? We have to apply a current at what frequency? You have to apply a DC current of say 1 ampere and what are we going to measure? Of what? Current where? Through the voltage source which is the same as that through R. So, this is i out.

So, what we need to do therefore is, we have applied a DC current into the output port of the adjoint and phi 1 hat, by the way, is basically, if this is 0, then this is going to be, this is d times t s. So, this is the advantage, this is the period of, this is the phi 1 hat which is the time reversed switch control waveform. And so, again, of course our assumption is that this RC is much greater than t s times d.

So, the first question that I would like to ask is, what must be the average output current? This, the DC current injecting into the capacitor is there for all time, it is a constant current, it is going on injecting regardless of whether the switch is open or closed. So what common can you make about the current through the, average current to the capacitor? Anywhere go to infinity. what comment can we make about the DC current through the capacitor?

It must be 0. So, if the average current through the capacitor is 0 on average, so therefore what comment can you make about the average value of i out? If i out is 0, so the average, the 1 ampere current that you injected can only flow through ground through two parts. One is through the capacitor and one is through the resistor.

The average current through the capacitor is 0. If it was not 0, the voltage across the capacitor would have blown up to infinity. Which is of course not possible and so therefore, there is the average current through the resistor is 0. I mean, through the resistor it has to be 1. So, therefore this means therefore that H 0 of j 2 pi 0 must be 1. Understand?

Now, what common can we make about the waveform, I mean, we need to find the steady state output current waveform and look at its Fourier coefficients and they will give us all the strengths of the, all the harmonic transfer functions that we want. Now, if you look at the voltage vc of t in the adjoint, during when the, when the switch is closed what comment can you make about the voltage? Will it fall or will it rise?

It will fall because the capacitor is getting discharged. And let us call the maximum voltage v 1 and the minimum voltage v 2. And when the switch is off, what common can you make about the capacitor voltage? From where? At what will it start and then? Yes, it will start at v2 because remember it is in steady state so the voltage across the capacitor is also periodic and therefore it will start at v 2 and reach v o.

Now, what comment can we make about the average value of v 1 and v 2? Let me ask a leading question. What comment can therefore can you therefore make about the current through R? If this is the voltage waveform across the capacitor, what comment can you make about the current through R? The current through R is non-zero only when the switch is on. Yes, correct.

So, during this period the current i out will be 0. So, when the switch is closed, what comment can you make about the current through R? It will decrease but what is it, we know the voltage waveform across the capacitor. v 1 by R to p 2 v 2 by R. And what comment can you make about the, if you go on increasing the value of the capacitance, what will change in this waveform? So, basically the capacitance becomes larger and larger. Well, the voltage v 1 and v 2 will become, come closer and closer to each other.

And this is d times t s, so what comment can you make about the average current? We know that the average current flowing through the output is actually 1. So, what comment can we make about the average value of v 1 and v 2? No, what comment can we make about the average value of v1 and v2? So, basically v1 by r plus v2 by R divided by 1 by d t s, times the average current is basically the pulse width, the average value of v 1 and v 2 divided by R, times d t s divided by t s, must be equal to 1. Does this make sense?

And as the RC time constant becomes larger and larger, what happens is that, so this C keeps increasing. As the as the capacitance keeps increasing, v 1 and v 2 become closer and closer to each other. And therefore, in the limit that RC is much much larger than t s, v 1 is approximately equal to v 2 and therefore, which is roughly, in which case this will be the v average, v average is nothing but 1 ampere times R by d. So, this is 1 ampere, so this is 1 ampere times R by d, so this is still dimensionally consistent.

So, if you look at the, but we are actually interested in the current waveform. So, the current waveform, basically in steady state will therefore be will look like this and if R is much larger than d. I mean, if R is, if the RC time constant is very very, large then this current waveform will be, for all practical purposes, is a rectangular pulse, what is the height of the rectangular pulse? It is simply be 1 by d.

The average voltage is 1 R divided by d, the average current will therefore be this divided by, I mean the current waveform will be this. So, to find the H sub minus l of j 2 pi times l f s, what would we need to do? We just have to find the Fourier series of this, this waveform. And how do you find the Fourier series? We know the Fourier transform of a rectangular pulse, we just sample the Fourier transform at l times f s.

So, the rectangular pulse starts, so you can see that you can think of this as a rectangular pulse with a delay of, this is 0 and this is t s, there is another pulse just before 0 because it is a periodic waveform. So, you can think of it as a rectangular pulse which is centered at 0 but then move to the left by t s by 8 because, not is by t s by 8, t s by 2 d, that is the that distance there.

The width of the pulse is t s by d and we moved it to the left by, sorry, d t s by 2. And therefore, so, if we first find the Fourier transform of the rectangular pulse move to the left by d t s by 2 and then sample, do 1 over t s of that and sample it at multiples of... So, the rectangular pulse is the, height is d 1 over d, the width is d t s, so a t sinc f times d t s. And what comment can you make about the Fourier series? And then you have to move it to the left by d t s by 2. So, this is multiplied by e to the j 2 pi, f times d t s over 2.

This is the Fourier transform of the rectangular pulse and this, you multiply by and this you have to sample, you have to multiply by, so the A sub l of this guy is nothing but 1 over t s. So, this is nothing but the Fourier transform of the rectangular pulse. And this is sampled at, needs to be sampled at l times f s. And therefore, this is nothing but sinc d times l times e to the j pi times l times t. 2, 2 goes away, f becomes l times f s. f s times t s is 1, so l times t.

So, A, the Fourier expansion is nothing but, A sub l is basically... And so, H minus l of 2 pi j 2 pi l f s is simply, you know you basically, we know that this is nothing but A. So, if you put l equal to minus 1, what do you get? So, H 0, of course, is sinc 0 times e to the j 2 pi l times 0 which is 1, that fortunately, that works out.

H sub 0 of j 2 pi 0 is sinc 0 times the j pi 0 which is 1. And so, we can verify with d equal to quarter. So, that is nothing but H sub minus l of j 2 pi times l f s, which is nothing but sinc l by 4, e to the j pi l by 4. So, l equal to 1, so you have sin pi by 4 divided by pi by 4, that is the sinc times cos pi by 4 plus j sin pi by 4 which is 1 by root 2 plus, by root 2, which is therefore nothing but 2 by pi plus j 2 by 5. And that is consistent with what we have gotten with... and so on and so forth. And you can verify that the other the other relations is also 4.

So, that, it shows another example where the we use the concept of adjoint to be able tom I mean, in this case actually the circuit was straightforward enough that we could get the answer with the regular method. Also, if you think about it in our regular expression, since the circuit was simple, we have done the analysis for multiple frequencies at the same time by just doing the math. But otherwise, typically you will need multiple input frequencies and then find out.

So, the adjoint, basically, all that you need to do is you just apply the output frequency and then look for the Fourier series of the input and that basically gives you the transfer functions in one shot.