Introduction to Time - Varying Electrical Networks
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Lecture 56
Reciprocity and Inter-reciprocity in LPTV Networks: Part 2, the transfer-function theorem

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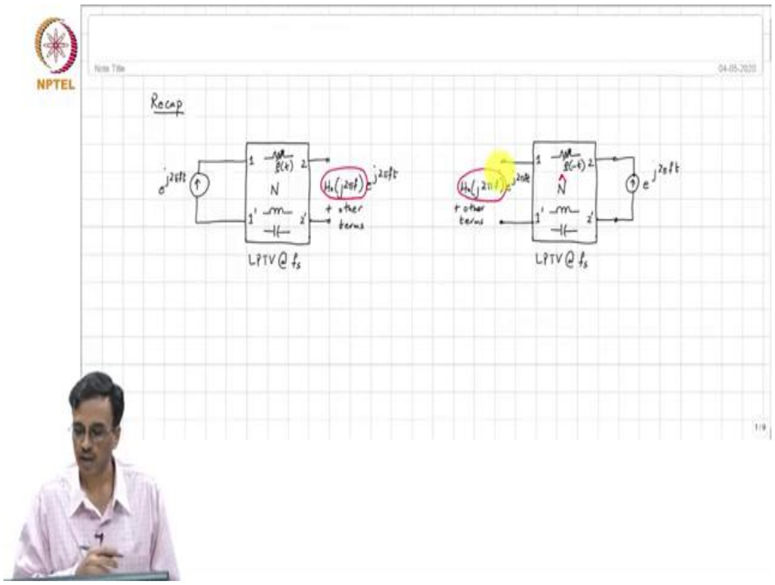

A quick recap of what we were doing in the last class, last class we were looking at the concept of reciprocity, inter reciprocity in linear periodically time varying systems. The network was assumed to be LPTV and varying at a rate fs, we assume that it only consists of time varying conductances, capacitors and inductors.

And we, yesterday we showed that if port 1 of n was excited with e to the j 2 pi ft and the output voltage at port 2 apart from having a whole bunch of tones at frequencies f plus integral multiples of fs it will also have a tone at f and that transfer function is H0 of j 2 pi f f t plus other terms.

What we saw yesterday was that if we formed, so if we call this g of t, if we formed the so called adjoint or the inter reciprocal network which we call n hat where all the inductors remain the same, all the capacitors remain the same and all the conductances are replaced by g of minus t, so this g of minus t is sometimes also called the time reversed version of g of t.

And you excite the output port with a current e to the j 2 pi f t and then it turns out that the voltage developed at cross port 1 is H sub 0 of j 2 pi f e to the j 2 pi f t plus other terms. The key point is to note that the zeroth order harmonic transfer function from the input port to the output port is the same as the transfer function from the output port to the input port in the adjoint network or the inter reciprocal network. We will see why this is significant in a little while but this is what I want to point out.

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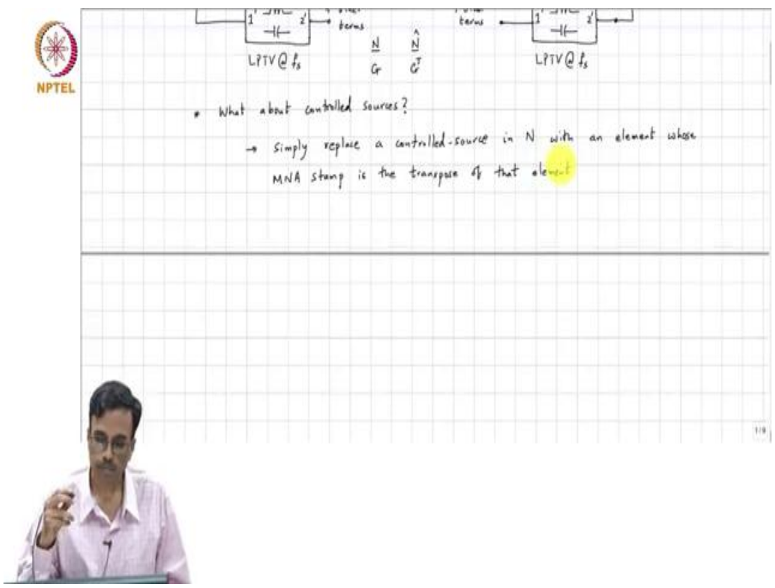

So, yesterday we couple of I mean we did not discuss control sources and remember that the key to realizing or deriving the adjoint network was to recognize that the MNA matrix of n has an MNA matrix g and n hat has an MNA matrix of g transpose. So, now the question is if you have control sources what do we do?

Well, g is simply formed by adding the MNA stamps of the individual elements. So, if you have controlled sources in the original network g, then you need to replace it with an element whose MNA stamp is the transpose of that element. So, again, so the idea is very simple you just replace a controlled source in n with an element whose MNA stamp is the transpose of that element.

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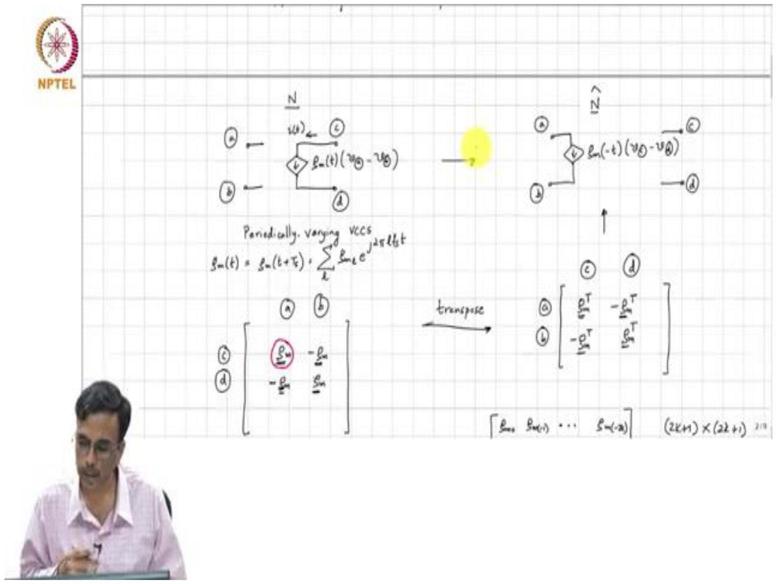

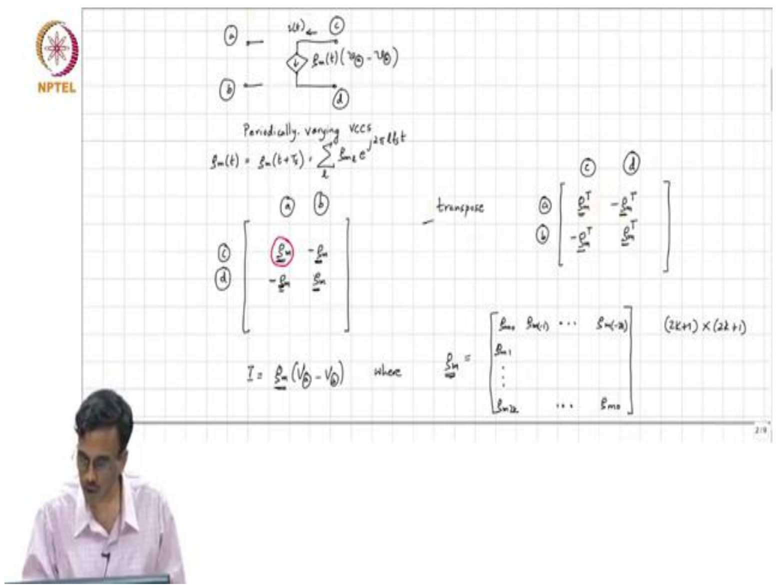

And so let us do this, so let us say you have n and you have a control source like this, remember that if you have a time varying network the gain corresponding to the controlled source or the trans conductance or whatever can also be time varying so we have four nodes a, b, c and d and this current is simply gm of t times v of a minus v of b, so this is a periodically varying control source, voltage control current source.

And gm of t equals gm of t plus ts and this can be expanded in a Fourier series gm sub l e to the j 2 pi lfs times t just like the time varying conductance. So, what do we do? Well, the idea is very similar the MNA stamp of the periodically time varying current source is going to look like this, so if this is the g matrix, this is the there is going to be something which only depend in row c and d because this the current is only flowing between c and d.

And between nodes in the columns a and b because that is the voltage dependence. And if we write and again if we write this in terms of the current in terms of the phasor vectors then the current flowing here is we call that I of t and its current corresponding phasor is I then this is nothing but the gm matrix times the phasor Va at a minus the phasor at b, where gm is a matrix which is of size 2k plus 1 cross 2k plus 1.

And this is gm0, gm minus 1, all the way up to gm minus 2k, this is gm1, blah, blah, blah, this is gm 2k and this is gmk. And so what will the MNA stamp be? This will be gm and minus gm, this will be minus gm and this will be gm. And this therefore is the MNA stamp of the voltage control of the periodically time varying voltage controlled current source.

What do we need to do in the, we now need to find its transpose. And what will the transpose look like? Well, if it appears in the ath and bth columns it must appear in the, you can see that this entry for instance appears is a block matrix but it appears in the cth row and the ath column, so now you must appear in the ath row and the cth column. So, this and it must be transposed.

Similarly, you must also have, so this must be c and d and you will have gm transpose and minus gm transpose minus gm transpose. So, in the ath row you have gm transpose and minus gm transpose and the bth row we have minus gm transpose and gm transpose. So, what comment can you make with regard to the element?

So, this is the MNA stamp so we started with the element and we wrote them in stamp, we transpose the MNA stamp, now we need to get back to the element and how does that look like? So, well this looks like a, it is a voltage controlled current source except that, yeah, very good, the ports are interchange, the controlling and current, control ports are interchanged that is one change so that is basically gm, the currents are flowing between nodes a and b.

And the controlling voltages are nodes c and d. So, this is gm of, I mean what comment can we make now the matrix corresponding to the time varying gain is the transpose of the original one, so what comment can we make it must be time reverse, so this is gm of minus t times vc minus vd, so this is the replacement that we must do in the, to get the adjoint, this is very similar to what we did in the time invariant case except that there is the additional step of having to time reverse the gain.

If the gain is fixed of course, then it is exactly the same thing, it does not, I mean if gm of t does not vary with time whether you flip it in time or not does not matter, but the key point is that you need to do two operations, one is to flip the controlled and the controlling ports and you time reverse the gain. So, that takes care of the voltage control current source, now I think you get the basic idea, I mean it is the rest of it is very straight forward.

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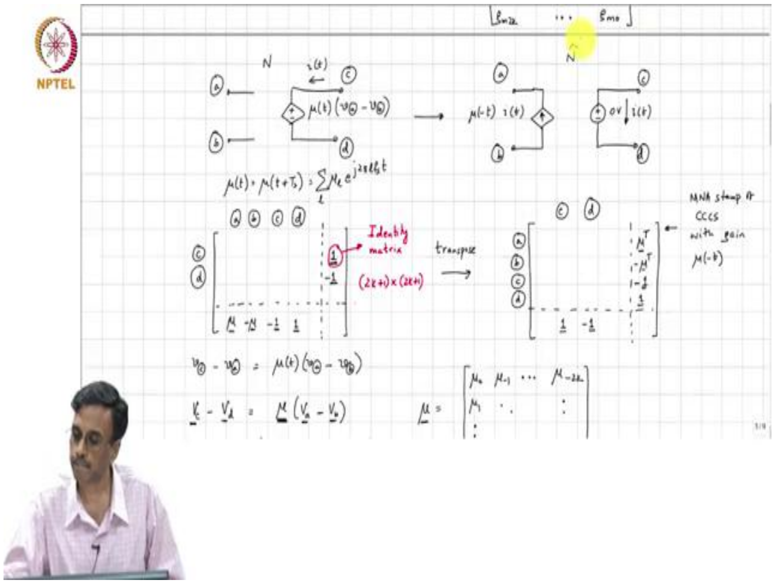

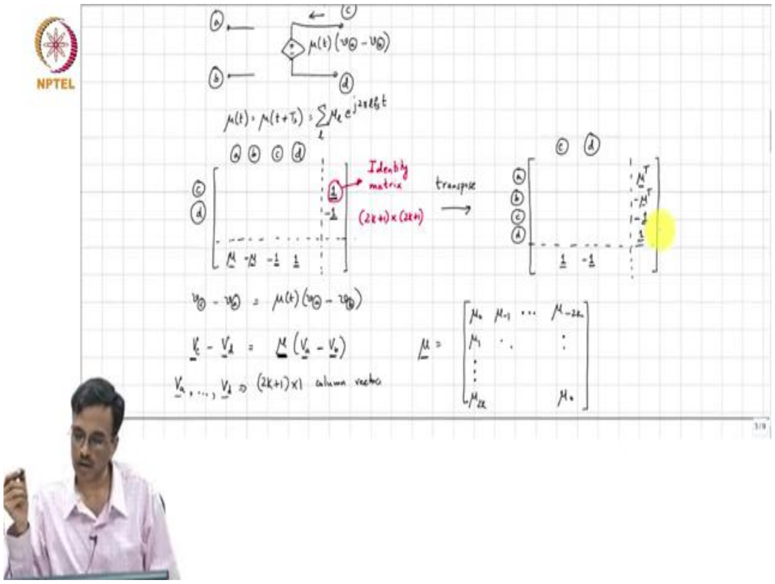

Let us say we have in n, we have a voltage controlled voltage source so this is a, this is b, this is c and this is d and this is mu of t times v of a minus v of b, mu of t is the same as mu of t plus t s which is again sum over l mu sub l e to the j 2 pi lfs times t. And what is the MNA stamp? So, you have I of t.

So, what do we do when we have a voltage controlled voltage source? We have to have an additional, so we add an auxiliary variable which is I of t and we have an auxiliary equation, so vc, the auxiliary equation is vc minus vd equals mu of t times v of a minus v of b. And therefore, in the phasor vector notation this will be vc, the vc vector minus the vd vector must be the matrix mu times va minus vb.

And remember that vc, vd so va, blah, blah, blah, vd are 2k plus 1 cross 1 column vectors and the matrix mu 2 k plus 1 plus 2, yeah is again mu 0, blah, blah, blah, mu minus 2k, we have minus 2 k there, yeah, so it should be mu minus 1 blah, blah, blah, mu 1 blah, blah, blah, mu 2k that is mu 0.

So, what is the MNA stamp? Well, in rows c and d, so basically we have this additional stuff for this variable which is I of t and that we basically we say we have now a matrix 1 and this is a matrix 1, so one, this one is what kind of matrix? No, see this I what comment can we make about the phasor vector corresponding to xi of t, this is a column vector with 2k plus 1 rows which basically quantify the strength of each harmonic in that, in I.

So, that I must be added as it is to each row so what must you multiply it by to get, you have, xi therefore is a identity matrix of size 2k plus 1 cross 2k plus 1. So, this is plus 1 and this must be minus 1, so the 1s in this matrix are basically 2k plus 1 cross 2k plus 1 identity matrices and what do we do about the last we need, so we have an extra variable and therefore we need an extra equation. And what is that extra equation?

That simply saying that mu times, so the vector must be in row a, row b, I mean column a, column b, column c and column d, so you basically have the matrix mu, the matrix minus mu, minus 1 and plus 1. And remember that all the 1s that you see in this MNA stamp are diagonal matrices.

So, I mean this is very similar to what you would get with the time invariant 1 except that now we have, now instead of dealing with the numbers you have matrices that is all. Even writing the

MNA equations very straightforward, once you generate the mu matrix it is just simply plopping that into the big matrix.

Now, so now, what do we need to do in the, if we need to generate the adjoint an element whose stamp is the transpose of this, so this is the transpose, to transpose what we do? Well, what will we get? Whatever is there in the row will become the column and vice versa. So, what do you expect to see?

Well, we need to have there will be some entries in the ath row, the bth row, the cth row and the dth row. And what will the entries be? There will be mu transpose minus mu transpose. Well, the diagonal matrix is anyway the same so minus 1 and 1. And what comment can you make about the last row? Well, that will have entries only in the cth and dth columns and that will be in c it will be 1 and d it will be minus 1. So, what is the last row now telling us? That last equation is telling us that vc equal to vd and the, it is telling us.

Student: Extra equation that is vc minus vd equal to (())(22:03) times v transpose.

Professor: No, no, no, I mean if you have an MNA stamp like this, what is this telling us? I mean see every time you add an extra column, you are drawing current from nodes a and a, b, c and d, so this is nothing but a current controlled current source where the, yeah, so basically this is vc is equal to vd and this is a 0 voltage source and this is a and this is b and the current that is flowing is current control current source.

So, this is the controlling current and what is this current, how is that current related to this control current? mu is varying with time, so mu of minus t, so this is mu of minus t times I of t, because this is the MNA stamp of a current controlled with gain mu of minus t. So, this is the substitution that you must make when you go from the real network to the adjoint network.

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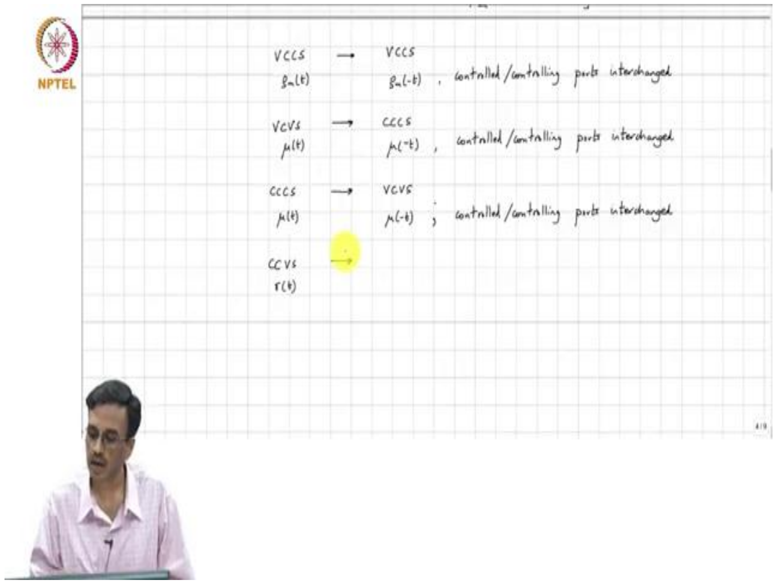

Now, so I am not going to derive the other two, what we will do is simply write down the final result. So, if we have a VCVS you have to replace it by a CCCS, if you have a. So, summary so far is voltage controlled current source gets replaced with the voltage controlled current source if this is gm of t this is gm of minus t and controlled and controlling ports are interchanged.

Now, a voltage controlled voltage source mu of t gets replaced by a current controlled current source mu of minus t. Again with controlled and controlled source controlling ports interchanged. Now, by the same token a current controlled current source with a gain of mu of t what will you do? Voltage controlled voltage source with the gain mu of minus t and again the controlling and control ports interchange.

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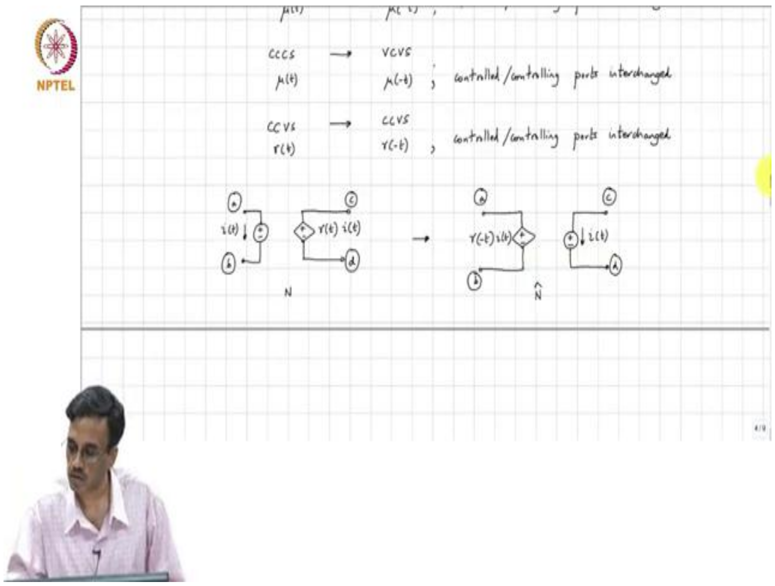

And finally the only thing that is left over is the current controlled voltage source with a gain which is the trans time varying trans impedance r of t so this is the, this is node a, this is node b, there is a current controlled, so this is I of t and this is c and this is d and this is r of t times I of t. So, this is the network in n what should you do in n hat? This is a current control voltage source so what will we have in the adjoint? This will be, it will be a CCVS that is, this is going to be r of minus t times I of t, this is n and this is what you will have in n hat.

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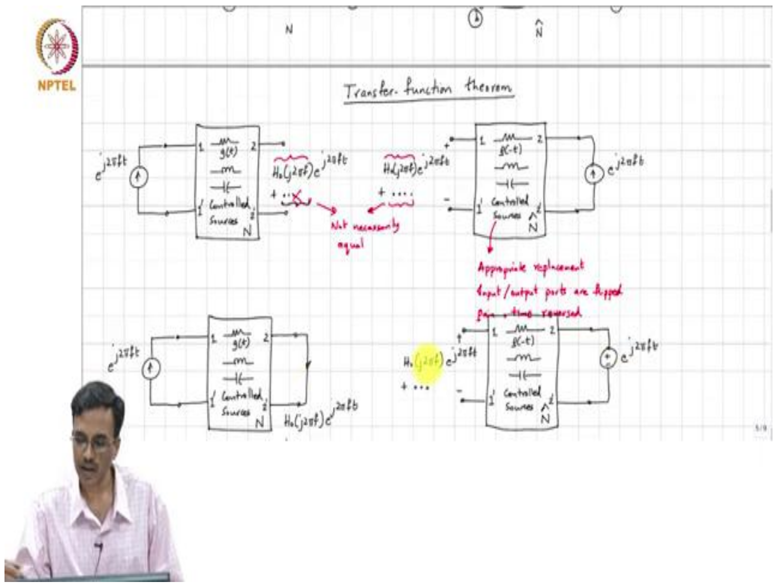

So, to summarize if n now has got g of t l c and controlled sources, this is n and we have the input port and the output port then what we will have in n hat? We know how to form n hat, this is g of minus t l c and control sources. And what is the difference between these control sources and those? Appropriate, yeah, appropriate replacement then input and output ports are flipped.

And gain or trans impedance or whatever is time reversed, this is 1, this is 1 prime, this is 2, this is 2 prime. And therefore, if I inject a current t I will get H sub 0 of j 2 pi ft H sub 0 j 2 pi f times e to the j 2 pi ft plus other terms and likewise if I interchange the location of the excitation what I will get here between ports 1 and 1 prime? I will get the same thing which is H0 of j 2 pi f e to the j 2 pi ft plus other terms.

But I would like to point out that these other terms and these other terms need not necessarily be, all that we have proved is that these terms are equal. So, this is what is called the transfer function theorem, it is not necessary that the exciting quantity is a current and the output is a voltage, you can also have other pairs of, you can have other pairs of, yeah, input sources.

And so for example, if the output was a current rather than a voltage, the output current would be H0 of j 2 pi f times e to the j 2 pi ft plus other terms so what comment can you make about, what we should do in the adjoint network? You have to apply? Yeah current controlled, I mean current gain in the original network will be equal to a voltage gain in the adjoint network, so this will be e to the j 2 pi ft and the voltage you will measure across port 1 will be H0 of j 2 pi f times e to the j 2 pi ft plus other terms, that make sense.

So, this is what you see if the and likewise if the original network has a voltage gain then in the inter reciprocal network it will be the current gain. And if you drive it with a current and you measure the, sorry, we have driven it with the current and measured voltage and current, we have driven it to the voltage measured voltage.

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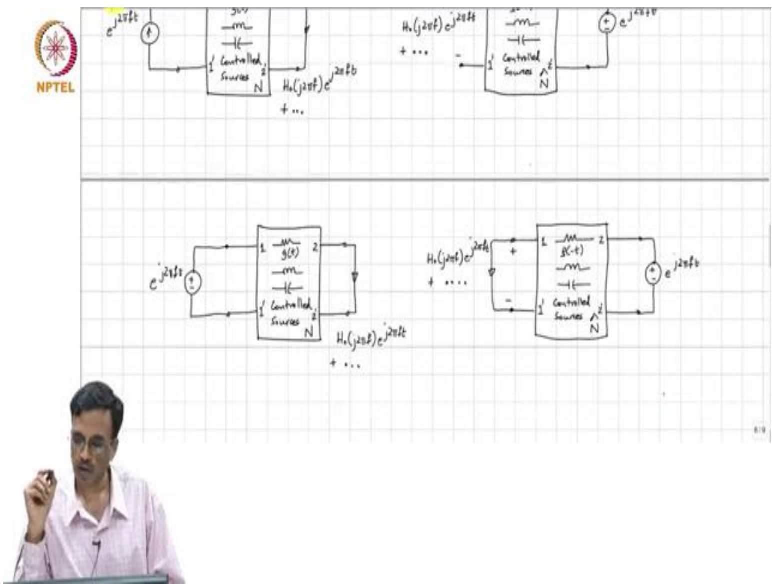

And if we the last thing is that you drive with a voltage. So, let us say you drive it with a voltage so this is e to the j 2 pi ft and you measure current, this is H0 of j 2 pi f e to the j 2 pi ft plus other terms. And what we do on this side? You drive with a voltage at the current variable this is H0 of j 2 pi f e to the j 2 pi ft plus blah, blah. That makes sense?

So, I mean this again there is nothing particularly new here we knew all this, I mean whatever we did, we only proved the result for this case but the same I mean you can do the same, run through the same proof for any of the other 4, other 3 types of excitations, excitation response combinations. So, that way there is nothing particularly new, the only key, I mean the key point is to observe that when you create the adjoint you have to time reverse the time varying components.