Introduction to Time - Varying Electrical Networks
Professor Shanthi Pavan
Department of Electrical Engineering,
Indian Institute of Technology Madras
Lecture 22
Bode's Noise Theorem - Frequency Domain

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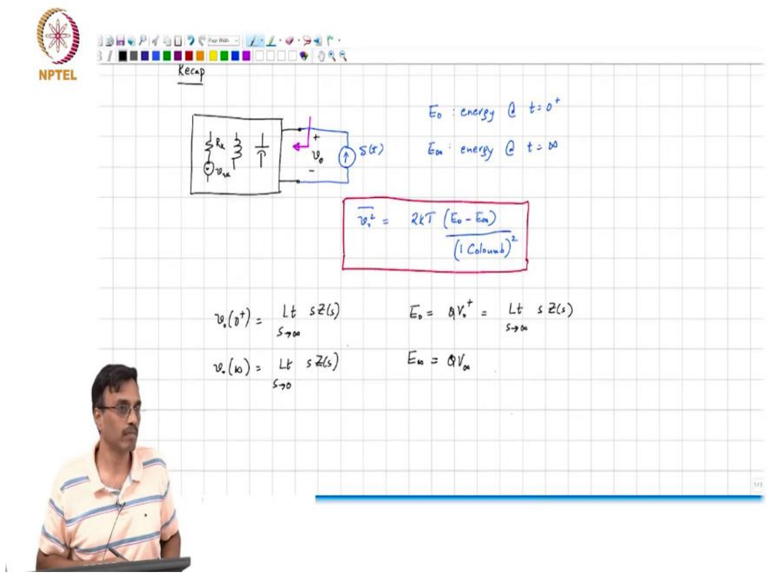

A quick recap of what we did in the last class, so what we said was, if you have an RLC network and you are measuring, you are interested in finding mean square noise at the output of the network and this is a calculation that often comes about in practice, and if you want to do this, the straight forward approach would be to as we discussed several times, every resistor R sub K is associated with a noise, voltage source with the voltage spectral density 4KT R sub K.

We find the transfer functions from every noise source to the output from which we can find the spectral density and you can then integrate the spectral density all the way from 0 to infinity, to obtain the V0 the mean square noise at the output, unfortunately, that is a very obviously a very laborious way of doing things and it does not make sense to do simply because, (())(01:33) working so hard to basically get all these transfer functions and all these spectral densities and finally we are throwing away, I would say 99 percent of that information because you simply integrating that whole thing from 0 to infinity.

So, yesterday we saw how one can use exploit reciprocity and the idea was the following, what you do is, to inject an impulse current into the network and let E0 be the network the energy stored in the network at t equal to 0 plus and E infinity be the energy at t equal to infinity, and we said that the mean square noise is nothing but 2KT times E0 0 minus E infinity by 1 coulomb square, I hope I got the spelling of coulomb right, and we saw yesterday cases where E infinity was 0 we also saw cases where E infinity was, what you call non-zero and this formula basically covers all the cases. Does that make sense.

Now, let us kind of see this from another pers the same result from another perspective, so when you inject a current into the network what comment can you make about the Laplace transform of the voltage developed across these two terminals? Pardon?

Student: (())(03:51)

Professor: It is simply the impedance, so if you inject a current impulse the voltage devel developed across the two terminals of the network has a Laplace transform which is simply given by the driving point impedance of the of the network, so, and what comment can we make about the initial energy stored in the network? How can we relate it to the initial voltage developed across the network? Or rather let me put it in other way, what comment can we make about the voltage developed across the network at t equal to 0 plus? Let me call that Vo, pardon?

Student: (())(04:48)

Professor: No, pardon.

Student: (())(04:53)

Professor: yeah, use and tell me what it is.

Student: (())(04:57)

Professor: Vo plus is simply nothing but, I mean what are all the those theorems for, the voltage wave form is Z of s, has the Laplace transform given by Z of s. So, what is the voltage at 0 plus?

Student: (())(05:16)

Professor: limit s tends to infinity, s times Z of s, and what common can we make about the initial energy delivered to the network you pumped in a current impulse, what comment can we make about the initial energy that is been delivered into the net into the into the network?

Student: (())(05:51)

Professor: Pardon,

Student: V0 times the charge.

Professor: yeah, it is V0 plus times the charge, so E0 therefore is nothing but,

Student: (())(06:05)

Professor: the charge is unity so what is this?

Student: (())(06:10)

Professor: E0 by one coulomb is nothing but is V0 plus, which is limit s tends to infinity of s times Z of s, and what is the E infinity by the same token? well E infinity is simply Q times V infinity, and what is V infinity?

Student: (())(06:57)

Professor: Limit, s tends to 0, s times Z of s. So, what is E0 minus E infinity? This is nothing but Z of s.

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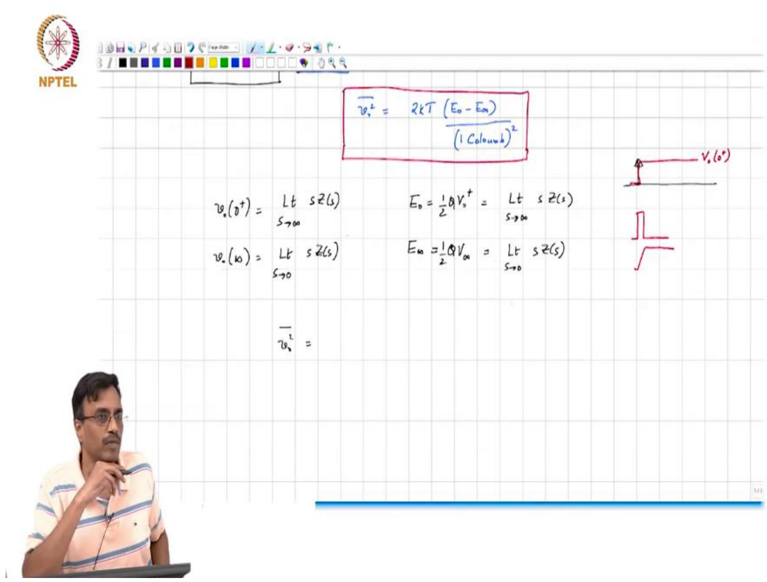

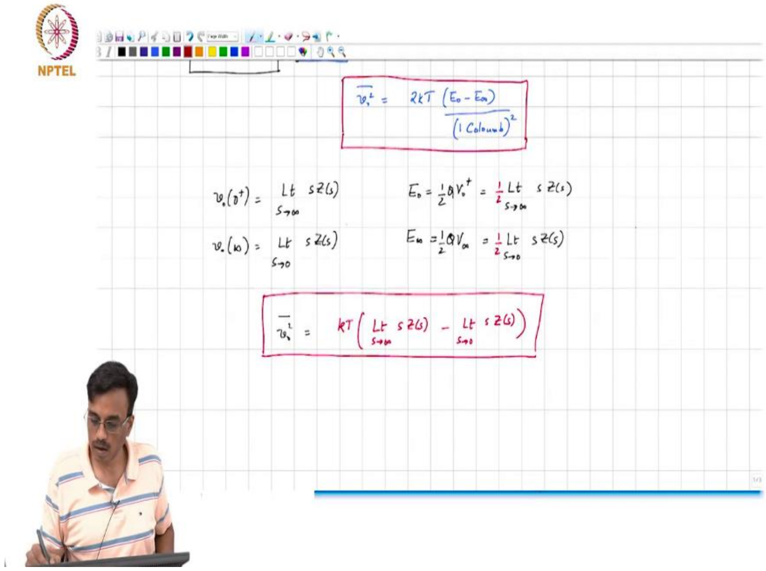

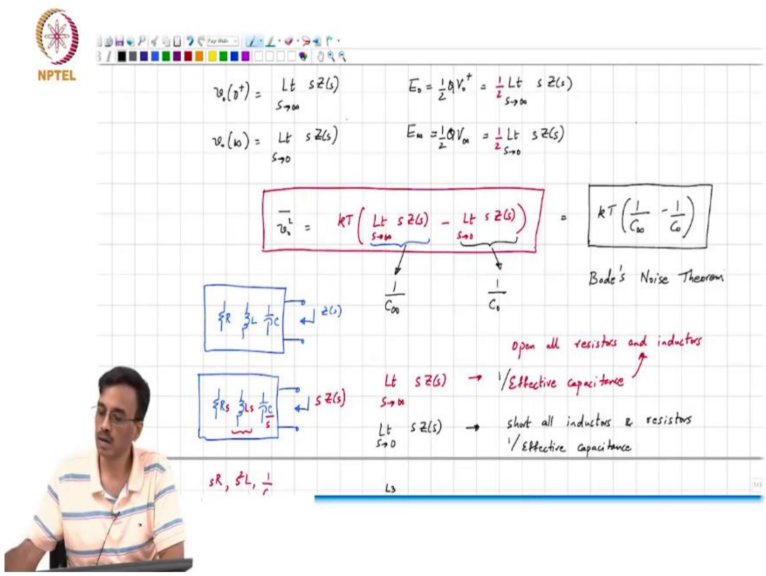

So, what is the equivalent to? so mean square noise is nothing but, oh I am sorry, actually we have missed a factor of 2, why?

Student: (())(07:53)

Professor: Remember, that we have injected an impulse and how much of and what is the energy delivered by the impedance, what is the energy that is going into the network the voltage across the network has gone from 0 to 0 to V0 plus, V0 0 plus, so what comment can we make about the energy in the network? what is the energy inside network?

Student: (())(08:23)

Professor: It is simply the integral of voltage times, the voltage waveform times the current waveform that has gone into the network and you can see that, I mean I mean there is a of course a discontinuity, but if you to interpret the impulse as being, say for instance a thin pulse, and the voltage waveform doing this, it is very clear that, the energy that is going in that there inside the network is half Q times V of 0 plus. So, this is 1 half times this, likewise, this is 1 half times s Z of s, s tends to infinity. So, the mean square noise is therefore, what is the mean square noise people? Yes.

Student: (())(09:30)

Professor: It is simply kT times limit s tending to infinity of s times Z of s minus limit as tending to 0, s times Z of s.

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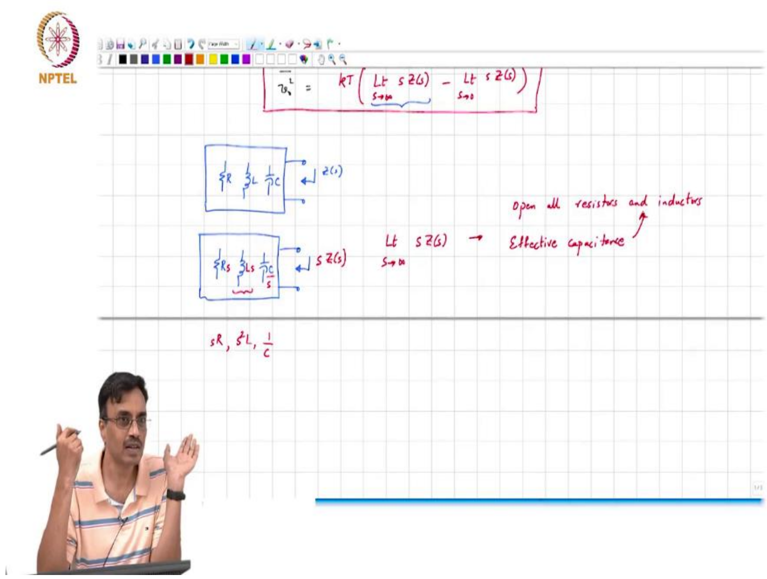

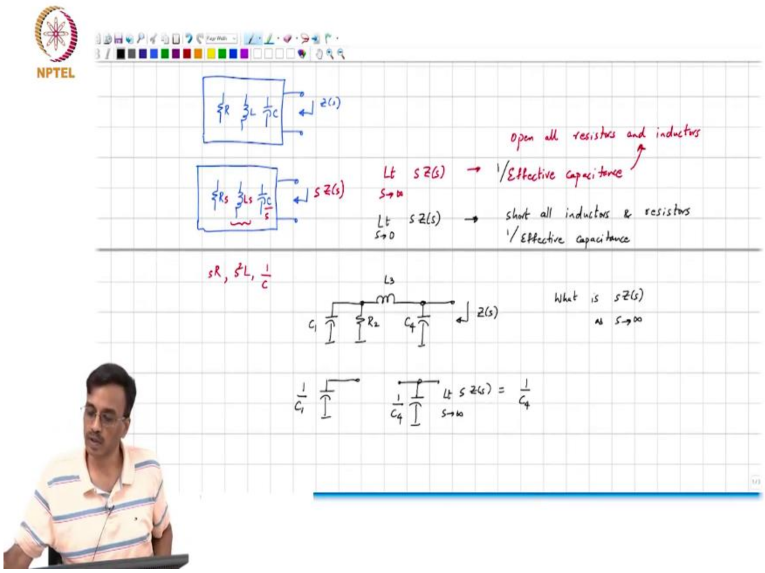

Now it turns out that, there is a simple way of interpreting this s times Z of s as tends to infinity, remember let us say you have a network and let us say you had the input impedance for some Z of s, now let us say this is R this is L and this is C, now if I multiply this, by k, by k and by k, oh sorry and, what will the impedance be?

Student: (())(11:10)

Professor: It will be k times Z of s, does make sense, all impedances have been scaled by the same factor and therefore the total impedance will also scale by the same factor k, now there is no necessity for that k to be real, it can be, it can be complex. So, for instance if I make k equal to s, so in otherwise I divide I multiply every impedance by the complex number s, what comment can we make about the looking an impedance?

Student: (())(12:04)

Professor: it will simply be s times Z of s, does makes sense people? now remember that s is the multiplying factor for every inductor, so what is in reality the impedance of this inductor now, at a certain frequency s?

Student: (())(12:27)

Professor: Remember that the impedance of the inductor was sL that is now being multiplied by an extra factor s, so every resistor has now become s times R, the impedance of the inductor has become s square times L, and the impedance of the capacitor has become, what is the difference the capacitor earlier? is 1 over sc, it is multiplied by its complex number s, so this is become 1 over c.

Now if we let s tend to infinity, how can you interpret this, as s tends to infinity s times R becomes infinite, similarly s square times L also becomes infinite, and therefore how do you interpret this this limit of limit as s tends to infinity of s times Z of s, you open all resistors and inductors, once you do that you will get a network with only with capacitors. So, yes, before how can you interpret s times Z of s? how can you interpret s time Z of s?

Student: (())(14:12)

Professor: Yeah, so basically it is the effective capacitance of what remains. Effective capacitance is just simply the capacitance, that number, that s goes away because, so let us say C1, R2, L3, so this is the Z of s, all right and the question is what is s times Z of s as s tend to infinity and one way of doing this would be to find the actual Z of s multiplied by s and let s tend to infinity the easier thing to do is, do it on a branch by branch basis as s tends to infinity and then look at the resulting network, so what will this become, the capacitance will become,

Student: (())(15:33)

Professor: 1 over 1 over C1, the resistance will become an open circuit the inductor will become an open circuit and what does this become, resistance of value 1 over C4, equivalently, so what is, so therefore what is Z of s times Z of s as s tends to infinity, staring at this, what do you see?

Student: (())(16:09)

Professor: It is 1 by 1 by C4 and I mean in English it basically means that, you open up all resistors and inductors you will get a network which only consists of capacitors, the equivalent capacitance that you see, is s times Z of s as s tends to infinity, pardon.

Student: (())(16:39)

Professor: Because the s times Z of s does not have dimensions of capacitance, I mean, s times Z of s has dimensions of capacitance not it is not resistance.

Student: (())(16:56)

Professor: This is one mega yeah, the effective capacity that you look it is 1 by 1 by the effective capacitance at (())(17:00). By the same token, therefore how do you interpret limit s tends to 0 of s times Z of s, what should you do? oh well that is straightforward, as s tends to 0, sR becomes a short circuit, s square L also becomes a short circuit, so you basically say short all inductors and resistors, and you look at the, the 1 over, actually this should be 1 over effective capacitance looking.

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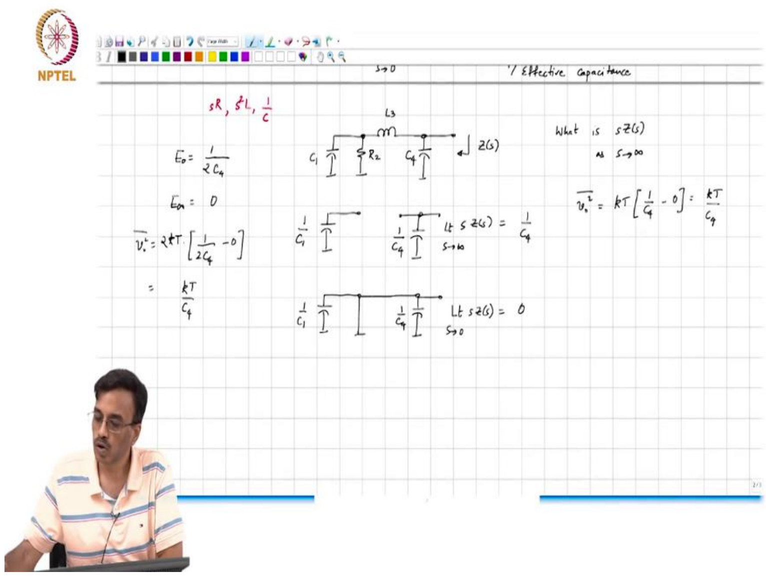

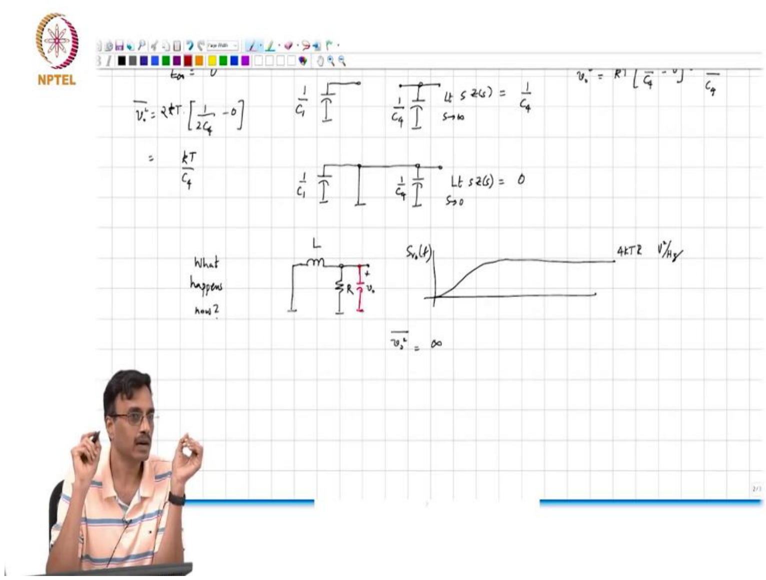

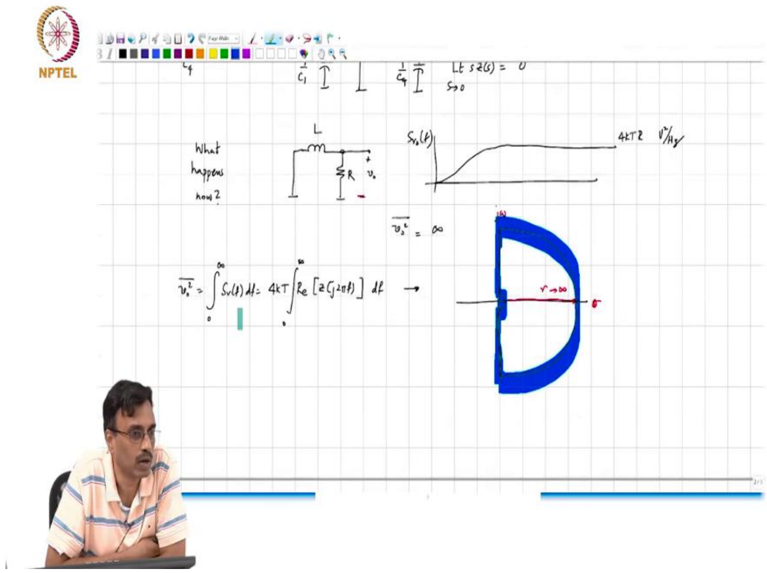

Now, let us do this for the for our example we have chosen, what should you do for our network, what happens to C1?

Student: (())(18:10)

Professor: It becomes resistor value 1 over C1, and then what happens to R2? R2 becomes a short circuit, L3 becomes a short circuit and C4 again, so what comment can you make therefore as s tends to 0 of s times Z of s, what is the effective capacitance looking in?

Student: (())(18:44)

Professor: Effective capacitance looking in?

Student: Infinite

Professor: It is Infinite, so basically because it is a dead shot, so therefore the, 1 over the capacitance looking in is, is, is 0. So, what comment can we make about the mean square noise looking, so what is the mean square noise? It is simply kT times 1 over C4 minus 0 which is kT over C, does make sense? And is this consistent with the energy formulation? Well, when you, what is E0, you inject an impulse, where does all that current go, it is it is on C4, because in the beginning all the inductors are open and what you call, the capacitance are short.

So, E0 is nothing but, 1 over 2 C4, E infinity what happens? There is some initial voltage in the capacitor C4, as t tends to infinity, well the inductors become short circuits and there is must be no current flowing through R2 that is what will happen at t equal to infinity, and therefore at t equal to infinity is very apparent that both C1, L3 and C4 are all discharged or disfluxed, and therefore E infinity is what is E infinity?

Student: 0.

Professor: 0, so mean square noise therefore is 1 half of KT 2KT times 1 over 2C4 minus 0 and this is basically KT over C, so this term basically has is often written as 1 over C infinity where 1 over C infinity is basically denoting, I mean, this is now just a symbol, that is all, 1 over C infinity is simply, what is C infinity?

Student: (())(21:17)

Professor: Is the capacitance that you see when you open up all the inductors and resistors, and likewise this is nothing but if you call this C in 1 over C infinity, what will you call this?

Student: (())(21:28)

Professor: 1 over C0, so this is written in, it is also written in this form, the books and this is what is called Bode's Noise Theorem.

Student: (())(21:59)

Professor: Come again,

Student: (())(22:10)

Professor: What do you need to do is the following, you to find C infinity what you do is, you open up all resistors and inductors you will get a network with only capacitors you find the effective capacitance looking in, using all your high school physics, if you are series parallel and, in some combination, you can do whatever you need to do and that effective capacitance is C infinity.

So, limit s tends to infinity s times Z of s, is one over that that effective capacitance that you see at infinite frequency, and likewise limit s tends to 0, of s times Z of s, is effective capacity you see at at when all the resistors and inductors are shorted and total noise is simply nothing but KT times 1 over C infinity minus 1 over C0. Yes.

Student: (())(23:15)

Professor: I mean, the equation I mean, I was hoping somebody ask this question. So, a very legitimate one, so, well what happens if you have a network like this, what comment can we make about this spectral density at at the output at this node? How does this look like, at low frequency this would be?

Student: (())(24:11)

Professor: What do would be at low frequency, 0 because it is a high pass filter, at high frequency what will happen, it will eventually tend to 4KTR volt square per Hertz. So, what is the total integrated noise of mean square value of, so what comment can we make about, it is simply the integral of this noise spectral density, and what is this telling us?

Oh well this is evidently infinity, because that spectral density is constant, and you keep integrating infinite frequency and, therefore. This basically for this network therefore for networks of this kind, you basically, theoretically have this integral does not converge, and again this is telling us that, oh well, if I simply put an inductor or resistor like that, I will get infinite mean square voltage which basically means evidently something again it is too good to be true.

So, in reality you will never have like this, because, oh well, every node is associated with Parasitic capacitance, and eventually the spectrum must fall off, and so, in other words this formula is only valid when the when the integral converges, which basically means in all practical networks this is this is valid.

Likewise, you can show, it is straight forward to see that if you have networks with RL and C and you are interested in finding the mean square, if you short circuit the network and find the mean square noise current, you will be able to show, I mean you will be able to write similar relationship involving L infinity and L0, for the mean square current noise integral across all frequencies.

So, so as I said, this is called Bode’s Noise Theorem and is a is a nifty little trick to know to basically solve or to look at a complicated network and estimate what the determine examine what the total integrated noise will be without going through any algebra (())(27:02) or complicated integrals.

This is not the original proof of the theorem, the original proof of the theorem, starts from Nyquist result, which is 4K to be Sv of f is 4KT times real part of Z of s, real part of j2 pi f, and we need to find the integral of, so what we need is basically integral Svf df, 0 to infinity which is infinity, and this actually turns out that you can think of it as a contour integral in the Z plane, this is X plane, this is the sigma and this is the J omega plane, so you take a a contour like this and you integrate, you go along the contour along that in the J plane, what comment can we make about poles and 0's of Z? let us say you have a passive impedance like this, what comment can we make about the poles of of Z?

Student: (())(28:22)

Professor: Poles are in the left half plane, what comment can you make about the 0's of impedance? Where are the, where are the poles and 0's of a passive impedance located? There must be in the left of S place, so you around this contour does not enclose any poles and 0's, so the contour integral around this entire contour must be, if you take Z of s, which is a complex function, and the contour does not enclose any poles or 0's, the integral around the contour must be, must be 0.

And, and it turns out that the you choose the radius, this contour is R, R tends to infinity, so, I will leave this an exercise for you to work out. But this basically it can be broken up into two par into three parts, one on that small circle whose radius tends to 0 and that is necessary to take care of potential pole at the origin and this guy here, and it is a, I do not know if you are able to see this, that contour integral is basically it will turn out to be, 1 contour, one of those circles, the large circle will turnout to be limit of s times Z of s as s tends to infinity.

And the small contour will turn out the small semi-circular contour will turn out to be limit s tends s times Z of s as s tends to 0, and along the axis it will be it will turn out to be integral minus infinity to infinity real part of Z of j2 pi f and you can relate the two and get the get the get the answer.

Contour integration is of course a very valid way of finding the proof, but it is a, at least to me it seems a lot more intuitive to basically look at the energy going into the networks and, it also turns out that this same energy based formulation approach also works when you as we see later in the course, when we have periodically operated switches inside the network where the network no longer becomes is no longer time invariant but becomes periodically time vary, then you will find that this notion of using energy is more general, because energy does not depend on, the concept of energy is independent of time invariant or time variant or linear or nonlinear, whereas when you say impedance you already mean that the network is time invariant.