Introduction to Time - Varying Electrical Networks

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Lecture 55

Reciprocity and Inter-reciprocity in LPTV networks Part 1

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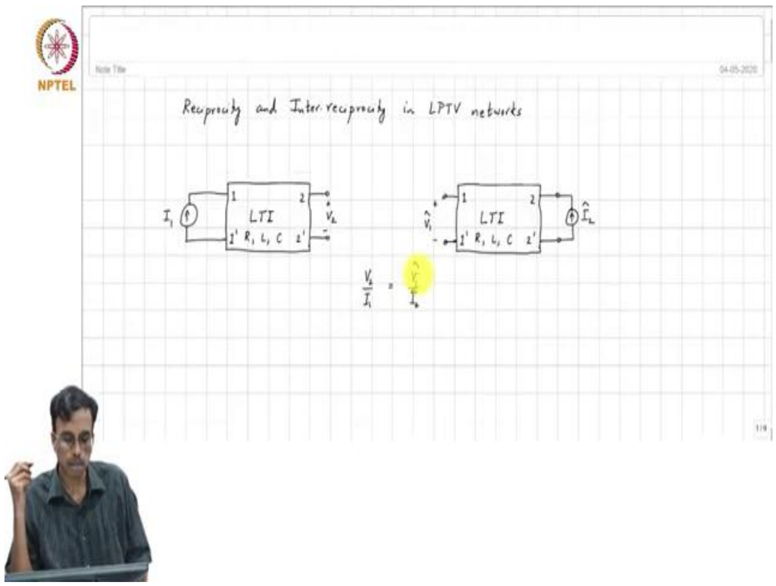

A new topic namely reciprocity and inter reciprocity in linear periodically time varying networks. So, when we were discussing time invariant networks we already have seen and understand what these terms mean. So, quick recap of what we did in the time invariant case, so that we can proceed in pretty much the same manner in the time varying case.

So, this is an LTI network and let us say this only consists of R, L and C and if you drive it with a current I and we observe I1 and this is v2 and all these refer to phasors they are all obviously the same frequency. And if you take this is 1, 1 prime, and this is 2 and this is 2 prime. Now, if we took the same network and interchanged the location of the excitation and the response, so this is I2 hat and this is v1 hat, then by reciprocity v2 by I1 equals v1 hat by I2 hat.

And if we had the network had control sources inside we know how to make this still hold, what you do is this is the network N and this is the network N hat, you would basically we derived rules which would basically make we can make N hat given and which has controlled sources.

So, for the time being let us just stick two networks with simply R, L and C and this is how, this is the result.

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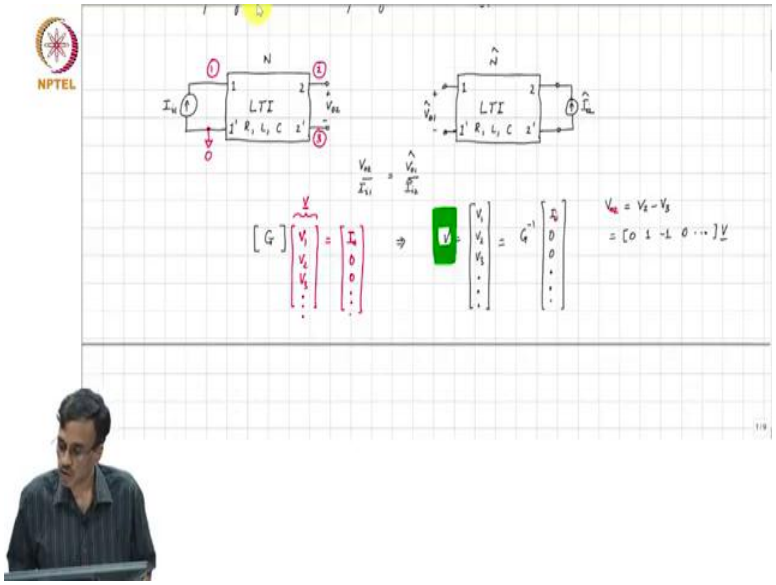

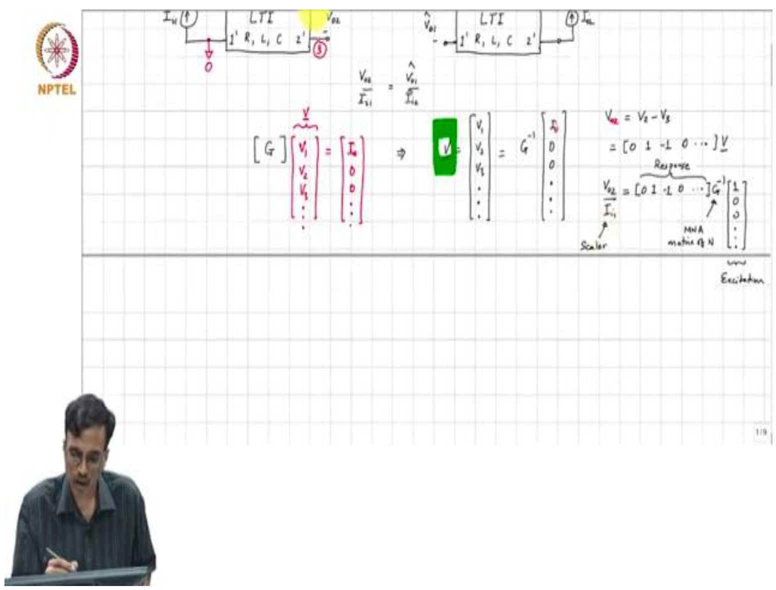

And how did we write this? Well, quick sketch of the proof, we wrote the G matrix of this network N and then we are exciting it, we arbitrarily number the nodes say this was chosen to be ground anyone of these nodes can be chosen as a reference, so this is ground, this is node 1, this is node 2 and say this is node 3. And the first job is to actually find the node voltages inside the network.

So, G we write the MNA matrix that will be G this element G times the input that means times the voltage vector that is basically v1, v2, v3, etcetera must be equal to the excitation vector and that will simply be I1 is going into node 1 and all the entries are 0. So, if we call this vector v then what did we see?

We, this is vector v which is given by v1, v2, v3 and so on is simply G inverse times input vector so that is basically I1, 0, 0. So, therefore if we want, so we sorry, I think I am going to call this vo2 not to get confused with so vo2 by Ii1, I will call this must be vo1 hat by Ii2 hat so this is going to be Ii1 and similarly this is going to be Ii1.

So, our output vo2 is nothing but v2 minus v3 and therefore this is nothing but 0, 1, minus 1 and all zeros times the vector v. Remember that v is this guy over here. So, therefore, vo2 over Ii1 is nothing but 0, 1, minus 1, 0, blah, blah, blah times G inverse times 1, 0, 0 and all zeros for. So, you can see what is happening this is the excitation, this is the MNA matrix of N and this is the response. Now, this is a scalar, this is a number which simply relates vo2 to Ii1.

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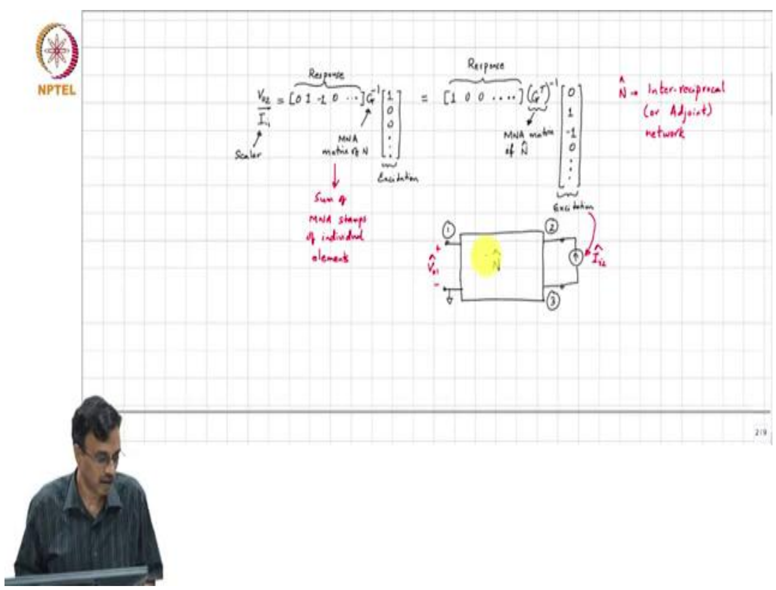

Now, this can, because this is a scalar can as well be a scalar is the same as its transpose, so this is the same as 1 0 0 times G transpose inverse times 0 1 minus 1 0 blah, blah, blah. So, now you can interpret this expression as taking a network N hat whose MNA matrix of N hat this is the, you can interpret this as being the excitation of a network N hat whose MNA matrix is the transpose of the MNA matrix with the original network and the excitation is that vector and this is the measurement response.

So, if you do this then if you draw the circuit corresponding to this, this is N hat and the excitation is, I mean the excitation is between, remember this is node 1, this is ground, this is node 2 and this is node 3. The excitation is between nodes 2 and 3, so that is the interpretation of that vector there.

And the measurement is basically saying, it is the voltage we are finding all the node voltages and we are measuring just the potential of node 1 so this therefore, so if this is I2 hat, this is going to be vo1 hat. And what this is telling us is that, we have interchanged the location of the excitation and the response and the transfer function remains the same.

Now, the key point is that N hat is the inter reciprocal or also called the adjoint network and this network is formed, remember the, how did, how do we get G? G is simply the sum of MNA stamps of the individual elements and now N hat is the transpose. So, in other words to form N hat you basically replace every element by an element whose MNA stamp is its transpose.

So, if you have a 2 terminal element like a resistor or a capacitor or an inductor, if they all remain the same with controlled sources you flip the orientation, we have seen, we have seen how to derive them. So, this is how we did this in the time invariant case. Now, let us see what happens in the periodically time varying case.

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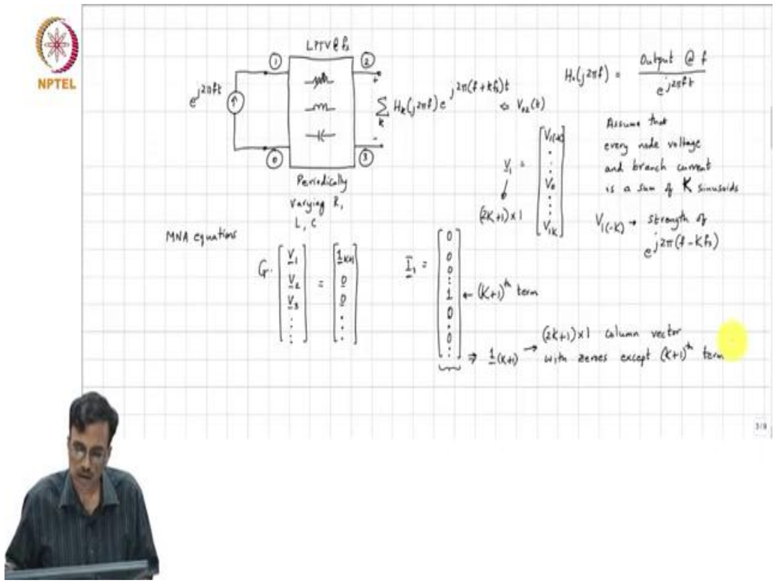

Again, to begin with we will restrict ourselves to networks with so this is LPTV at fs and again this is node 1, I will choose this to be node 0, this is node 2 and this is node 3, I will excite it with an input current which is e to the j 2 pi ft and I will measure the voltage at the output and the voltage as you know is not only consists of f but all components which are variable.

So, this is H sub k of j 2 pi f e to the j 2 pi times f plus kfs times t. Now, suppose we are interested in finding H naught, H naught of j 2 pi f, so is nothing but the ratio of, so this is that let us call this, this is nothing but vo2 of t. So, is nothing but the component ratio of the output at the frequency f to the input current at frequency f, sorry this is yeah, so the output at f will be some H0 of j 2 pi f times e to the j 2 pi ft, where e to the j 2 pi ft in the numerator, denominator goes away and this is basically the ratio.

Again there is not much of a difference it is just simply the ratio of the phasor corresponding to the frequency f in the output to what we apply in the input. And as I said we restrict ourselves to networks with R, periodically varying R and fixed inductors and capacitor. And a special case of periodically varying R as we have already seen in this course is a periodically operated switch. Now, this is the transfer function H 0 of j 2 pi fs is what we are interested in.

So, how would we find that out? Well, again we will write the MNA equations. Now, what do we, when we write the G matrix the, again we get a set of equations, so we will write the MNA equations, so G times the v matrix, the v matrix itself now consists of, is a each v is basically a column vector which contains information about the strengths of every frequency in that particular node.

So, let us call that, so this is v1, v2, v3 and so on and remember v1 is nothing but, the vector v1 is nothing but we assume that there are that every node voltage and branch current can be expressed as, is the sum of capital K sinusoids. So, this v1 basically will consist of a phasor which is v1 minus k blah, blah, blah, v0 and v1 of plus k, so this is a 2k plus 1 cross 1 column vector.

So, likewise every branch current and every node voltage can be expressed by a bunch of 2k plus 1 numbers where each number quantifies the strength of that particular tone. So, this is, so for example v1 of minus capital K is basically the strength of the tone e to the j 2 pi f minus capital K times fs and in practice this k can be chosen to be sufficiently large so that the entire waveform is captured with sufficient precision.

But since we are only worried about a theoretical development here, we not worry about the practical issues about what how large that capital K must be and so on. So, G times, the G matrix times the voltage vector is equal to the current excitation and the current excitation also now we have to at every node there is a current being injected and that current can also be having components which are of the form f, f plus, minus fs and so on.

So, each current excitation term will also be a vector, yeah, so it is a column vector, so let us call this capital I1 and in our case it will be all zeros. And what is capital I1? Is simply the vector of components of a current which are exciting node 1 and they are all 0 except for and basically so this will be 1, so basically this is the Kth or K plus 1th term.

So, rather than write notation like this, what I will simply a vector which is got a 1 in the k plus 1 term I am going to call this as 1 sub capital K plus 1 and that is just for us to understand that only the K plus 1th term in there in that vector is 1 all the other terms are 0. So, basically 2k plus 1 cross 1 column vector with zeros except the k plus 1th term.

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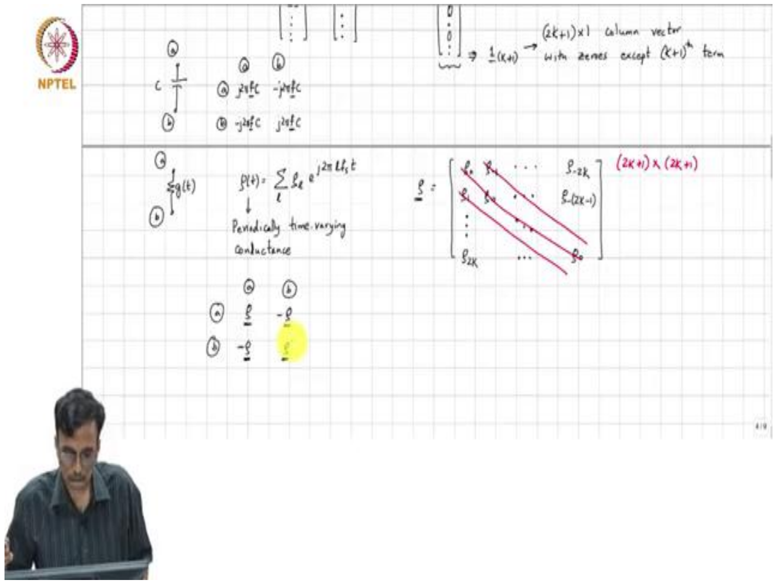

So, and what is G? G is again the sum of the MNA stamps of all the elements and what are the MNA stamp of a capacitor connected between nodes say a and b, a and b yeah, very good, so what will we do? Well, in the matrix you have this j 2 pi fc minus j 2 pi fc minus j 2 pi fc and j 2 pi fc.

And what is f? f is nothing but that diagonal vector which goes from f minus k times fs all the way up to f plus k times fs. So, this is the MNA stamp of a capacitor and likewise you will have a resistor stamp also, I mean an inductor also will have the, will have a similar stamp. And a periodically varying conductance say this G of t, we will have we first form a matrix G remembering that G of t is nothing but sigma over l of G sub l of j e to the j 2 pi lfs times t.

So, this g of t is a periodically time varying conductance. And these numbers can be put in a matrix like this so this is g0, g minus 1 blah, blah, blah g minus capital K g1 g0 g minus k minus 1 and all the way down here you get gk this becomes g0. So, it will turn out that all the diagonal elements are the same, all these elements will be the same, these elements will all be g1 and so on and such a matrix is, this is a 2k plus 1 cross 2k, sorry, this must be minus 2k g minus 2k g minus 2k minus 1 and this becomes g.

So, this is a 2k plus 1 cross 2k plus 1 matrix. And the idea is that if you apply a voltage across a and b which is got a given harmonic structure then you multiplied with G and you will get the resulting harmonics in the current waveform. So, the MNA stamp of g of t therefore as we have seen earlier is that we have just like what we did for a in the time invariant case is given by this matrix g minus g minus g and g.

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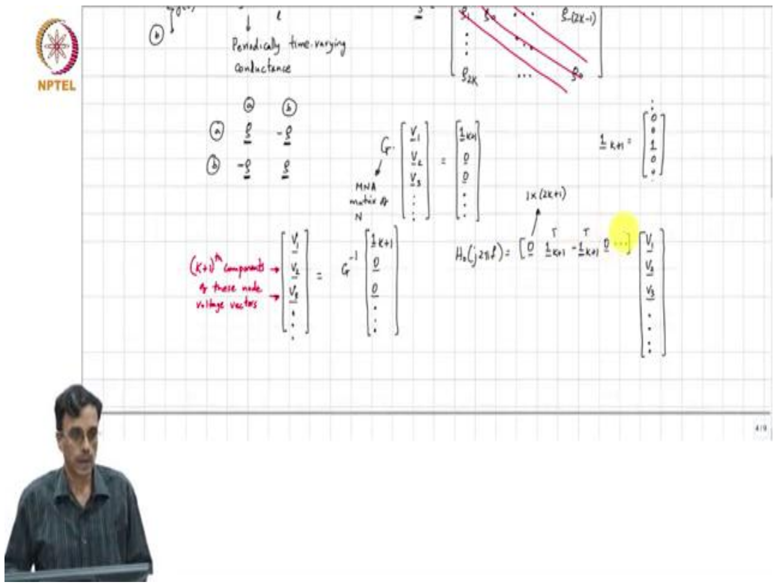

Now, so what is H0? So, how do we go about finding H0? We first we write the MNA matrix of N, we get the voltage vector I mean this times the voltage vector is the current excitation vector, as we did with the time invariant case we will first write v1, v2, v3, blah, blah, blah must be G inverse times 1 capital K plus 1 0 0 zeros everywhere.

So, this will give us if you solve this set of equations this will give us the information about all node voltages and which basically means that at each node it is telling us what the strengths of the various sinusoids are. Now, but we are not really interested in all the node voltages. Which node voltages are we interested in?

We are only interested in v2 minus, we are interested in these two voltages and remember that v2 vector and v3 vector basically each one of them contains, it contains 2k plus 1 components so we are not really interested in all those numbers either, we are only interested in the k plus 1th component of these nodes.

So, what do we do? So, our output so H0 of j 2 pi f is nothing but what do we do? We should take v1, v2, v3 and so on and multiply it with a vector 0, this vector 0 is a this is what kind of vector now, is it a row a vector or a column vector? This v1 is a column vector it has to be a row vector with, so this is 1 row and 2k plus 1 columns.

And so we are not interested in v1 we are interested only in v2 and which component of v2 are we interested in? k plus 1th component, so we would have to put here a row vector which is 1 in the k plus 1th term. So, but we already have a notation which says 1 k plus 1 is a column vector with 1 the k plus 1th position and zeros everywhere. So, this will then therefore be if you want a row vector you have to write the transpose. So, this is 1 k plus 1 transpose. Does that make sense? So, what will be the next term?

Student: Minus 1 k plus 1.

Professor: Very good, it will be minus 1 k plus 1 transpose and then the rest of it will all be zeros. Now, sorry I do not know, this is fine the this one we know is nothing but G inverse times the excitation vector, so this is 1, blah, blah, blah, so let us write that because it is important we will write that separately.

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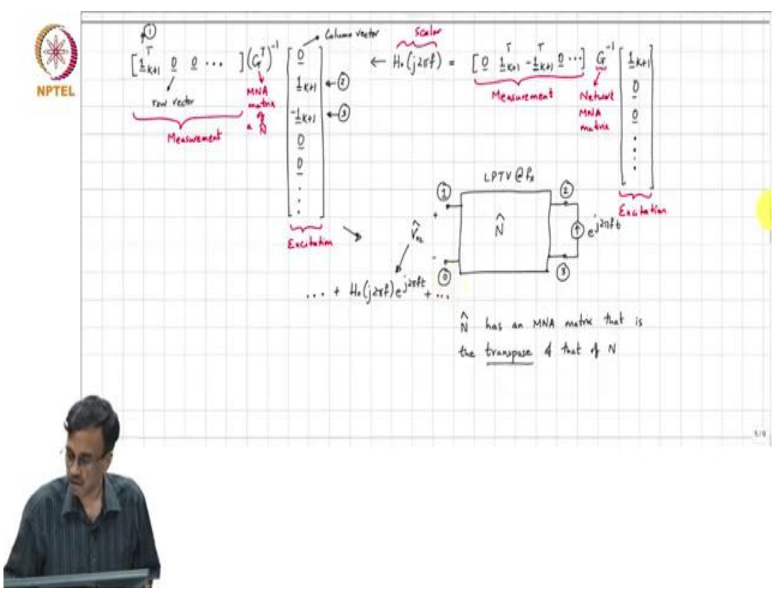

So, this is nothing but so H0 of j 2 pi f is this times G inverse times the excitation vector which is 1 k plus 1 0 0 and so on. So, let us take some time to interpret this, this is the excitation G is the network MNA matrix and this is the measurement response. And because and this is scalar. Now, because it is a scalar it must be the same as its transpose.

And therefore, it can be rewritten as 1 k plus 1 transpose 0 0 blah, blah, blah. So, this is a row vector again times G transpose inverse times 0 1 k plus 1 minus 1 k plus 1 0 blah, blah, blah, and this 0 is a column vector. So, this can be interpreted now therefore as this is the excitation, this is the measurement and this is the MNA matrix can be interpreted as the MNA matrix of a network N hat.

So, if you want to draw a picture which depicts these equations what should it be? Well, we have our network this is N hat, this is 1, this is 0, this is 2 and this is 3 and where are we exciting the network? Well, the current source is between nodes, so this is, this corresponds to excitation at node 2, this corresponds to excitation at node 3 and at what frequency are we exciting the network?

At f because only the k plus 1th term is 1 so this must be e to the j 2 pi ft. And where are we making, what are we measuring as our output? We are measuring this corresponds to node 1, so we are interested in voltage at node 1 but which harmonic? Exactly, so we are only, this is a, we are only interested in that k plus 1th term of that node voltage phasor vector. And therefore, this is we are interested in the output at, so this is vo2 hat and we are only interested in the component at f.

So, what this is saying is that if you come up with a network this is also LPTV at fs and N hat has an MNA matrix that is the transpose of that of N. And therefore, if you take such an network and you excite its output port by a current then this vo2 hat will have H0 of j 2 pi f times e to the j 2 pi f t plus blah, blah, blah. Does make sense?

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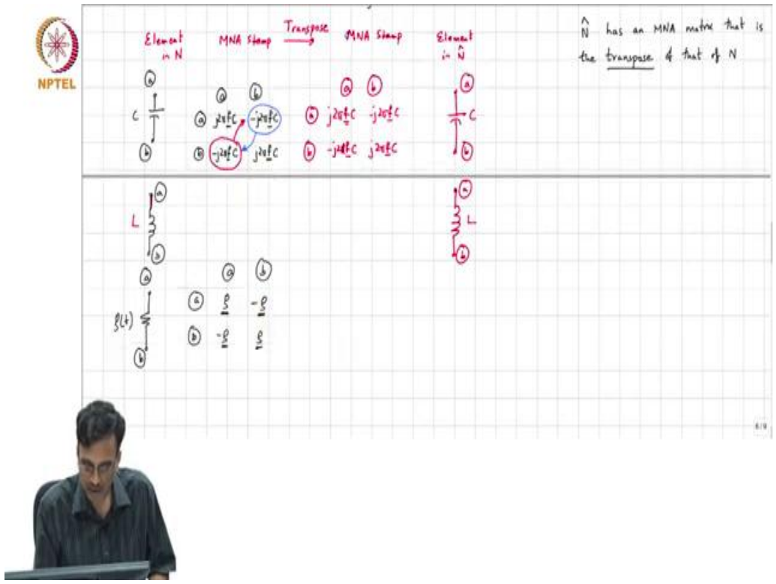

So, if we, now how do we generate N hat? Well, we take the individual elements that are there, we take the individual elements that are there, well, this is the original element or l, let me call that element in N, this is the MNA stamp of that element. Now, what do we, remember, that the G matrix is simply the sum of the MNA stamps of the individual elements.

So, transposing the G matrix is equivalent to saying I will form a network where I replace each element in the original network by an element whose MNA stamp is the transpose of that of the original. So, if I transpose the MNA stamp, what will I get? f is a diagonal matrix, so if you transpose it, it remains the same.

And therefore, that will remain so j 2 pi fc minus j 2 pi fc minus j 2 pi fc and j 2 pi fc. See the interesting thing is that when you transpose it, now remember all these are matrices, I mean they are not numbers, so when you transpose it you must not only the, in this case the f will get transposed and even though in this case, in the case of a capacitor things look pretty harmless.

This guy where will he go in the transpose when you take this MNA stamp and transpose it where will he go? So, not only are you transposing the blocks in the block also you must transpose the matrix. Fortunately for us with the capacitor because everything is diagonal there is no real issue.

So, this is going there and this fellow is coming here but fortunately f is a diagonal matrix and therefore we have no issue. So, what is the element which corresponds to this MNA stamp therefore, so this is the MNA and then the therefore we need the element in N hat and that also happens to be the same capacitor connected between the same two nodes, inductor also is the same thing, so there is no so I am not going to do the math here now.

So, this is a, this is b and in the adjoint or the inter reciprocal network you will have this is a and this is b, this is l, this is also l. Now, comes the interesting part. So, what other elements do we have? We have periodically time varying conductances, so this is a, this is b and this is g of t and what is the MNA matrix of this character? a, b, this is a and b this is that matrix g, this is minus g, this is minus g and this is plus g.

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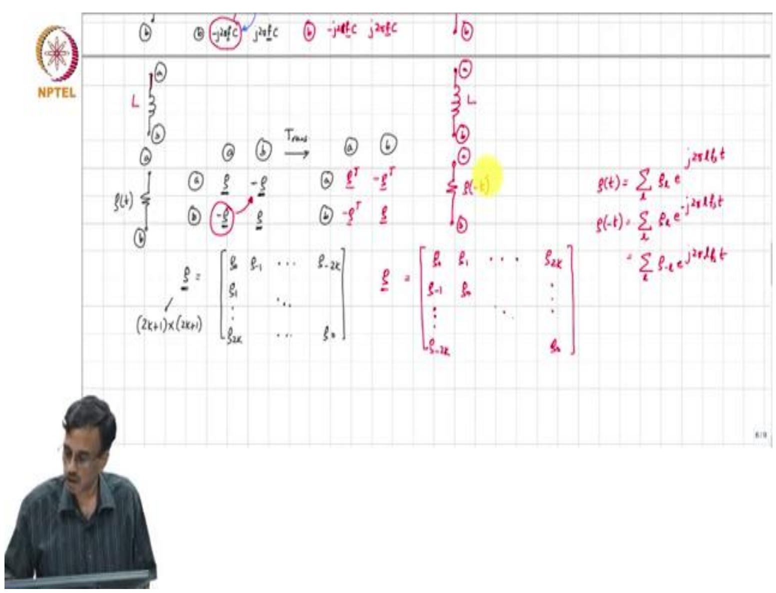

And recall that G matrix is a 2 is the 2k plus 1 cross 2k plus 1 matrix which is g0, g minus 1, blah, blah, blah, g minus 2k, this is g1, g2k, this is g0. So, this is the MNA stamp so we need to transpose it, what do we get there? This is a, this is b, a, this is b, so if you transpose the MNA stamp, what do you get?

You have to transpose this, so this g transpose and this is minus g transpose and that minus g transpose is not transposing of this matrix, it is basically this guy is going there and is getting transposed and likewise let me write this in red this is minus j transpose and this is g. So, this corresponds to a conductance so g transpose is basically g0, g1, g2k, this is g minus 1, g minus 2k, g0 blah, blah, blah.

And remember g of t I think we have already done this in the earlier class long ago is sigma l g sub l of e to the j 2 pi lfs times t, so g of minus t therefore must be sum of l g sub l of e to the minus j 2 pi lfs times t. So, if I, so basically this is saying that if I flip the g of t wave form in time then whatever was g, the first Fourier series, the strength of fs in g of t will become the strength of minus fs in g of f minus t, so this is equivalent to saying I mean this is equivalent to saying sum over l of g of minus l e to the j 2 pi fs, sorry, j 2 pi lfs times t.

I mean do you get this? Because this g sub l, I mean if you change l to something else uniformly it remains the same. So, what is the element therefore that corresponds to this MNA stamp, it basically is g of minus t. So, the key difference between a time invariant network and a periodically time varying one even if the network has got only resistors inductors and capacitors is that the time variation of the periodically time varying conductances must be flipped in time.

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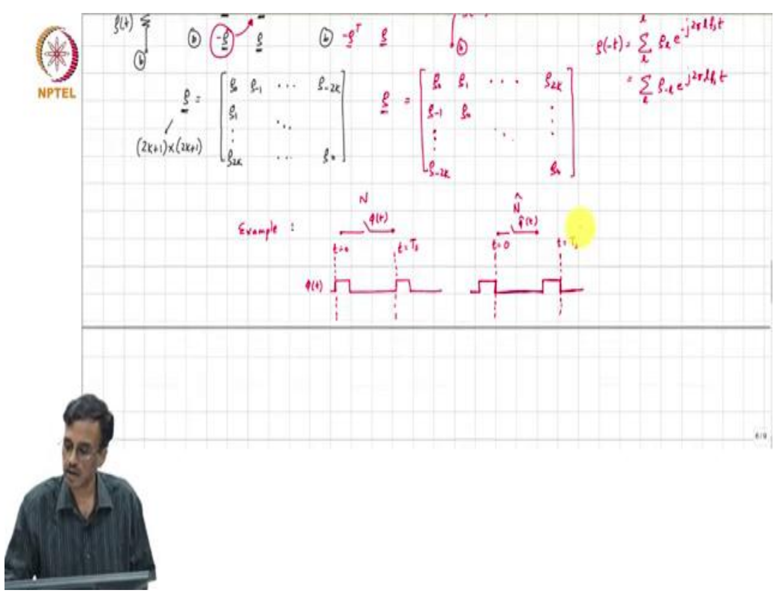

So, for example, let us say you had a switch in N which was controlled by some phi of t which did, so let us say this is t0, this is t equals ts. Now, what would the, so in N hat, you would have the same switch controlled by phi hat of t, so this is phi of t and phi hat would basically look like this so this is t equals 0, this is t equals ts. If you have multiple switches you basically do flip for everything.

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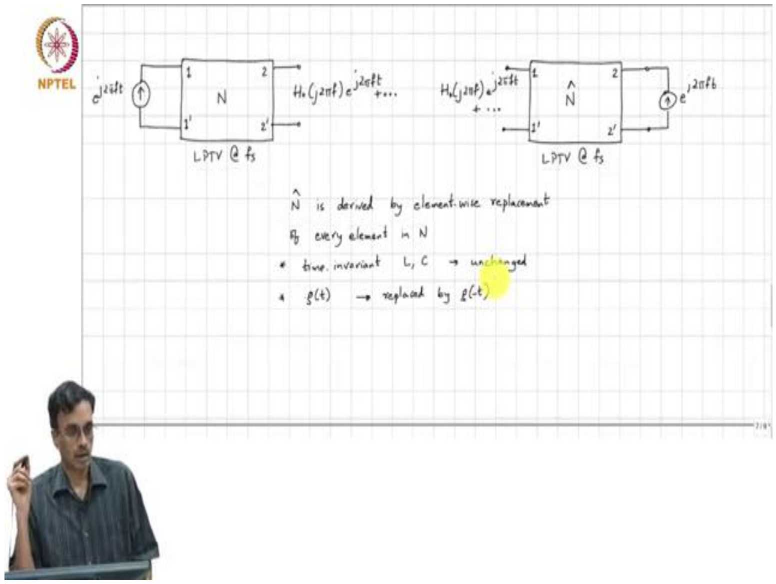

So, to summarize therefore, we have an original network N that is LPTV at fs we excite it with a current e to the j 2 pi ft we get a voltage at the other port which is H0 of j 2 pi f times e to the j 2 pi ft plus a whole lot of other terms what this is telling us is that, what this is telling us is that you interchange, so this is 1, 1 prime, 2, 2 prime, you excite the second port of the adjoint with e to the j 2 pi ft and what you will get is the same H0 of j 2 pi f times e to the j 2 pi ft plus plus plus a whole bunch of other terms.

And how do we go from N to N hat? So, N hat is derived by element wise replacement of every element in N. C and L so linear, as long as they are linear and time invariant let me write that up L and C are remain unchanged. Periodically time varying conductance is replaced by g of minus t. Now, this basically explains I mean basically discusses reciprocity networks with only with time varying resistors, inductors and capacitors, we have to take a look at control sources tomorrow.