Linear Dynamical Systems
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Week – 02
Stability
Lecture – 09
Lyapunov Stability (Part II)

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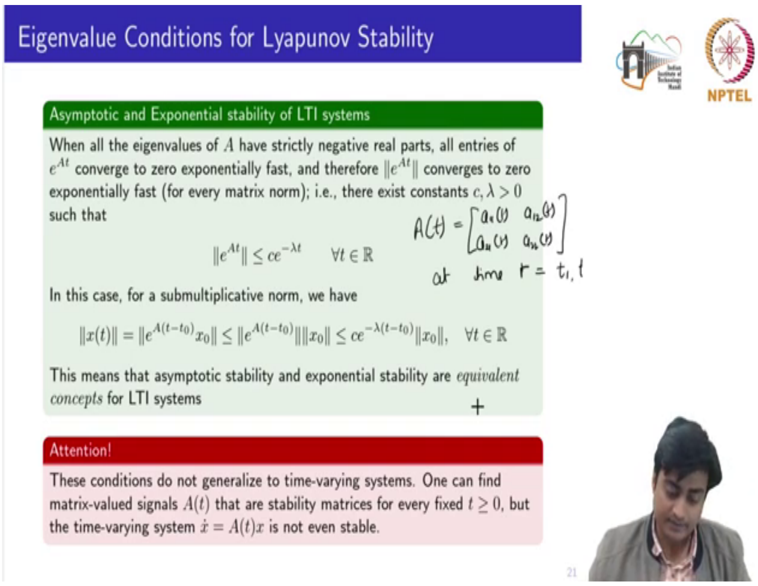

And, next we will see what is the equivalence between the asymptotic stability and exponential stability particularly for the LTI systems. So, when all the eigenvalues of the matrix A have strictly negative real parts, all entries of the e to the power A t would converge to 0 exponentially fast.

And therefore, the norm of that exponential matrix converges to 0 exponentially fast for every matrix norm, you can take from 1 norm, 2 norm, infinite norm or the frobenius norm. Which also meaning to say that there exist two positive scalars such that, the norm of this matrix is always less than equal to c in to e to the power minus lambda t for all t belonging to the R.

So in fact, you can also see this equivalence from a logical proof also, since we know that the asymptotic stability particularly for LTI system because we know at the outside that for linear systems, the response would be exponential in nature. So, even if it is decaying or diminishing asymptotically towards to 0 it will decay exponentially.

So, we have this asymptotic and exponential stability equivalent particularly for LTI system. So, you can see a technical proof of also of this that for a sub multiplicative norm if I compute the norm of the signal x which is given by this e to the power A t minus t naught for some initial condition x naught.

So, the norm of this quantity would always be less than equal to the multiplication of their individual norm, right and this one we can replace from here by replacing by that t minus t not their exist two scalar c and lambda and finally, we obtained this which is basically the definition of the exponentially stability for LTI systems. So, these two definitions are equivalent for LTI systems.

So, the next questions arises whether these two definitions are also equivalent for the time varying systems or linear time varying systems which is not true. We will see a certain a couple of examples later in the lecture that these conditions do not generalized to time varying systems. Meaning to say that one can find matrix valued signals A of t that are stability matrices for every fixed t greater than equal to 0, but the system itself is not stable. Meaning to say, let say I have A of t consider a one of particular example a 1 1 of t, a 1 2 of t, a 2 1 of t, a 2 2 of t ok. Now, at a fix time t let see at time t is equal to let consider three different time instance t 1 t 2 and t 3.

So, if I compute the matrices at these three different time the matrix A might be a stable matrix meaning to say the all the eigenvalues of this matrices, of this matrix computed at three different times are stable, but the system itself is not stable, ok. So, we in that case we cannot say these two definitions are equivalent for the LTI system, but we will see more concretely when we will consider a couple of examples.

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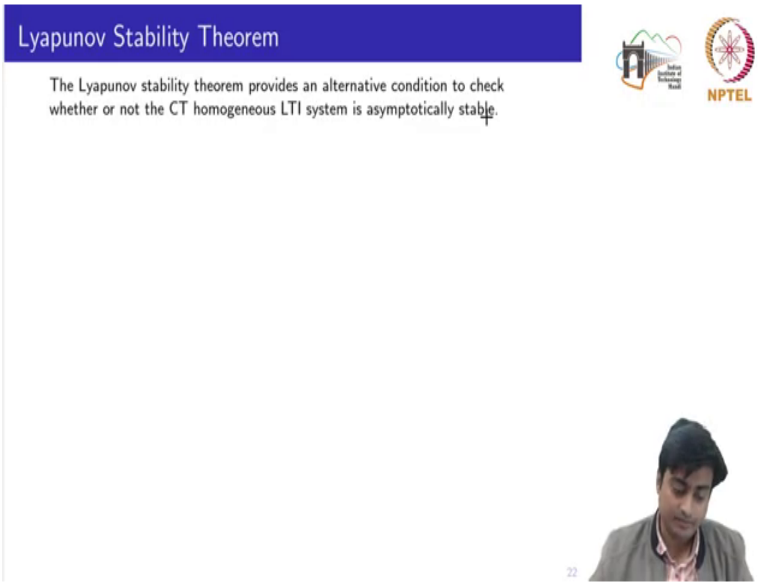

So, this is the key important theorem in the analysis of stability for linear system, the Lyapunov stability theorem provides an alternative condition to check whether or not the continuous time homogeneous linear time invariant system is asymptotically stable. So, the first condition or the first test we saw that you compute that given the system. You compute the solution; you compute the solution basically you compute state transition matrix and then seeing the solution you could see whether the state or the output signal is bounded or not right, all right

Second test is the eigenvalue test where given the system only for LTI system; this eigenvalue test is not valid for the ltv systems; so, if we have an LTI systems you can do an eigenvalue test. But, even if do not want to compute the eigenvalues this Lyapunov stability theorem provides a test, whether you need to ensure the existence of some matrix which in fact give you the test for the stability. So, here we do not need to compute the solution, we do not need to compute the eigenvalues, but we will see some other different criteria which is this one.

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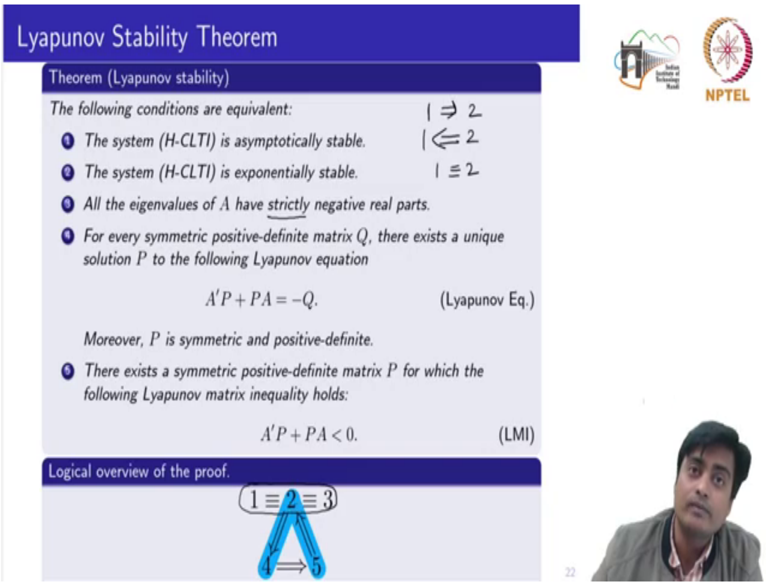

So, the following conditions are equivalent, the system homogeneous CLTI is a asymptotically stable, which is equivalent to saying that the system is exponentially stable.

Which is again equivalent to staying that all the eigenvalues of a have strictly negative real parts. The fourth statement reach for every symmetric positive definite matrix.

So, symmetric matrix we know if we talk about the matrix Q that the Q transport should be equal to Q, positive definite that the quadratic form with the weight matrix Q is greater than 0. There exist a unique solution P to the following equation which is also known as the Lyapnov equation that is a transpose P plus P into A is equal to minus Q, more over P is symmetric and positive definite.

Which is equivalent to saying that there exist a symmetric positive definite matrix P for which the following Lyapnov matrix in a equality holds, that A transpose P plus P into A is less than or 0. So, here LMI is basically linear matrix in equality not the Lyapnov matrix in equality, this LMI term has been generally used for linear matrix in equality, right.

So, the next we would see the proof of this important theorem because this like we are seeing in the statement of this theorem, that here we without even computing the solution or the eigenvalues just by saying the existence of the symmetric positive definite matrix P we can determine the stability we can comment on the stability of the system right.

So, since there are five statements this is how we would going to proceed to prove all these five statements. So, when we say that the statement 1 is equivalent to the statement 2, we mean to say that 1 implies 2 meaning to say if 1 is true then 2 is true right also the other way round then if the statement 2 is true then it implies that the statement 1 is also true.

So, when these two implications are satisfied we say which is equivalent to say that 1 is equivalent to 2, both ways implication is being satisfied. So, 1 is equivalent to 2 that is exponentially stable and asymptotically stable, this equivalence we have already seen for the LTI system that the exponential stable system and their asymptotically stable system is basically the same thing.

Now, by the eigenvalue test, we can have the equivalence between the second statement and the third statement that all the eigenvalues of A have strictly do not forget this part strictly there should not be any eigenvalue with 0 real part ok, then strictly negative real part right. So, this three equivalence we have already seen.

Now, the equivalence which we will prove in the next couple of minutes that 2 implies 4 and 5 implies 2, basically we wont to closer loop. Now, the 4 implies 5 is quite trivial mean to say that if 4 is true. If the statement 4 is true then the statement 5 is also true and this it could be done by choosing Q as an identity matrix. Once you put Q as identity matrix for the same dimension in the state in the Lyapnov equation you would directly obtain this LMI. So, 4 to 5 is pretty much straight forward. So, the next you would see 2 implies 4 and 5 implies 2.

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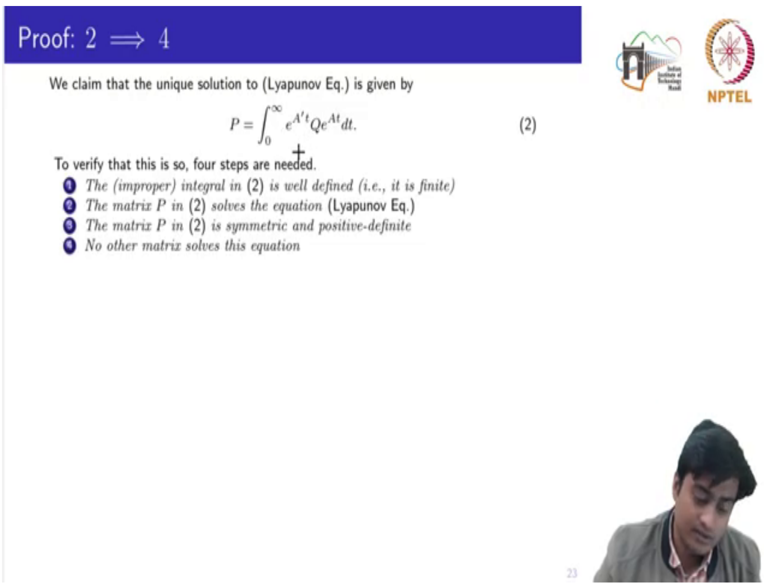

So, first we will see that 2 implies 4 meaning to say that if the system is exponentially stable then given a symmetric positive definite matrix Q, this symmetric Q is given then the Lyapnov equation is satisfied for some P which is symmetric and positive definite, ok. So, in this implication it is already assumed that the system is stable is asymptotically stable. So, to start with the proof we claim that the unique solution to the Lyapnov equation is given by this.

P is a definite integral from 0 to infinity given by this quadratic form A to the power A transpose t into this matrix Q which is given the matrix A is also given in. So, basically you can compute matrix P in to A to the power At dt. So, to verify this; so, we are claiming that this is a solution.

So, to verify that this is actually a solution to the Lyapnov equation being symmetric and positive definite we need four steps to be proved. These four steps that the integral defined in the in this equation 2 is well defined that is it can be computed there is it has a finite value. The matrix P solves the equation Lyapunov equation, the matrix P is symmetric and positive definite and finally, that there does not exist any other matrix, then defined by this which solves the Lyapnov equation ok, because this is a value, this is an improper integral from 0 to infinity.

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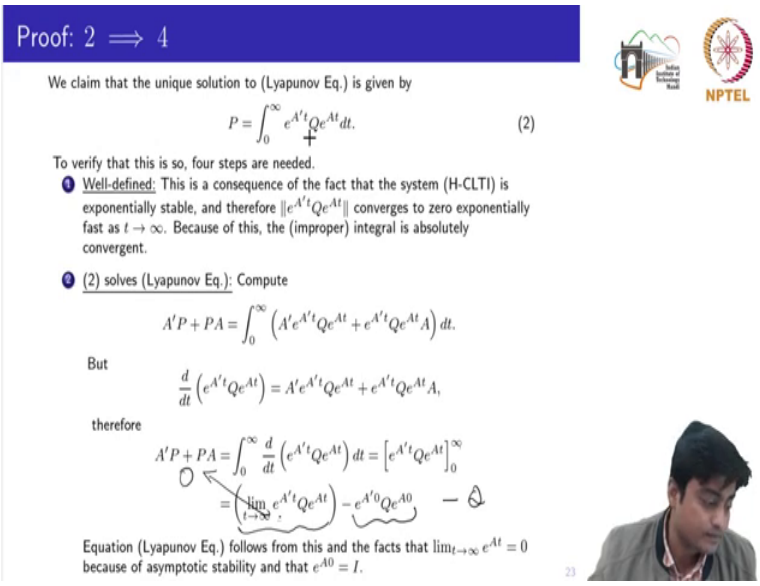

So, let see the well defined first. So, this is a consequence of the fact the system is exponentially stable. So, we if it is already given that the system is stable; so, now, if you see that the system is stable meaning to say all the eigenvalues are on the left hand side strictly, on the left hand side. So, this exponential will decay this exponential will indicate. So, finally, this entire part would go towards to 0.

So, this integrand would as t tends to infinity would go towards to 0. So, there exist a value, you can actually compute the value. If this integrand with t tending to infinity is approaching towards to infinity, then you could not compute any value and the value would be infinity.

Basically, you are computing the area under this signal, ok. So, because of this that these are exponential stable or both these integral would die out as t tends to infinity the integral is absolutely convergent. So, we would have a certain value second that this equation solves the

Lyapnov equation. So, this you can verify by putting this P into the Lyapnov equation itself and then verifying whether the left hand side is equal to the right inside.

So, in the left hand side we put this P which is given by A transpose P plus P into A. So, I put this P integral from 0 to infinity here. So, since this matrix is A transpose and A are time in variant matrices, I could take inside the integral without affecting the value of the integral, ok.

So, this finally, I would obtained, but on the other hand if I compute the derivative of this quadratic form if I take the derivative of this quadratic form this would come equal to this one which is nothing but equal to this thing, ok. So, I can replace this whole term by the derivative of this quadratic form, ok.

So, therefore, this I can write the left hand side of the Lyapnov equation I can write the integral 0 to infinity this part, because this part has been replaced by this derivative of the quadratic form. Hence, this you can compute by taking the value at 0 by computing the value at infinity. So, this I can read by limit t tends to infinity of this part minus at t is equal to 0, ok.

So, see now that we already know that the system is asymptotically stable. So, I know that these exponential as t tends to infinity would go towards to 0. So, this part would entirely would become 0, right and this part since e to the power A transpose 0 is identity. So, finally, this one would become minus of Q. So, left hand by putting this P on to the left hand side of the Lyapunov equation, I actually obtain the right hand side meaning to say, that this P satisfies the Lyapnov equation.

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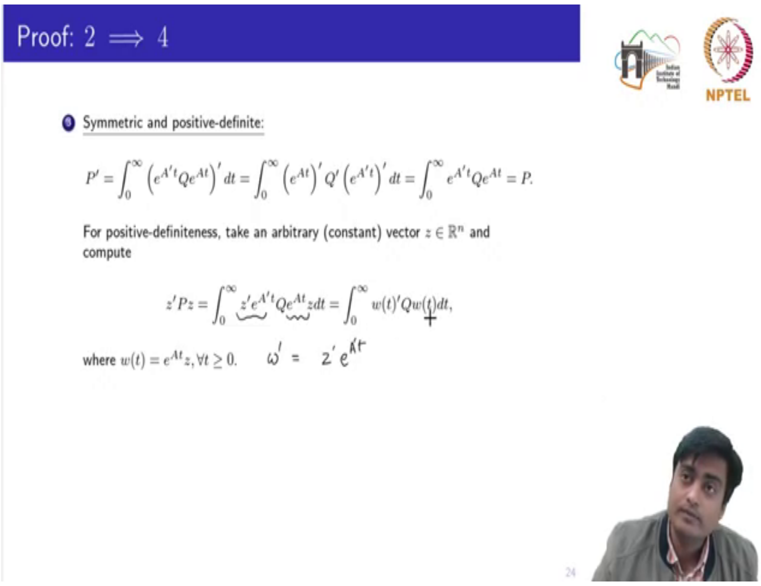

The third point is about the symmetric and positive definite. So, first you will see the symmetric we know the definition of the symmetric matrix that the transpose of that matrix should be equal to the matrix itself. So, first of all we take the transpose of that matrix P transpose. Now, carry forward it to the integral inside we can compute the, we can simplify the transpose in this manner. So, by using the transpose property it would become e to the power A t transpose this term would become first Q, Q transpose and the first time would be go into last to the transpose, ok. So, finally, we would have e to the power A transpose t Q is a given symmetric matrix.

So, we know that Q transpose is equal to Q, I can replace this Q transpose by Q and the A transpose, transpose would become would nullify right. So, I can write this e to the power A t. So, if you notice that this in fact, become equal to P. So, we have verified that the P is a symmetric matrix now, for the positive definiteness we define an arbitrary vector z of the same dimension n is given by z transpose P z. Again by putting the P matrix, the P integral defined earlier into this part we obtained this.

And finally, I replace this part z transpose e to the power A transpose to t or to be precisely e to the power A t into z by another signal w t. So, see if you take the transpose, if you take the w transpose you would obtain z transpose e to the power A transpose t which is this part. So, I can write this as w transpose Q into w, ok.

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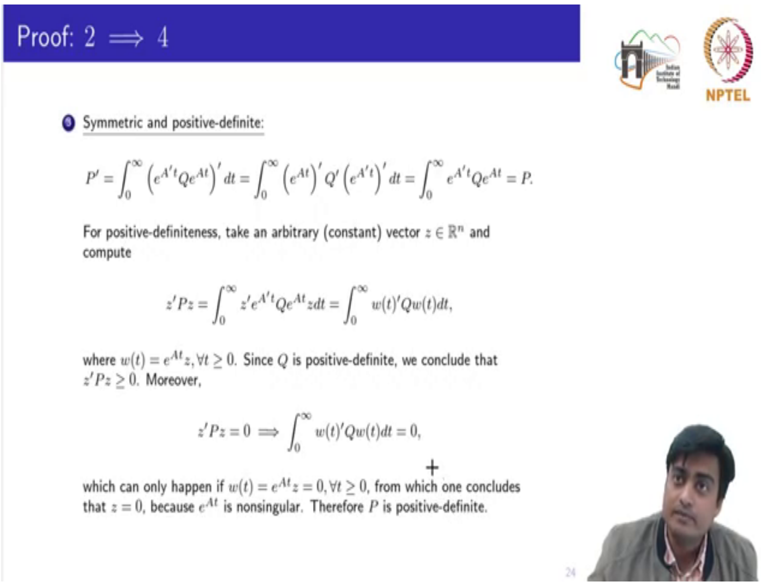

So, we know that Q is already a positive definite matrix. If Q is a positive definite matrix then I know that this part would be greater than or equal to 0, we will come on to this part will equal to 0 we know for positive definiteness the equal to parts should not be 0. Since the z could be any arbitrary vector. So, we are not sure whether it would be greater than only greater than equal to 0 or greater than equal to 0; so, let us see.

So, we know that it would be 0 that z transport P z would be equal to 0, if this part is equal to 0, right. Now, note is here that this part, this is a quadratic form that this part would go to 0 only if this w t signal is 0, because Q is a positive definite matrix which can never become equal to 0. w t equal meaning to say e to the power A t z is equal to 0 and e to the power A t is a nonsingular matrix already. So, this will become 0, if and only if z is equal to 0 right from which one conclude that is equal to 0, because e to the power a t singular therefore, P would always be positive definite matrix, ok.

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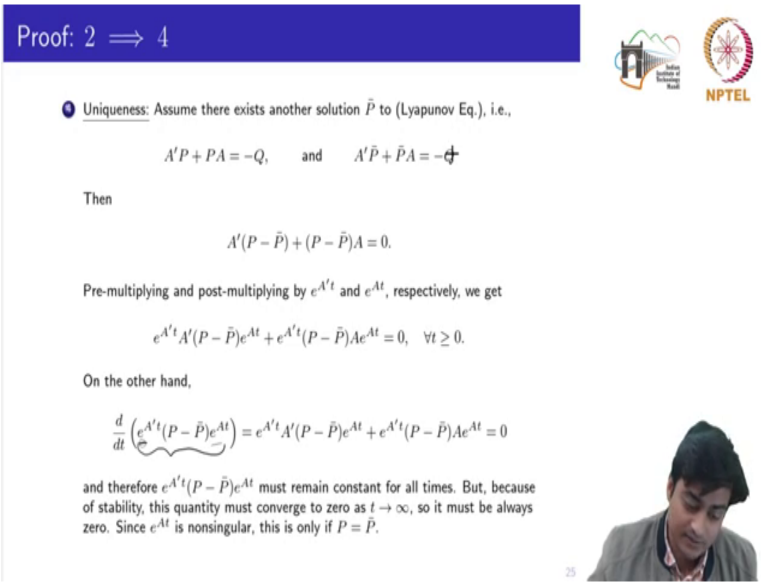

Now, the fourth one is the uniqueness property which is meaning to say that there exist only that P and does not exist any other P. So, let us assume that there exist another solution given by P bar which also satisfies the Lyapnov equation, and this P bar is different than the P defined in the equation two. So, here we write the P in the Lyapnov equation and the P bar which was satisfy the Lyapnov equation for the same Q.

So, if we subtract these 2 equations, we obtained this equation A transpose P minus P bar plus P minus P bar into A is equal to 0 because a Q matrix remains the same. Now, if I multiply this equation pre-multiplying and post-multiplying by e to the power A transpose t and e to the power A t, I can do this because both these matrices are nonsingular matrices, ok.

So, I can pre multiply by the e to the power A transposes t and post multiply by e to the power A t, the right hand side would remain as it is equal to 0, ok. Now, if you pay a close attention to this one, it is actually the derivative of the this part, of this quadratic form e to the power A transpose t P minus P bar in to A to the power A t which becomes equal to this. So, what it says that the derivative of this one is equal to 0.

So, meaning to say that this term would either be equal to 0 or would be equal to some constant value and this is what it seems that must remain constant for all time, but because of stability this quantity must converge to 0, because A is having all the eigenvalues on the left hand side.

So, this part or these two parts would go towards to 0 as t tends to infinity, meaning to say that this form would become equal to 0 only if this weight matrix become equal to 0, right. It is meaning to say that P minus P bar equal to 0 meaning to say P would always be equal to P bar. So, there does not exist any other matrix than P. Certainly, for different Q there would be different P, but for the same Q there could not be two different P, this is how we defined the uniqueness, ok.