Linear Dynamical Systems
Prof. Tushar Jain
Department of Electrical Engineering
Indian Institute of Technology, Mandi

Week – 02
Stability
Lecture – 08
Lyapunov Stability (Part-I)

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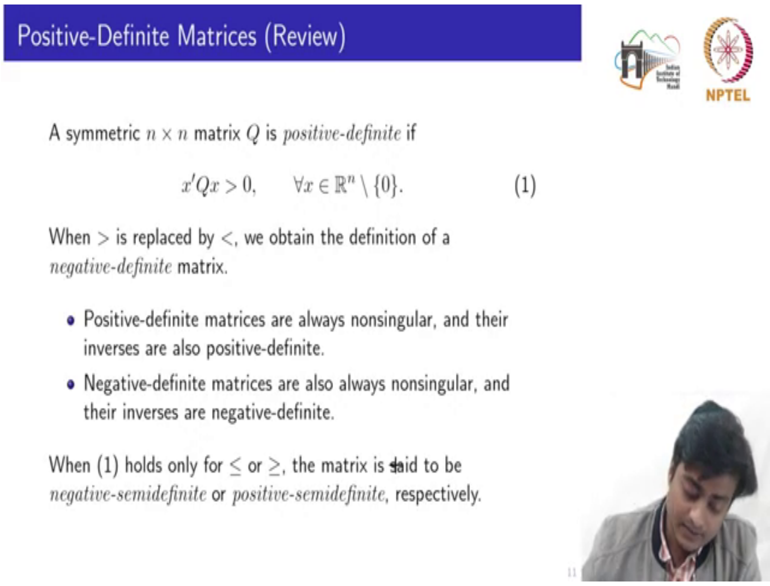

So, the next is the positive definite matrices. So, a symmetric n cross n matrix Q is positive definite. If this quadratic form that is x transpose where x is a vector, Q is the symmetric square matrix; symmetric matrix meaning to say that the transpose of that matrix is equal to the matrix itself, right. And, this is always greater than 0 for all x belonging to the n dimensional space excluding the null set because right if x is 0 then this inequality would not hold. So, when this greater than sign is replaced by less than sign, we obtain the definition of a negative definite matrix.

So, both these positive definite matrices and the negative definite matrices are always nonsingular matrices and their inverses are also positive definite and negative definite respectively. So, now, when this greater than sign changes to either greater than or equal to or less than or equal to then those matrices are defined as the positive semi definite or the negative semi definite, right.

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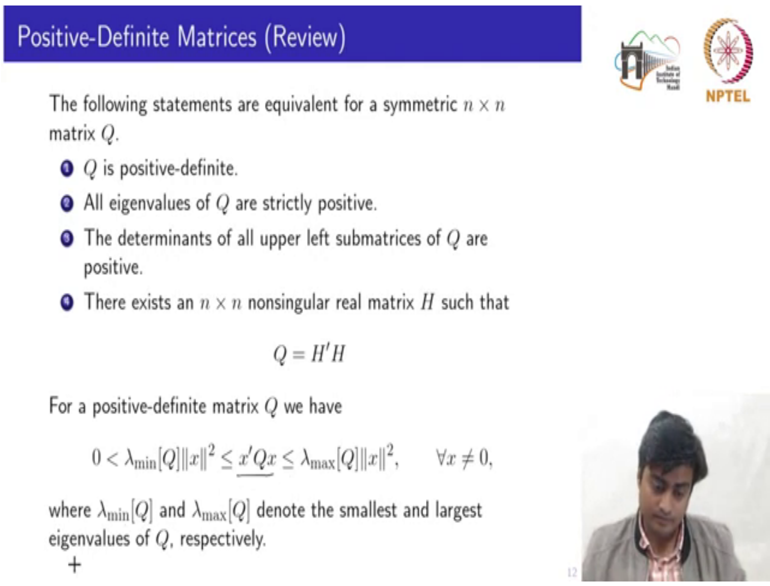

So, this is one of the important results with respect to a positive definite matrices. So, the following statements are equivalent for asymmetric square matrix of dimension n, then Q is a positive definite matrix which is equivalent to saying that all eigenvalues of that matrix are strictly positive.

So, if Q is negative definite then all eigenvalues of that matrix are strictly negative, right which is equivalent to saying that the determinants of all upper left sub matrices of Q are positive and vice versa. If the matrix is a negative definite which is again equivalent to saying that there exist another square non-singular real valued matrix H; such that the matrix Q is basically the multiplication of the transpose of that matrix with itself, ok.

Another result we have that for a positive definite matrix Q we have let us say this is the relationship what we have been studied. Now, the minimum value of this quadratic form is given by lambda min of Q and lambda min defines the smallest eigenvalue of the matrix Q multiplied the squared norm of x to vector x. The largest value would be, the or it would be less than or equal to the largest eigenvalue of the matrix Q multiplied by the squared norm of the vector x.

And, since we know that Q is a positive definite matrix, this would always be greater than 0; this is in fact, definition of the positive definite. So, this 0 would be the lowest value, ok.

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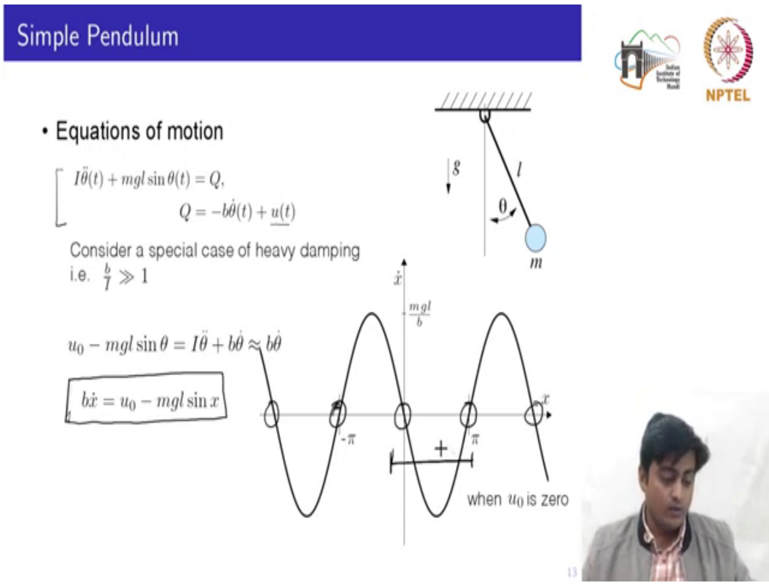

So, let us consider one basic example before moving to the definitions of stability. So, this is a simple pendulum what we have been seeing in you your wall clocks particularly. So, here in the first; so, these two equations defines the basic equations of motions of a simple pendulum where the torque are balances out where Q is the damping torque influenced by the external force u of t, ok.

Now if you are considering a special case of heavy damping that is b by I is very greater than number 1, then this relationship can be simplified to this equation. Now, here since u is the external force applied to the simple pendulum if we consider u naught is equal to 0, and we plot the graph between the x dot as the y axis and x is the x axis is the variable x itself, then this is how the motion is defined because it is the pure sinusoidal function scaled by the vector mgl by b, ok.

Now, if we want to determine the stability of the system how we can do this analysis. So, since it is pretty much straight forward to see that this system is non-linear system because of sinusoidal function. So, here we determine the stability around the steady state values, by the steady state values we mean to say at those points where x dot is becoming 0 in these equations of motion.

So, if you see a close look at this picture, all these points are the points where x dot are is becoming equal to 0. So, we are interested in seeing the stability of these steady state or at the equilibrium position. So, around let us say if we want to limit our time interval within which we want to do the analysis, let us say it is given by 0 to pi.

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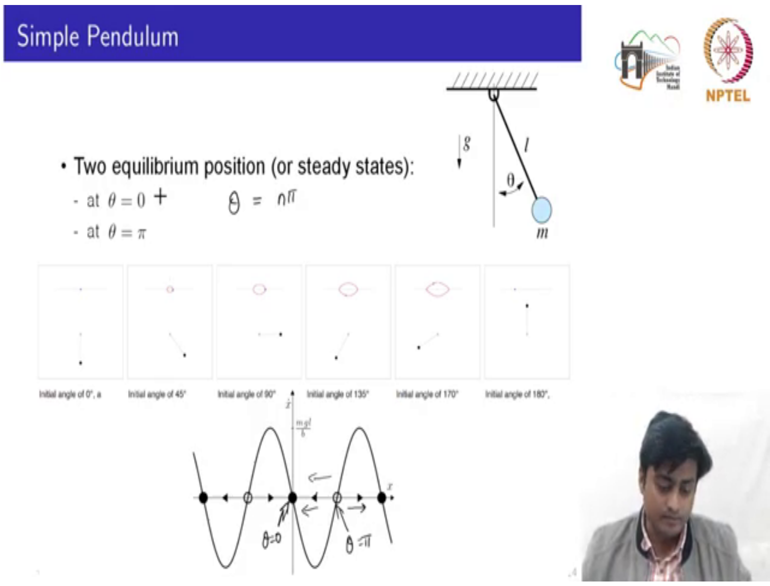

So, in this interval we have two equilibrium points, theta is equal to 0 and theta is equal to pi. In fact, the equilibrium points are over the entire axis is theta n pi where n is from minus infinity where n is an integer from minus infinity to infinity, and the even multiples of n basically defines the 0 and the odd multiples of pi basically is given by these approach, right.

So, let see the motion with respect to different initial conditions, let say if the initial angle of this pendulum is at 0 degree. So, this will not move this will always be stay there. Now, if I give if I initialize this motion of the pendulum let say at an angle of 45 degree it feels certainly makes such kind of oscillation of smaller amplitude in the x axis and then finally, it will settle down at theta is equal to 0.

Now, again if I keep increasing to 90 degree by possibly it would make a larger deviation, but ultimately it would settle down at theta is equal to 0; similarly, 135, 170 up to just before 180 degree, right. Because, if I initialize the pendulum at an angle of theta is equal to 180 degree, it would not stay at theta is equal to 180 would certainly fall down and come to theta is equal to 0. Meaning to say that theta is equal to 0 is basically the point where if I initialize between let say 0 to 1 7 or 0 to just before 180 degree, it will always settle down at theta is equal to 0, but it will never settle down at theta is equal to pi.

This we can also see in this figure as well that this point is theta is equal to 0 and this point theta is equal to pi. So, if I initialize at any point just before 180 degree the motion will always settle down in this direction. But, if I initialize at 180 degree and if I see larger interval either it would go towards here or go towards their with respect to this point, but it would never rest on at this point.

So, the question arises here among these two points, which point is the stable point and which point is the unstable point? Now, how we can define this formally is what we are going to see the next.

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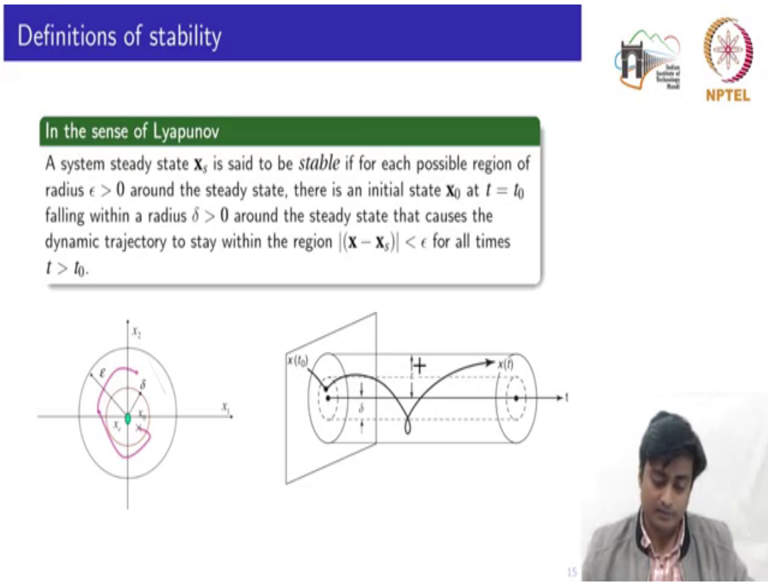

So, in the sense of Lyapunov we can define a system steady state x s for the simple pendulum, we have 2 steady states in the interval 0 and pi, right is said to be stable if for each possible region of radius epsilon a positive epsilon around the steady state. There is an initial state x naught at t is equal to t naught falling within a radius delta which is again a positive number around the steady state that causes the dynamic trajectory to stay within the region. The absolute value of the difference between the x and the steady state is less than the epsilon, the given epsilon for all time t greater than t naught.

So, if we want to visualize this definition, let us say consider a second order system we mean to say having 2 state system x 1 and x 2, and we draw 2 circles of radius one epsilon and another one is delta inside this epsilon, ok. So, if I initialize my system at this value x naught which is inside this delta then for the whole time axis that is t tends to infinity, my trajectory will not leave this outer circle it will always stay in the circle.

Now, if we are wondering about the time axis, the time axis is perpendicular to this plane. The cross sectional view is given by here, here where you are initializing at x t naught inside this in a circle sorry, but for the whole motion that is t tending to infinity, this trajectory would never leave the outer cylinder of radius epsilon, ok.

So, this is the same phenomena which we had seen in the simple pendulum example that if I am trying to see at the steady state x is or theta is equal to 0. The same kind of motion I would going to see, but if I am trying to visualize this definition at theta is equal to pi, it would certainly leave out the outer rings outer cylinder with the radius epsilon.

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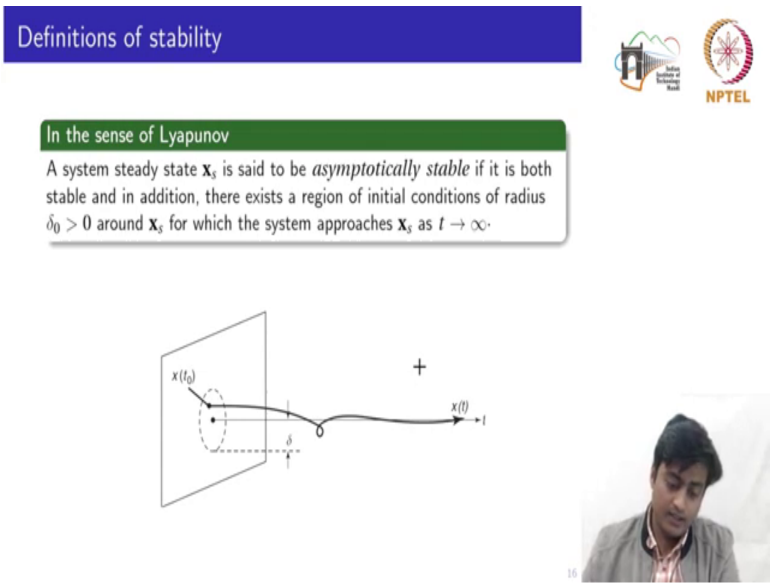

Now, another definition is about the asymptotic stability that if it is both stable in the sense we have defined in the previous slide and in addition. There exist a region of initial conditions of radius delta naught which is again positive around the steady state. This is the steady state point for which the system approaches the steady state as t tends to infinity; now, this steady state could be 0 also.

So, like in the case of the simple pendulum one of the steady state was 0. So, basically we are finding the region of the initial conditions of radius delta naught, that if I initialize my system with those in with the initial condition line in that circle it will always reach towards to the steady state value.

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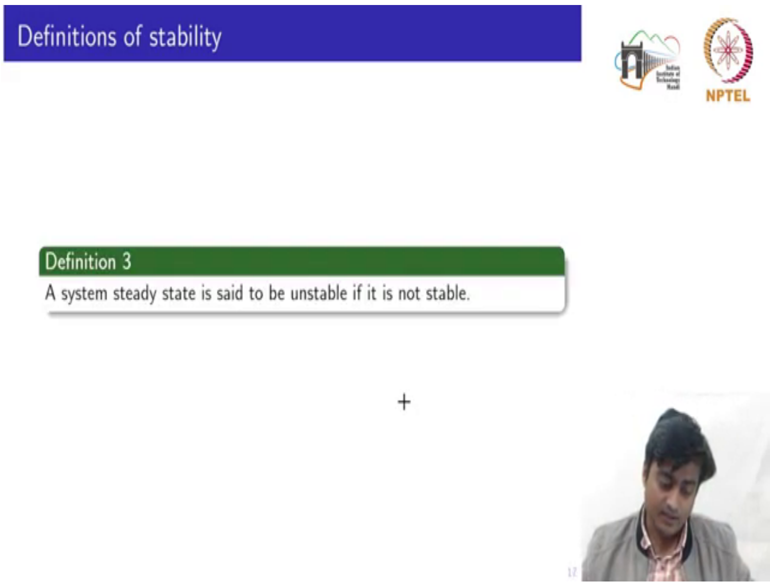

The last definition the system steady state is said to be unstable of course, when it is not stable.

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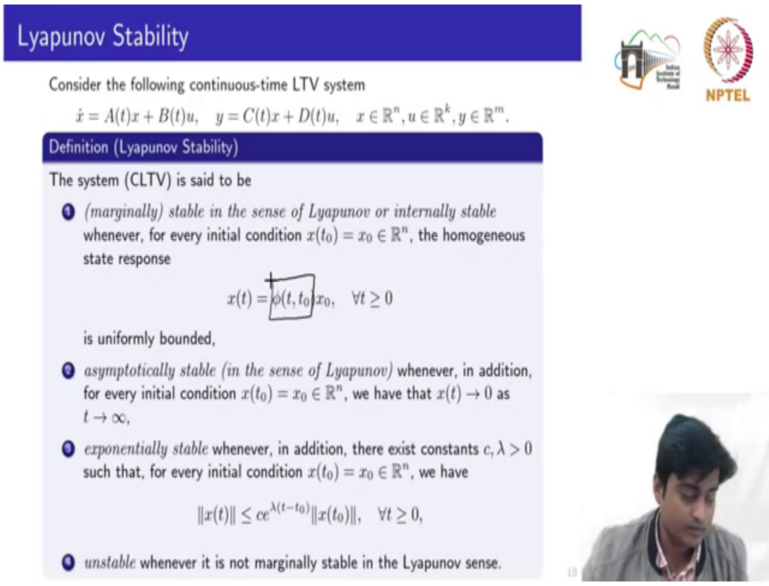

Let us see how what do we mean by that in the sense for the linear state system we had seen in the first week. So, consider the following continuous time linear time invariant system where the linear system is given by the A B C D matrices which are time varying and the dimension of this x u and y are given by n, k and m respectively,.

So, we define for this linear system that the continuous time linear time varying system is said to be marginally stable in the sense of Lyapunov or internally stable whenever for every initial condition x naught. The homogeneous state response given by this is uniformly bounded their phi is the state transition matrix initialize at t naught and we are seeing the response at time t, right. It is the system is asymptotically stable again in the sense of Lyapunov we need to say the definition, what we had seen in the last slide that whenever in addition to the above the first point for every initial condition x naught we have that x t tends to 0.

Now here, I have chosen the steady state value to be 0 at the outside as t tends to infinity, ok. Exponentially stable that whenever in addition there exist constants c and lambda both are the positive scale of values. Such that for every initial condition x naught, the norm of the signal x of t is always less than equal to c into e to the power of lambda t minus t naught and the norm of the initial condition.

So, this exponential stability we had not seen in the definition earlier, but the, but for the linear systems we know already that the response is basically an exponential response; particularly for the linear system irrespective of whether are you in the time may be in case or the time varying case. So, this is the reason we have included about the exponential stability since the response is kind of an exponential behavior ok. Again, I am stable when it is not marginally stable in the Lyapunov sense.

So, all those definitions we had seen in the last couple of slides those definitions can be set specifically to the linear time varying system in this particular sense. So, here you would see in the first point that I need to compute the response of the system and by computing the response. I need to basically compute this part which we had already seen in the last week which is quite difficult to compute the state transition matrix if it is a time varying case.

So, either I compute the solution because if the once the solution is available to me for the time starting from the initial value to up to t tends to infinity if the trajectory is bounded or the trajectory is reaching towards to 0, I can speak about the stability, but for that I need to compute the solution. So, here we will see that without even computing the solution, how we can determine whether the system is stable or not.

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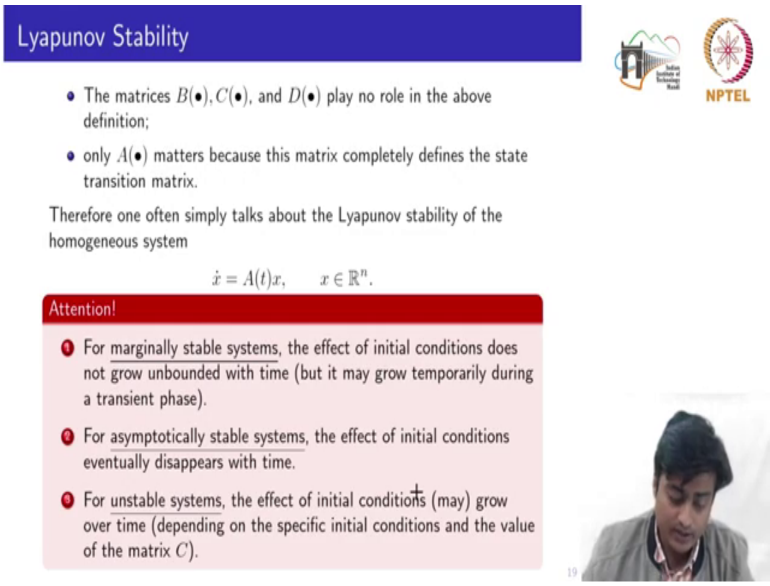

So, one important thing to notice here is that although we have defined the system in terms of the A B C D matrices. So, the rest of the matrices except the A matrix played no role in the above definition, only the A matrix because this matrix completely defines the state transition matrix. Therefore, in the director of the slide we will going to talk about the Lyapunov stability only of the homogeneous system.

Because, if the matrix A which is completely defines the state transition, matrix A is stable than the B C D matrices generally played no role in determining the stability. So, there are; so, if you tried to relate those definitions with the example we had considered that for marginally stable systems. What we are trying or what are what we are interested in that the effect of initial conditions does not grow unbounded with time right, but it may grow temporarily during a transient phase.

So, when we were computing the response of the system, the system start with some transient and settle down that some steady state. And, the definitions what we have introduced only speaks about at the steady state whatever is happening at the transient state we are not the stability in general is not concerned with that, ok.

Now, for the asymptotically stable systems, the effect of initial conditions eventually disappears with time and for unstable system. The effect would grow overtime depending on the value of the C matrix that how fast and how slow it is growing, ok.

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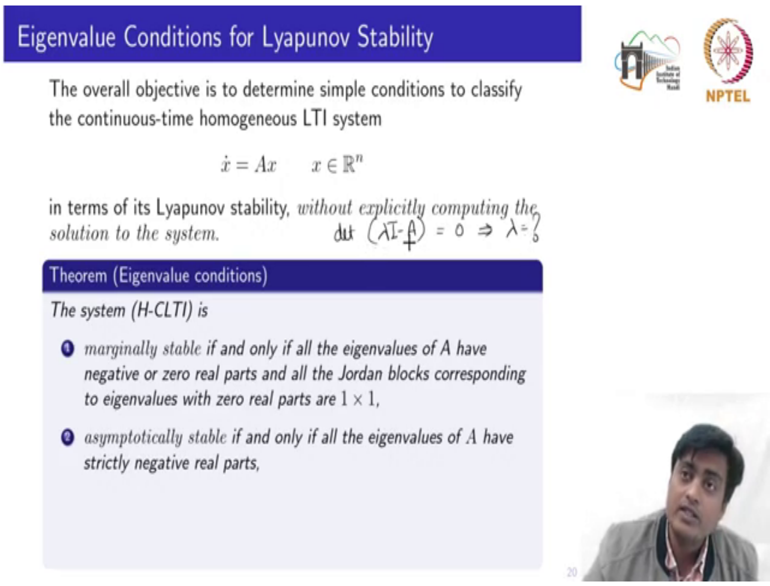

So, let us see the eigenvalue conditions for the Lyapunov stability, this is in fact, the first test of determining the stability where you do not need to compute the solution system. So, it is said already that we starting for the linear time invariant system where A matrix is no longer are time dependent. So, we want to compute the stable or determine the stability without explicitly computing the solution of the system, ok.

So, this is one of the important results and possibly you have seen in the ug control course that the system homogeneous continuous time linear, time invariant system is marginally stable if and only if all the eigenvalues of A have negative or 0 real parts, ok. Now, if some of the eigenvalues are having the 0 real parts then those eigenvalues should be simple eigenvalues or simple roots meaning to say they must not be more than at most 1, more than 1, ok.

So, the system is asymptotically stable if and only if all the eigenvalues of the matrix A have strictly negative real parts; now, just to recall about computing the eigenvalues of the matrix A. Let us say we have this matrix a then you can compute the eigenvalues by lambda I minus A, where I is the identity matrix of the same dimension of what A is having and then computing the determinant of this matrix. Then, once you have the polynomial in terms of lambda equate it to 0 and then you compute from here all the lambdas. So, all these lambdas are basically the eigenvalues of the matrix A, ok.

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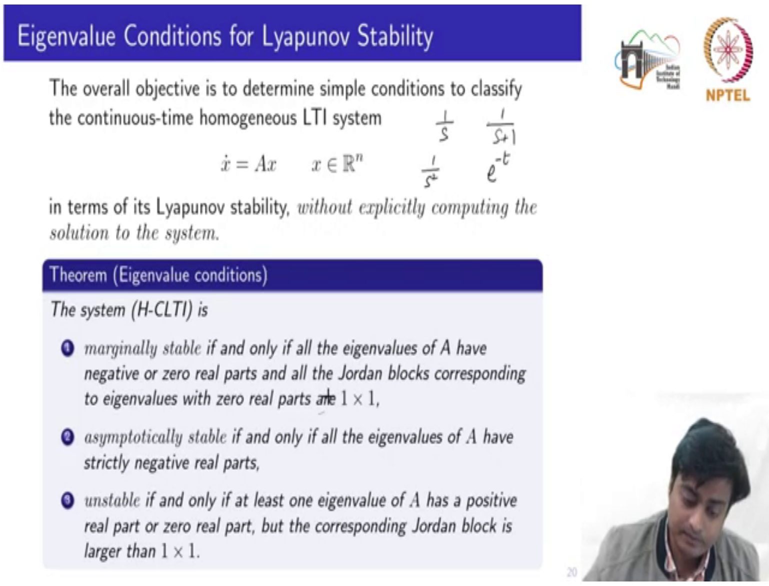

The third is unstable if and only if at least one eigenvalue of A has a positive real part or 0 real part, but the corresponding Jordan block is larger than 1 cross 1. Again, if you have zero eigenvalues either it could be a stable or unstable, it would be stable if it is simple, if it is not simple then it could be unstable, ok. So, here one important thing if you want to related with the transfer function, let us say we have a transfer function given by 1 by s, a simple integrator.

Now, for any constant input you for any constant input you would see that the response is rising is would integrate, right; the response could integrate for any initial condition the response would become constant. So, what does it mean that the if I am using the integrator with respect to the initial condition then it would be bounded right, but it would not be asymptotically stable because the response will never go towards zero.

Now, let us say if I have a transfer function 1 upon s plus 1, I know that the response is exponential that would be in the time domain, it could be e to the power minus t. So, as t tends to infinity the response would die out. So, this such kind of system having this transfer functions are asymptotic stability.

Now, consider if we have 1 upon s square irrespective of any initial conditions or any input the system will always be unstable, right. So, that is why we say that if we are having an eigenvalue or the roots or the poles of the transfer function at 0, then it should be a simple root. But, here the roots are not simple because here at 0 we are having 2 roots or in other sense we have Jordan blocks corresponding to eigenvalues with zero real parts more than 1 cross 1, ok.