Linear Dynamical Systems
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Week - 02
Stability
Lecture - 07
Introduction to Stability Analysis

So, welcome to the second week of the course Linear Dynamical Systems on Stability.

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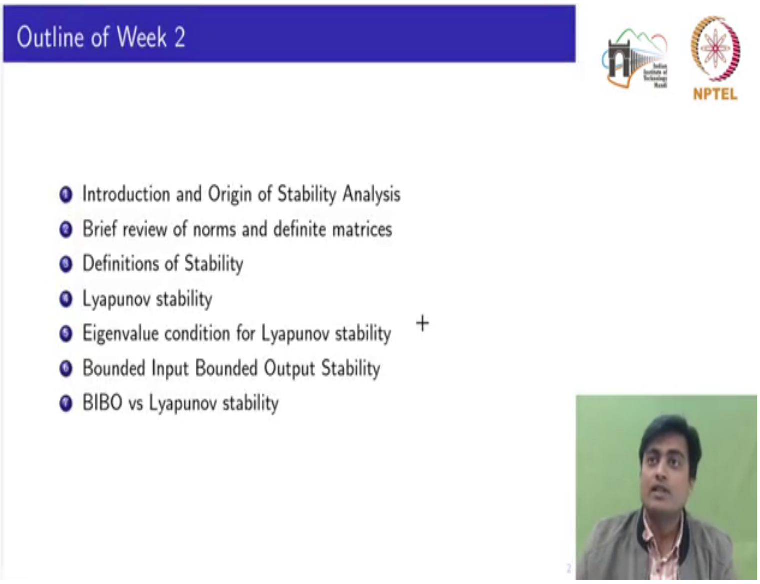

So, in this week, so, this is the overall outline of the week 2, where we will start with the Introduction and origin of the Stability Analysis. Then, we will see a Brief review of norms and definite matrices. These tools would be useful for the rest of the topics in this module.

Then, we will see the different definitions of the stability, we define and also determine the stability test in the sense of Lyapunov.

We will also see the eigenvalue condition with respect to the Lyapunov stability and finally, we will see the bounded input and bounded output stability and its connection with the Lyapunov stability.

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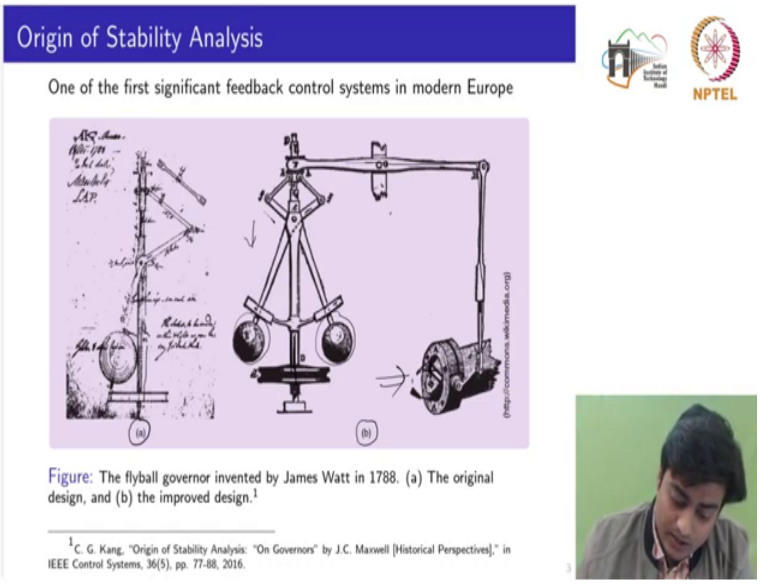

So, here in this picture you see the governor; the flyball governor. So, on the left hand side that is in the figure a, this is the original design of the flyball governor. So, basically the governors are the devices which are used to control the motion of the machine. So, here the machine is basically that engine, then later on the improved design of the governor you are seeing in the figure b. So, the role here is to basic basically governor is the controller, where we want to control the speed of the engine.

Now, here why first of all why do we call it the flyball governor? Here, you see the two balls. So, when and this is shaft, basically this shaft is connected to the engine. So, whenever the engine is rotating at some speed, these two balls either will fly apart or they will come closer depending on the motion of the engine.

So here, with the extended shaft, this is connected to the supply of the fuel. Here, you see the opening of the fuel; either the fuel is being supplied to the engine or the fuel is not being supplied to the engine. So, let us see the motion of the governor.

Let us say there is some desired speed which we have fixed for the motion of the engine. If the engine is rotating at a higher speed, then that is desired speed. So, these two balls will fly apart and because of the motion of these fly balls, this shaft will be brought down.

Once it comes down, it goes up; meaning to say it will close the supply to the engine, supply of the fuel to the engine and once the supply is stopped the engine will rotate at a lower speed. So, basically it is to control the speed of the engine. So, this is in fact, the first significant feedback control systems in the modern Europe.

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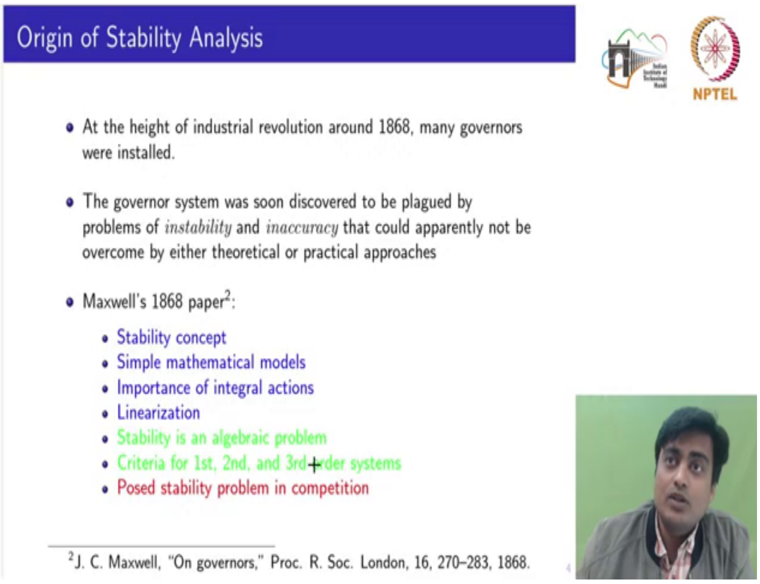

So, at the height of the industrial revolution around 1868, many such governors were installed. In fact, there were many improvement in the design of the governors was established, but all those governor's systems were highly plagued by the inaccuracy and the instability of their motion.

So, there was a very famous paper by Maxwell in 1868. So, this paper is sited here. You could also see this paper is freely available on the internet, if you would like to go a bit deep into what this paper talks about. So, this paper brings many important things theoretically and also the from the practical point of view.

First of all, it introduces the concept of the stability. Second this paper describes the Simple mathematical models of the engine and the governors. It also highlights the importance of the integral actions which you have possibly studied in your CG, control course when you studied the specific part of proportional integrator and the derivative controller.

Fourth is the Linearization. Fifth which is fifth and six which are quite important that stability is basically an algebraic problem and the second last is in this paper some criteria for the 1st, 2nd and 3rd order systems were also given that under what conditions the system would be stable or the system would be unstable.

So, basically this paper posed the stability problem in competition, that how one can define the stability and for the higher order systems or for the complex systems the stability can be assessed.

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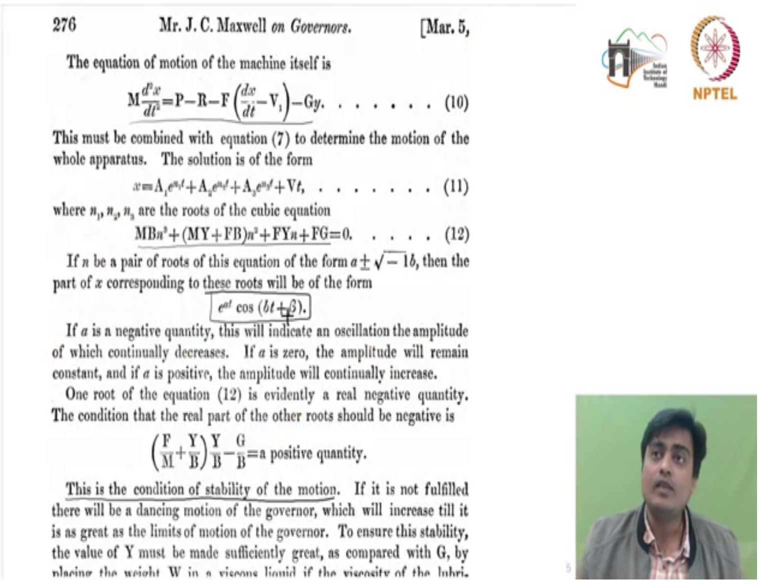

276 Mr. J. C. Maxwell *on Governors.* [Mar. 5,

The equation of motion of the machine itself is

$$M\frac{d^2x}{dt^2}=P-R-F\left(\frac{dx}{dt}-V_1\right)-Gy. \quad (10)$$

This must be combined with equation (7) to determine the motion of the whole apparatus. The solution is of the form

$$x=A_1e^{n_1t}+A_2e^{n_2t}+A_3e^{n_3t}+Vt, \quad (11)$$

where n_1, n_2, n_3 are the roots of the cubic equation

$$MBn^3+(MY+FB)n^2+FYn+FG=0. \quad (12)$$

If n be a pair of roots of this equation of the form $a\pm\sqrt{-1}b$, then the part of x corresponding to these roots will be of the form

$$e^{at}\cos(bt+\beta).$$

If a is a negative quantity, this will indicate an oscillation the amplitude of which continually decreases. If a is zero, the amplitude will remain constant, and if a is positive, the amplitude will continually increase.

One root of the equation (12) is evidently a real negative quantity. The condition that the real part of the other roots should be negative is

$$\left(\frac{F}{M}+\frac{Y}{B}\right)\frac{Y}{B}-\frac{G}{B}=\text{a positive quantity.}$$

This is the condition of stability of the motion. If it is not fulfilled there will be a dancing motion of the governor, which will increase till it is as great as the limits of motion of the governor. To ensure this stability, the value of Y must be made sufficiently great, as compared with G, by placing the weight W in a viscous liquid if the viscosity of the lubri-

So, there is a snapshot of this paper which I have uploaded here. So, this is. So, here the important thing is in this basically it is a continuation of some couple of equations starting from 1 to 10. So, here he defined the motion of the machine itself by this complete differential equations and then, he computed the solution of the differential equation which we had seen in the week 1 of this course and then, he and then he computed the roots of certain equation which is basically polynomial of degree 3.

So, the important thing here he computed the solution of such kind which we have studied that using the exponential solutions of the linear systems and after computing the roots, he basically gave the conditions of the stability of the motions.

So, initially for the first, second and third order systems, he was able to write the conditions in terms of the eigenvalues or the roots of the characteristic problem. We will see a couple of things we have already seen in your UG course and are much more detailed analysis in the sense of Lyapunov, we will try to discover in this module.

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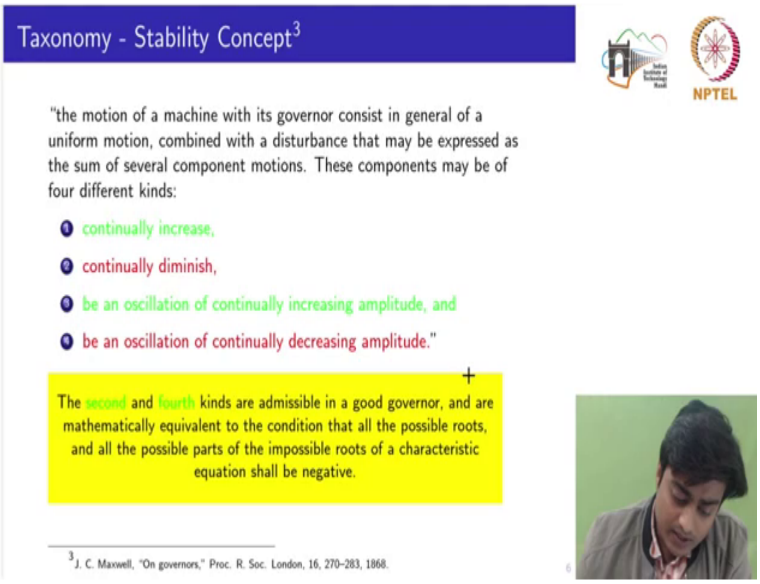

So, this is how he defines the stability concept that "the motion of a machine with its governor consists in journal of uniform motion, combined with a disturbance that may be expressed as the sum of several component motions. These components may be of 4 different kinds; continuing either it could be continually increasing, diminishing or be an oscillation of continually increasing amplitude and the oscillation of continually decreasing amplitude."

Basically, the idea is to see how the motion is been governed by this particular solution. So, from your UG control course, we you already know that the second and the fourth kinds are admissible in a good governor, because the response is diminishing to 0 and are mathematically equivalent to the condition that all the possible roots and here, the possible roots are the real roots and all the possible parts of the impossible roots, if the root is impossible meaning to say the root is complex; then, the real part of that complex root of a characteristic equation shall be negative.

We will see all of the connection of this negative and the positive eigenvalues or the roots of the characteristic polynomial with the in the sense of Lyapunov Taxonomy.

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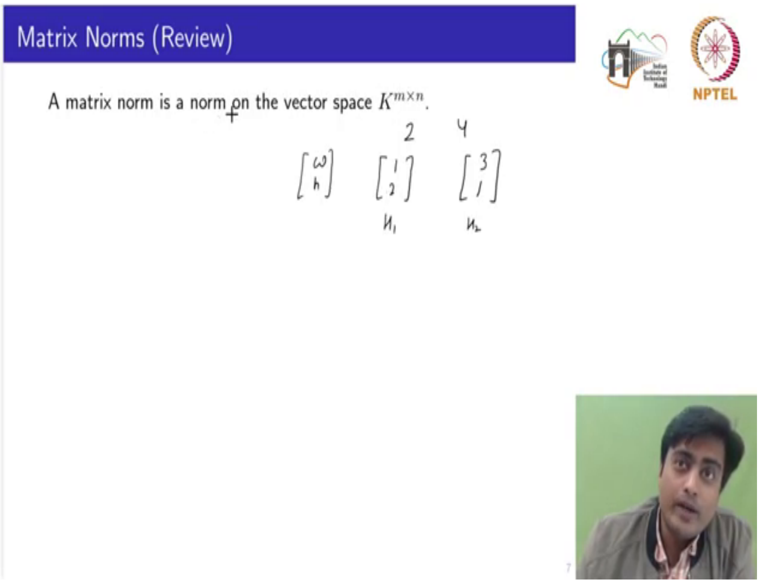

So, proceeding further with the with knowing the origin of the stability, here we will introduce a brief or the matrix norms and the metric vector norms and also the positive and same negative definite matrices. So, matrix norm is a norm of the vector space K of dimension m cross n right. So, first of all we you should be able to understand that why do we need a norm of a matrix or a vector for comparison purposes say for example, if you have some scalars value, let us say 2 and 4.

If some scalar values are given to us, we know that the number 4 is greater than number 2 or vice versa right. But if let us say a vector is given to you, in the sense of 1, 2 or 3, 1. So, these vectors would be a representation of many things. Let us say if you talk about the size of the human body. So, the first element could be the width and the second element could be the height.

So, let us say here are 2 humans; H 1 and H 2, whose width and height is given in this format. So, if we want to know the size of the human for the comparison purposes, how can we make this comparison? Because here we have two numbers. So, the idea of the norm is to obtain a scalar value either for a vector or a matrix so that you can compare two or more of the matrices or the vectors.

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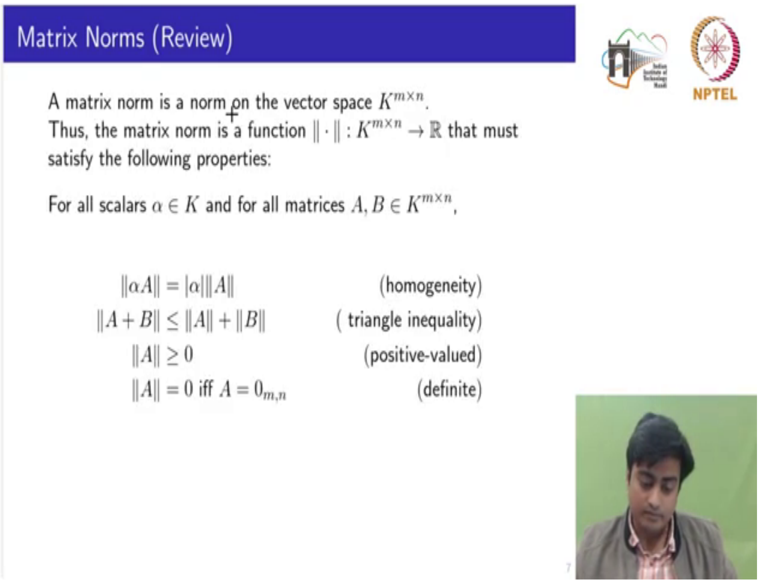

So, thus the matrix norms is a function which is defined by this such that an m cross n dimensional space can be mapped into a scalar value that must satisfy the following properties. For all scalars alpha belonging to the peal K and for all matrices AB belonging to the dimension of m cross n. If that matrix is satisfying the property of homogeneity meaning to say that the norm of the multiplication of the scalar alpha and the matrix is equal to the absolute value of the alpha and the norm of the matrix.

Second is the triangular inequality that the norm of the summation of 2 matrices is either less than or equal to the summation of their individual norms. Positive value that the norm is always positive or 0 and it would be 0, if and only if the matrix itself is a zero-matrix of the same dimension right which we call the definite.

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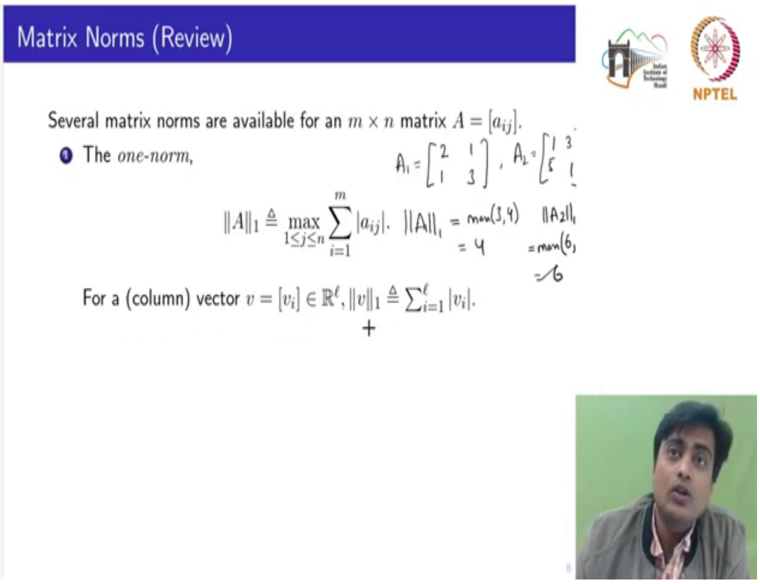

So, several matrix norms are available for an m cross n matrix A which are defined by this a i j elements. So, a i j elements is i is defines the number of rows and j is the column of the matrix. So, the first norm, we define is the one-norm. One-norm is defined by let us say a matrix is given to you A.

So, this basically mean that I will sum up all the elements along the columns and then, I will compute the maximum of that single row say for example, if I and let us see if I have two matrices; 2 1 1 3; let us call it A 1 and another matrixes 3 5. If I want to compute the one-norm of this matrix A 1. So, it would be given by the maximum of I will sum all the elements along the columns so that the sum is 2 plus 1 is 3 and 3 plus 1 is 4 and the value is 4 right.

Now, similarly if I want to compute the norm of this matrix A 2, it would be given by the maximum of 5 plus 1, 6 and 3 plus 1, 4. So, the number is 6. So, here I know that the metric the size of the matrix A 2 is greater than the size of the matrix of A 1. Now, if we have a column vector, then the column the one-norm of the column vector is basically given by the summation of all the elements of that vector.

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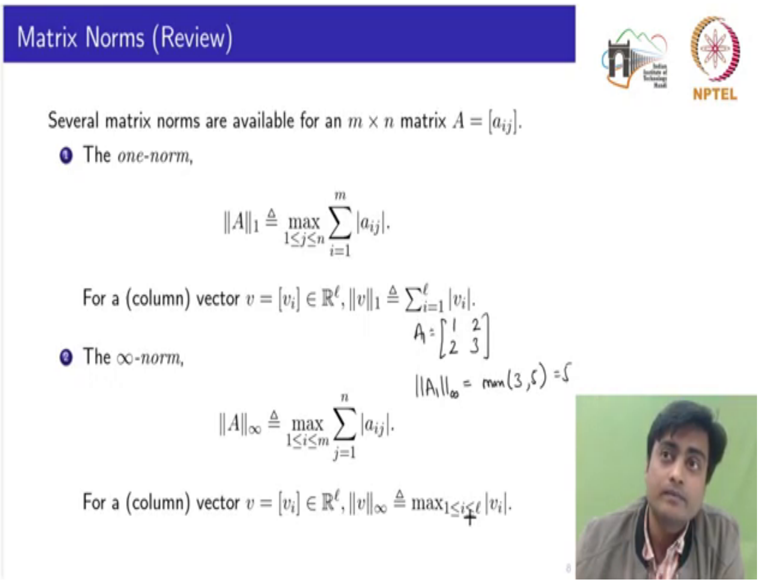

The second norm is the infinity-norm. Infinity-norm is basically the exactly the opposite of how we have computed the one-norm. So, in the infinity-norm, we sum the elements along the rows and then, we take the maxima out of it right. So, if again considering the example A 1 1 2 2 3.

So, the infinity-norm of this matrix A 1 is basically given by the maximum of summing along the rows. So, here 1 plus 2 would become 3 and 2 plus 3 become 5. So, the value would begin by 5 ok. Now, if we have a column vector, then the infinity-norm of that column vector is basically the maximum element of that vector which is given by this one because here we do not have the second column to sum up the elements.

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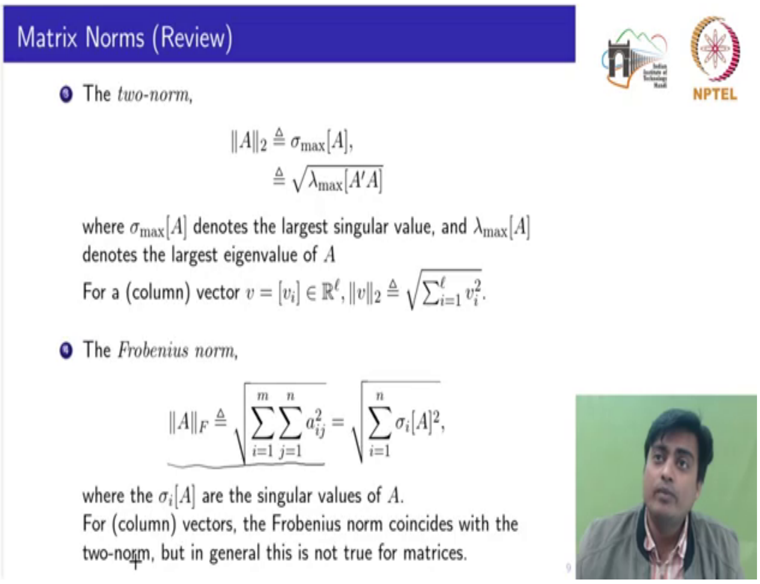

Another norm is the two-norm. So, two-norm is basically computed by the computing the largest singular value of the matrix A, since we have not studied about this how to compute the singular values. You can also compute the two-norm by computing the largest eigenvalue of the matrix which is given by the multiplication of the transpose of that matrix by itself and then, taking the under root will give you the two-norm. Now, for a column vector, for a column vector, it is given by the square root of the submission of squared elements of that vector.

Another norm we have the Frobenius-norm. So, the Frobenius-norm you can compute by this formula, where again sigma i denotes the singular values of the matrix A. If you do not want to compute the singular values, you can also use this formula to compute the Frobenius-norm.

So, for column vectors the Frobenius-norm coincides with the two-norm meaning to say if you compute the Frobenius-norm of the vector and the two-norm of the vector the values would certainly be equal, but in general this is not true for the matrix ok. So, keep this thing in mind. So, there are a couple of norms which possibly we would be requiring to do our analysis later in this module.

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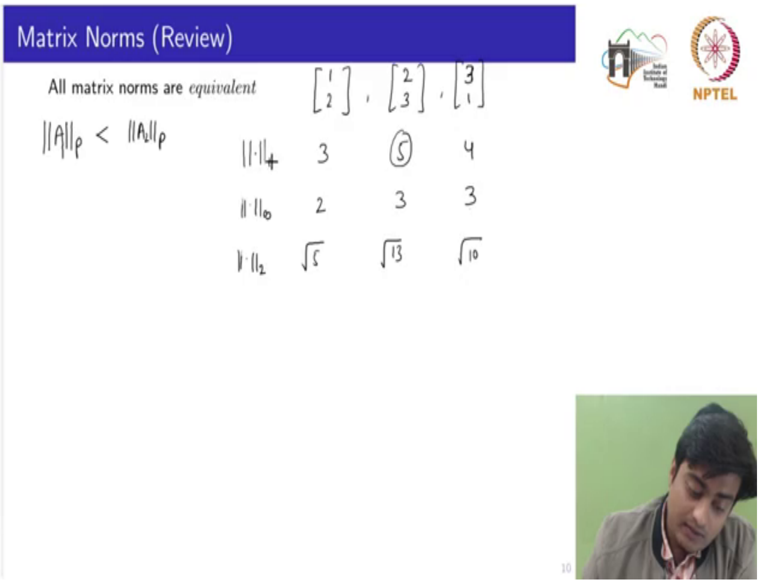

So, the most important thing here is that all matrices, all matrix norms are equivalent, you can use any matrix norm to compare two or more matrices. Say for example, let us talk about the vectors. Let us say we have one vector 1 2; another vector 2 3 and another vector let us say 3 1 ok.

Now, if I want to compare the that which vector is of the largest size than the other of them other of the vectors. So, if I compute the one-norm; the one-norm would be given by the summation of all the elements; basically 3 and this one will need 5 and this one will need 4. So, if I compute the infinity norm, it would be the mixing of element 2 3 and 3.

If I compute the two-norm, the two-norm is given by under root 5 for this one; under route 13, 3 squared 9 plus 2 square 4, under root 13 and for this one, it would be under root 10. So, if you compare all the values of the norms, you would see that the second vector is having the largest size, although when you are computing the infinity norm of that vector this its value is equal to the third vector, but it cannot go lower than the third vector right.

So, when we say all matrices norms are equivalent, we actually mean to say let us say I compute any norm, let us say called the p norm of the matrix A 1 and the p norm of the matrix A 2.

Now, if for any norm this relationship is satisfied, then for any of the p norm p 1 to infinity and the Frobenius norm, this relationship will always hold ok. Although the value might change as we have seen in this example here that for the second vector, the one-norm, infinity-norm and the two-norm are having different values with respect to the first and the third vector right.

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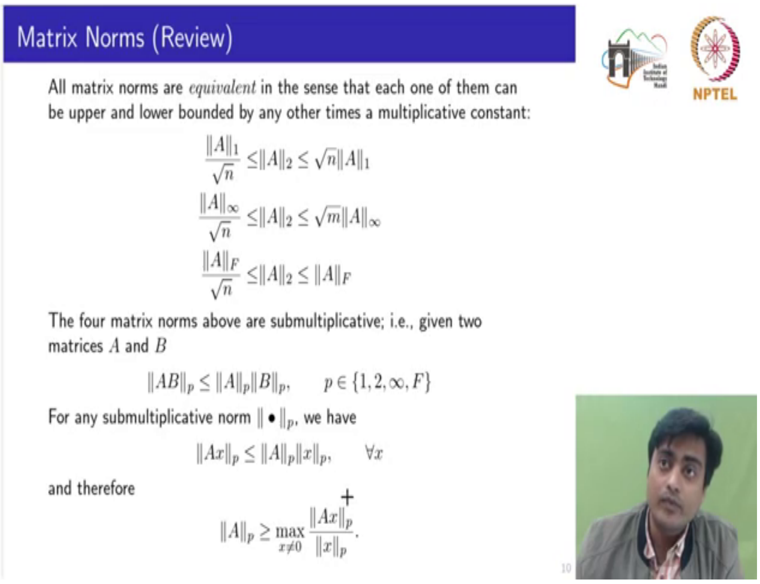

So, if you also want to compute the relationship between the different matrix norm, this relationship you can use to compute those values, where m and n basically defines the dimension of the matrix. So, the 4 matrix norm, what we had discussed above are some multiplicative in the sense that the norm of the multiplication of 2 matrices is basically less than equal to the multiplication of their individual norm and I wrote this p because p belongs to 1 to infinity and all including the Frobenius-norm.

So, for any sub multiplicative norm, we have this relationship if I replace B by the vector x, then I would compute the p norm of the matrix using this relationship as well, which would be always greater than equal to the maximum value of the ratio the p matrix norm of the A x and the p vector norm of the x, where x is not equal to 0.