Linear Dynamical Systems
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Lecture – 43
Tutorial on week 7 and 8

So, now, we will see the combined Tutorial of the week 7 and 8. So, in the week 7, we discussed about the observability and in the week 8, we discussed about the state estimate design. So, since both the concepts are pretty much related. So, we will take the we will go with the combined tutorial.

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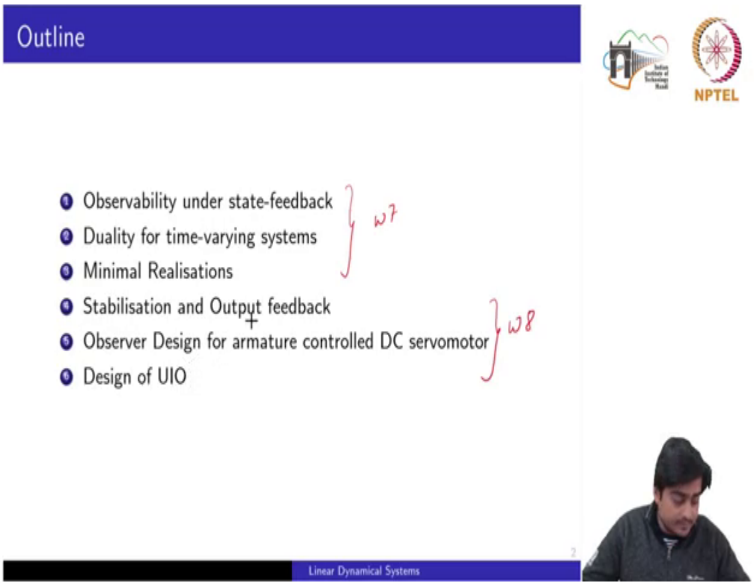

So, these topics would be covered for solving the numericals. So, the first three problems are basically from the week 7 and the next three problems are basically you can relate with the week 8. So, we in from the week 7, we will focus onto the observability under the state feedback. So, one of the concepts we had studied during the controllability week, that once you have design a state feedback controller, then the controllability is invariant. But we cannot ensure the observability under the state feedback.

So, we will see one of the problem on that aspect. Second, we shall see the duality for time-varying systems. For the LTI system, we had seen a detailed proof that the about the duality; but we have left this question, unanswered at that time that if there exist any duality for the time-varying case similar to the LTI case.

The third problem is about the minimal realisation, where we will try to correlate the concepts of controllability and observability with the infinite with the many realizations. So, the fourth, fifth and sixth problems is are basically about the output feedback, observer design and the unknown input observer design.

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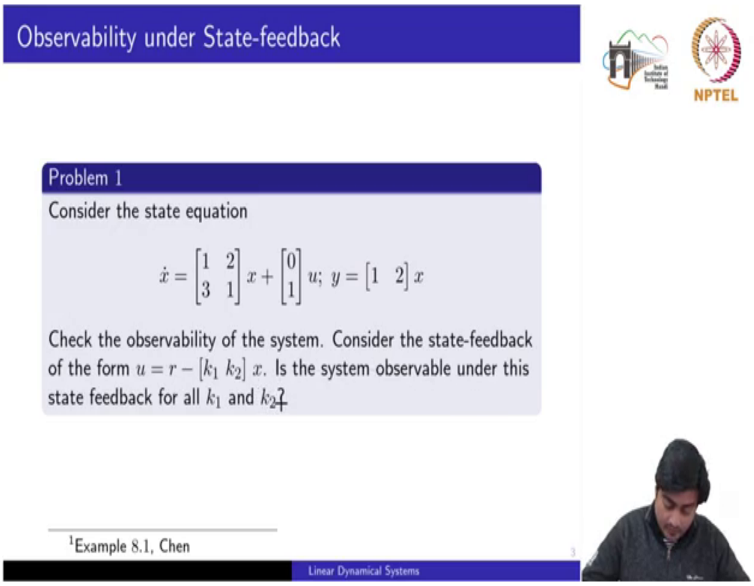

And the first problem, we are considering the state equation x dot is equal to x plus b u y is equal to c x, where a, b, c matrices are given here. So, this problem was taken from the example 8.1 of the one of the references mentioned on the course page. So, first we need to investigate about observability of the system. Second, we shall design the state feedback of this form which is r minus k x; where, k is a vector composed of k 1 and k 2. So, we need to investigate whether the system is absorbable under this state feedback for all k 1 and k 2 ok

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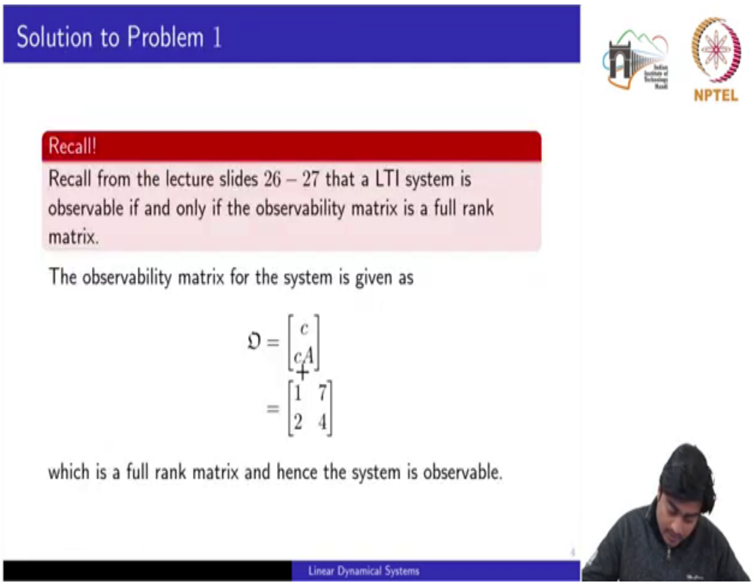

So, the first part is pretty much straight forward that you compute the observability matrix given the a and c matrices and finally, check the rank of that controllability, observability matrix. So, here it is a full rank matrix.

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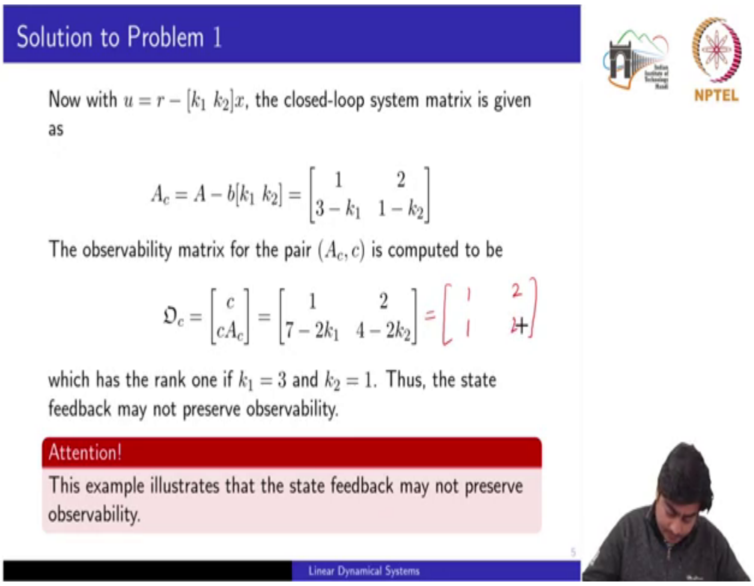

So, the system is observable, but the interesting part is to see whether the observability is invariant under this state feedback. So, let us put this controller into the state feedback system in that case the closed loop system matrix could be given by we denoted here by A subscript c is equal to A minus b k. So, substituting the corresponding values of A b and taking k 1, k 2 as the parameters. So, we computed the observability of the closed loop system in terms of k 1 and k 2.

Now, if you put k 1 is equal to 3 here and k 2 is equal to 1 here. So, from here we would get by putting k 1 is equal to 3, we will get 1 and putting k 2 is equal to 1, we will get 2. So, this is straight forward to see that the rank of this observability matrix is not 2, but 1; meaning to say that the system is not observable under the state feedback.

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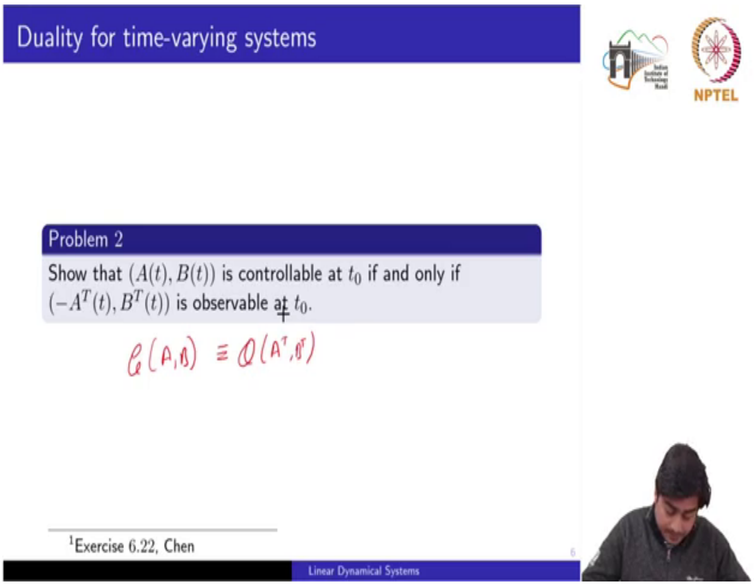

The second problem discusses about the duality. So, given the pair A and B which is time varying. So, we need to show that the controllability of this pair is equivalent to the observability of the pair which is minus A transpose and B transpose ok. Now, if you recall the results for the LTI system, if A and B are timing variant then the let us say the controllability of the pair A comma B is equivalent to the observability of the pair A transpose B transpose ok.

But here, we have a slight change here, we are having the negative sign. So, we need to investigate whether we can have some duality in for the time-varying case.

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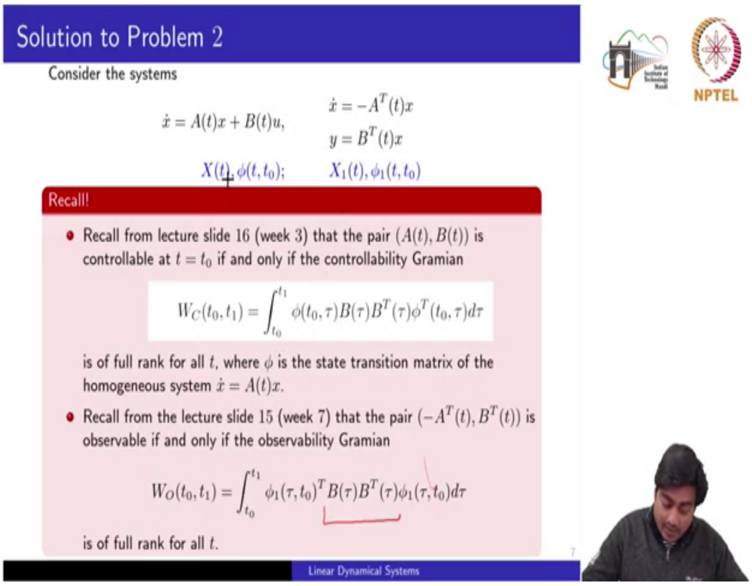

So, to solve this problem, we considered two systems. The first system is x dot is equal to A t x plus B t u. Another system is where we have replaced the A matrix by minus A transpose and the output matrix, we are taking as B transpose right. We have deliberately write these state space equation into these forms because we will investigate the controllability of this system and for investigating the controllability, we only need the A B pair.

Now, for observability, we only need the pair the state matrix and the output matrix. So, we have not consider any input term here. Corresponding to these two systems, we are defining two important parameters which are required to solve these time-varying systems. First is the fundamental matrix which we have denoted by x for the system 1 and the fundamental matrix denoted by x 1 for the system 2.

Similarly, this phi is the state transition matrix for the system 1 and phi 1 is the state transition matrix for the system 2. These are the two important parameters which we have discussed in the first week, when we had seen the solution of the time-varying systems. Now, we will recall first of all how we have defined the controllability and the observability. In terms of the Gramians. So, these concepts are basically related to the week 3 and week 7, where we have shown that the controllability of the pair A comma B is equivalent to the full rank of this Gramian which we have defined as the controllability Gramian.

So, here we are taking the directly the straight transition matrices this phi and the B transpose remain as it is of the input distribution matrix. For the observability, since these are the state and the output matrices, we have directly written the Gramian in terms of these pair which is supposed to be full rank, if we want to ensure the observability of this pair ok.

Now, the logical idea behind the proof is if we are able to show that W c is actually equal to W o and if we ensure that this is of full rank, it is necessary that this would also be a full rank ok. So, this W c we have defined for the system 1 and W o we have defined for the system 2. So, we only need to show the equivalence of this W c and W o.

Now, we pay attention to the on to the right hand side, the terms which are appearing in the at the middle of this integrant remains the same ok. The difference lies only in the state transition matrices. So, the next idea is if we are able to show the some kind of relationship between this phi or the state transition matrix of the system 1 and the and that of the system 2, then we can directly show the W c is actually equal to W o ok.

Now, with a clear look we only need to show that phi t 1 coma tau should be equal to phi 1 tau comma t naught transpose, similarly here ok. So, now, to show the equivalence between this W c and W o, we need to show the equivalence of the state transition matrix and the solution of the state transition matrix basically hinges upon the fundamental matrix ok.

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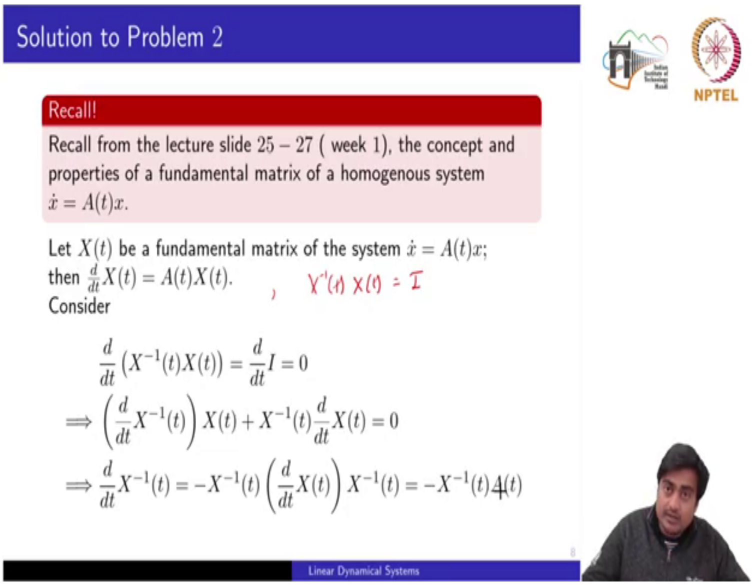

So, let us see if there is any relationship between the fundamental matrices of both the systems and then, we will come directly onto the state transition matrix. So, let X t be the fundamental matrix of the system. We only need to consider the homogeneous system, then we know that the fundamental matrix actually satisfy this equation. So, we can write the derivative of capital X is equal to A times capital X.

Another property which we had also seen during week 1 that X inverse t and X of t is actually equal to the identity and X is a non singular matrix. So, taking the derivatives both sides of this equation which is this one, we can or and further simplifying, you will end up to this relationship where it says the derivative of the inverse of the fundamental matrix is actually equal to minus inverse of X and A right.

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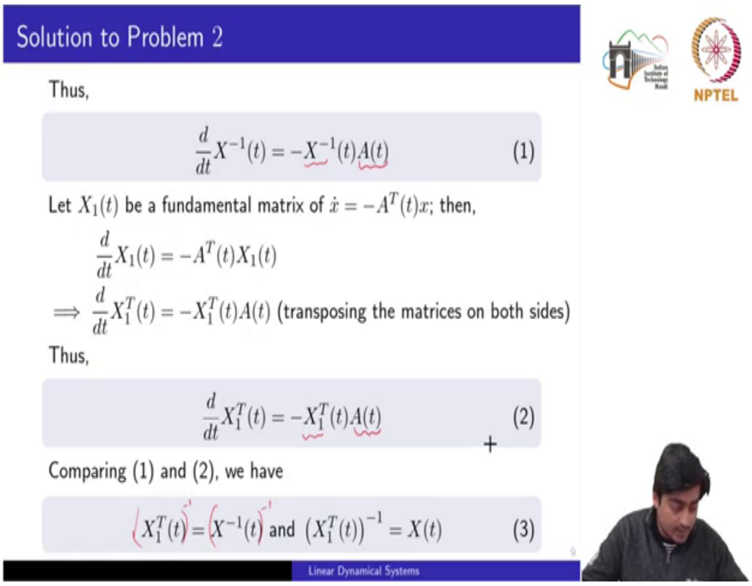

So, this is a key equation and we will put it as equation 1. Now, consider the second system whose fundamental matrix was X 1 and the state matrix was minus A transpose. Now, similarly what we had seen in the previous slide that X 1 again would satisfy this state equation and we would obtain directly this equation as equation number 2.

Now, if you compare both these equations 1 and 2, you would see that the A matrix is at the same place and now, X 1 transpose actually become equal to X inverse ok. Now, under this condition both these system would be the same. Similarly, if I take the inverse of the X 1 transpose, it would become equal to X. We just need to take the inverse both side ok. So, we get this equation. So, both the systems are related by the some relationship between their fundamental matrix.

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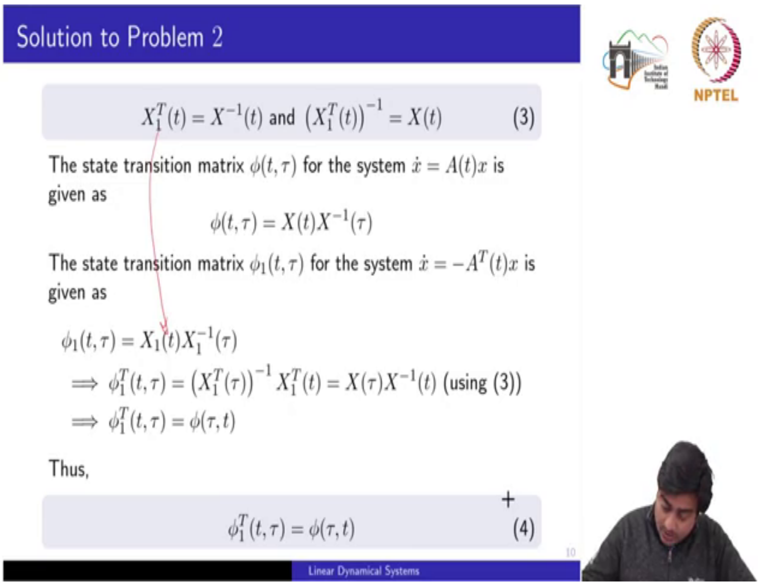

So, now let us see the state transition matrix which we had seen in the week 1 as well is given by the multiplication of the matrix x and its inverse at some time tau ok. This is how we define this phi t comma tau. Now, for this system we can write the phi 1 as again this one. Now, here X 1, we have replaced from the equation 3, what we have obtained in the previous slide.

Further simplifying it in terms of the only X, we obtain that the transpose of the state transition matrix of the second system is actually equal to the state transition matrix of the first system with the time reversal ok. Phi 1 transpose t comma tau is equal to phi tau comma t. So, this is again key equation which we will going to use it later to obtain the result. So, let us proceed again with this equation 4.

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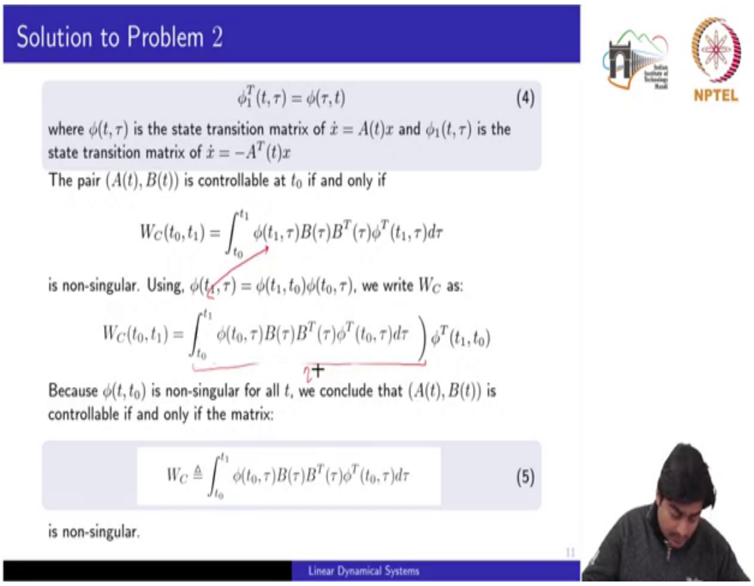

So, we had seen that the pair A, B is controllable, if this matrix is a full rank. So, again we are using one of the properties of the state transition matrix that if I take some time t naught in between t 1 and tau, then I can write this phi t 1 comma tau as the multiplication of 2 phi's as t 1 comma t naught and t naught comma tau right. Now, substituting this into this W c and simplifying we can take a pre multiplying factor phi t 1 comma t naught as the common part and the post multiplying factor as phi transpose t 1 comma t naught.

Now, from the property of the state transition matrix that it should be non-singular. So, for W c to have a full rank this term should be positive definite or should be in fact, definite ok; should only be definite. Definite in the sense that this matrix should be non singular which we have defined A as eta, this common part eta ok.

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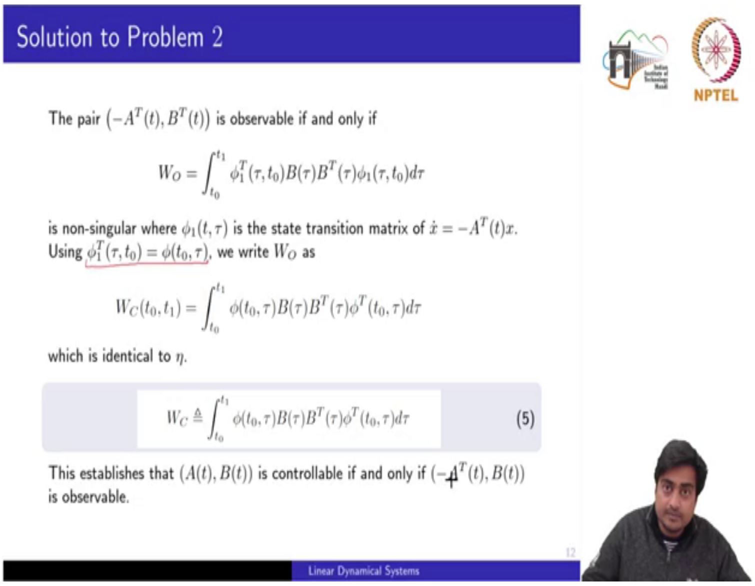

Now, coming onto the observability Gramian of this pair and using the relationship which we had obtained in the equation 4, I can replace all these phi 1 by this phi with the time reversal of course. So, directly by replacing the phi 1 by all phi, we obtain this W which is nothing but equal to eta actually. So, here we had shown that the observability matrix of the second system is actually equal to the controllability Gramian of the first system ok. So, this establish the duality which you would realize that which is slightly different from the LTIs.

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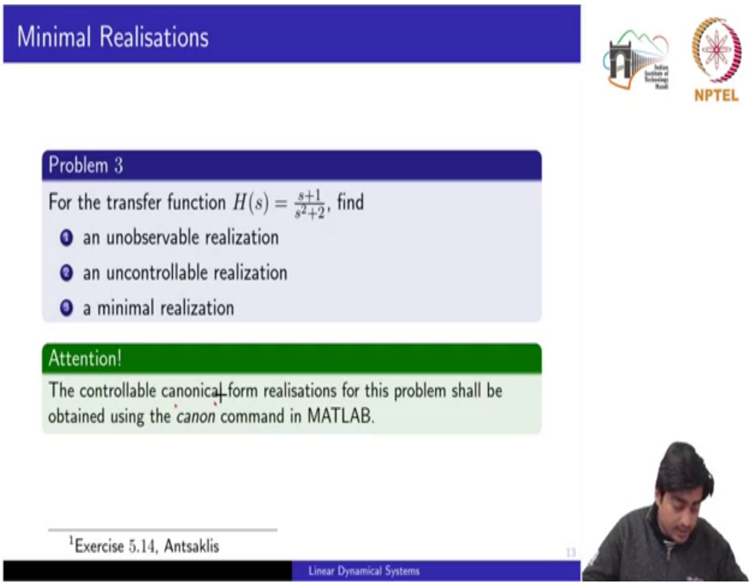

The third problem is about the minimal realization. So, we are given some transfer function which is a single input single output transfer function which is s plus 1 over s square plus 2. So, here we need to we want to find three realizations; first one should be unobservable, but controllable; second uncontrollable, but observable and third a minimal realization which is equivalent to saying that the that realization is both controllable and observable.

So, here we will going to use directly the this MATLAB command to obtain the realization which you can also compute by going through the procedure defined in the slides.

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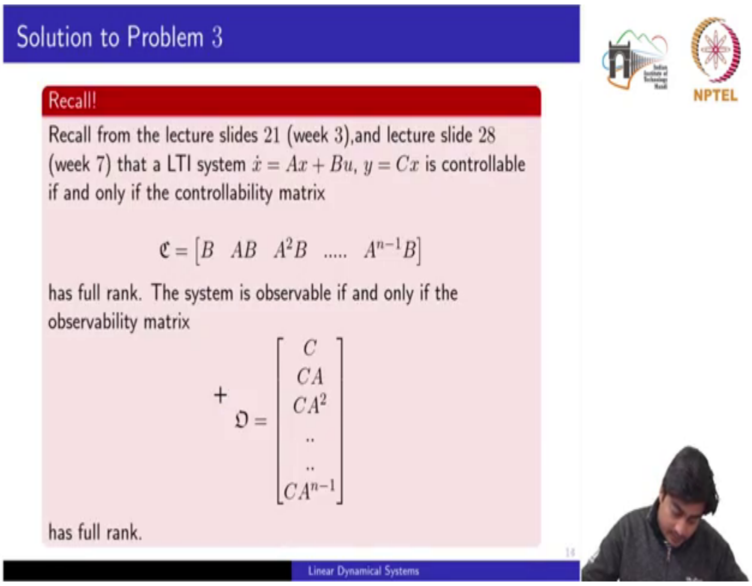

So, the controllability or the full rank of the controllability matrix and the observability matrix ensures that the system is controllable and observable both.

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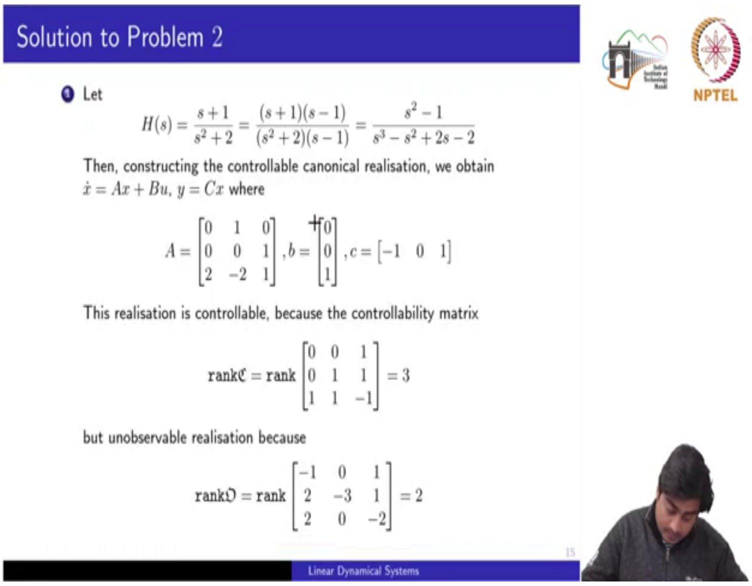

So, to obtain the unobservable realization what we have done here. So, in this transfer function we have added a one polynomial factor both in the numerator and in the denominator ok.

So, opening this polynomial, we would obtain this one. Now, for this transfer function, we obtain directly the controllable canonical realization in form of this A, b, c matrices and if you compute the controllability of this matrix it is a full rank, but it is unobservable because the rank of the observability matrix of A of the pair A comma c is only 2 ok.

So, this realization is an unobservable realization. Why? One insight of this one is that because we have added a common vector both in the numerator and the polynomial which we are seeing only the external signals that is to say input and output signal won't be the transfer function would not be able to capture it.

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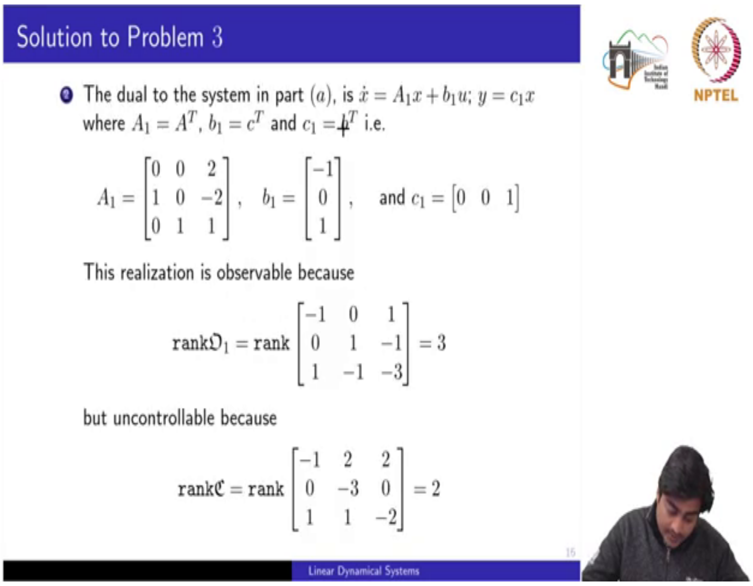

The second part, we could solve straight forwardly by taking the dual form of the system A that is replacing the A matrix by its A transpose, b matrix by the transpose of the output matrix of the first system of the first equation and the output matrix by the transpose of the input matrix of the first system. So, these are the matrices what we have obtained. Now, if we compute the observability or the rank of the observability, then it shows that the that this realization is observable. But again it is not controllable and this is what implicitly the concept of duality says.

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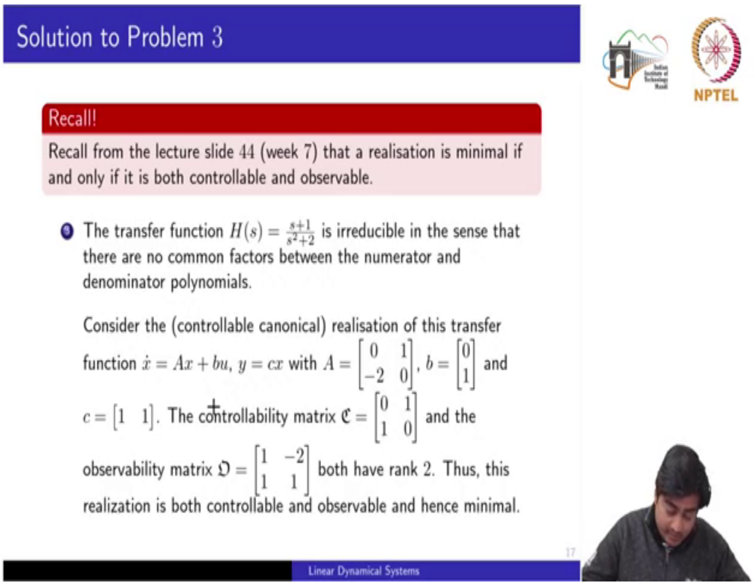

Now, lastly is the realization which is supposed to be minimal and we had seen detailed proof that the that realization should be both controllable and observable. Now, one of the inherent meaning of this is that both the numerator polynomial and the denominator polynomial are co-prime, that is there are no common factors in those polynomials ok.

So, now directly writing the or computing the A, b, c matrices, we obtain these ones for which you can compute the rank of the controllability matrix and the observability matrix which is supposed to be 2. So, this realization A, b, c is the minimal realization and this minimal realization basically in terms of the dimension of the vectors the state vector.

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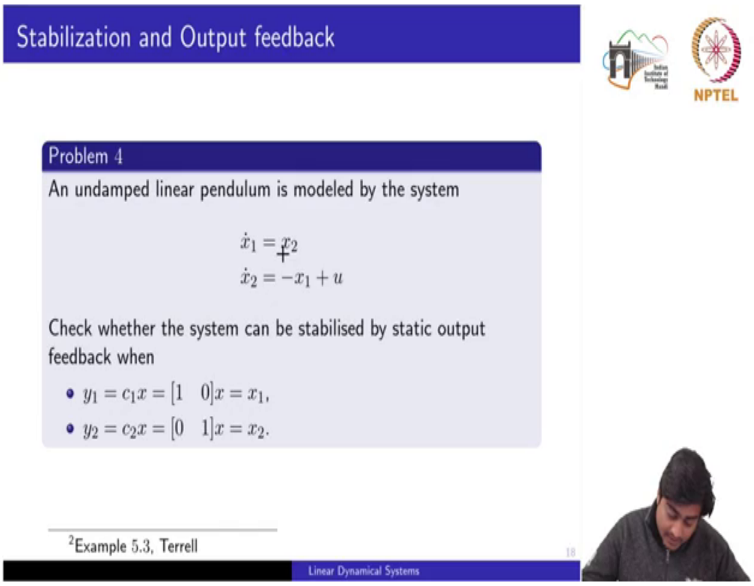

In the next problem, we want to design a feedback controller, but here there is a slight difference. So, let us see first. So, given this simple system which is a second order system with 2 states and 1 input which is acting on only on to the second state dynamics. So, here we want to check whether the system can be stabilized by aesthetic output feedback under these two scenarios. So, here we are considering a two different scenarios in which our y 1 is x 1. So, we are taking x 1 as a direct measurement and in the second case, we are only measuring x 2 into the output.

So, if you recall the concepts of the controllability, we had discussed earlier I can directly obtain the A, b, c matrices or these two equations. Now, if you check the controllability of the pair A comma b, it would be controllable ok. It would be controllable. Now, what we want to see that whatever the concept, we had studied about the controllability is there some relationship between the controller state feedback or the output feedback controller design.

Because do not forget that the concept of controllability, we had discussed is only about the state. So, basically, we should refer it as the state controllability ok.

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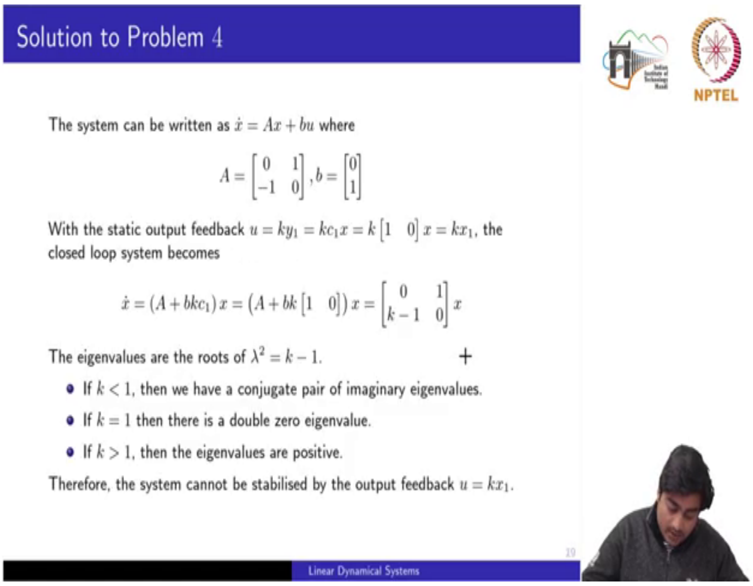

So, let us see. So, I can. So, the A, b matrix for the system is given by these matrices and we want to apply this feedback which is given by u is equal to k y 1 and y 1 is nothing but c 1 x and c 1 is 1,0; where, I am taking actually the state 1 the first state. So, if I see the actual controller is this now u is equal to k x 1 meaning to say I am feeding only the first state and want to see whether the system can be stabilized by feeding only one state or only the first state.

So, if we check the state matrix of the closed loop system, I will obtain this as A plus b k c 1 and substituting the value of A, b k and c 1, basically we want to analyse for k. So, after substituting all these matrices, we obtain the state matrix as this one. So, we can analyse that for what values of k the system can be analysed by computing the eigenvalues of this matrix. So, which basically aims to solve this quadratic equations for the lambdas.

Now, from this equation, you would see if k becomes equal to 1 right; k becomes equal to one then the eigenvalues would be a conjugate pair of imaginary eigenvalues, meaning to say that the system would be oscillatory. Now, if k is a becomes equal to 1, then we would have a double integrator meaning to say that the system is again unstable and if k is greater than 1, then all the eigenvalues would be positive that is on the right hand side.

So, for all the values of k the this or using this control law as u is equal to k x 1 would never give you a stable system that is so, that the system cannot be stabilized by this output feedback.

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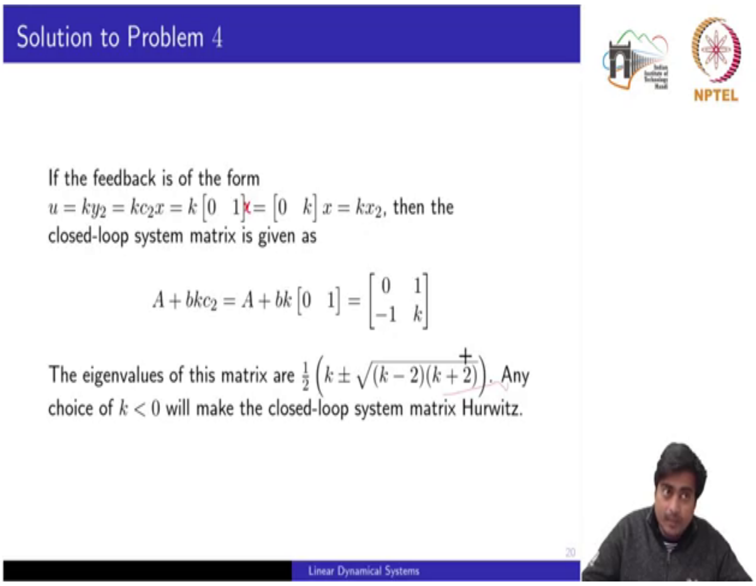

Now, consider the second case, where instead of feeding back the state 1 or the first state, we want to feedback the second state ok. So, let us see. So, now, the control law would become u is equal to k x 2 finally. Because the c matrix or the c vector has been changed to 0, 1. So, here we should have x ok. So, for this case the closed loop matrix would be given by this one; A plus b k c 2 and in terms of k, this would be a parameterize a state matrix. Now, computing the eigenvalues, we obtain this one and you would see that any choice of negative values of k, these eigenvalues would be onto the left hand side ok.

So, this means that if I feedback both the states, if I feedback both the states, so I know from the controllability property of the pair A comma b, the system can be stabilized ok. But now if I see that instead of feeding back both the states, if I feedback the first state the system can never be stabilized. But by feeding the second state, we see that the system can be stabilized for the negative values of k right. So, the concept of the state controllability what we had discuss is basically different from the what we are seeing in this numerical.

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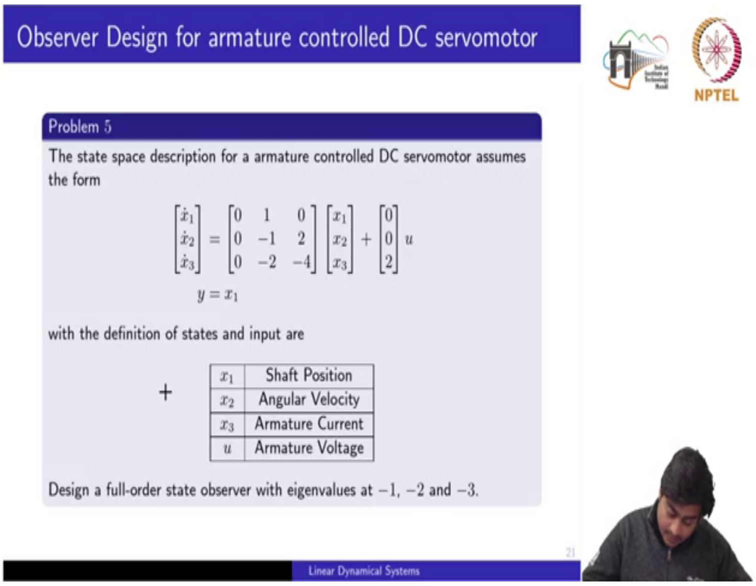

In the problem 5, we have some armature controlled DC servomotor, whose a, b, c matrices are given by these values ok. So, here your c would be a vector containing only first non zero element, the other two elements would be 0 of the c matrix. So, the definitions of the station inputs are given here. We want to design a full order state observer with eigenvalues place placed at minus 1, minus 2 and minus 3 ok.

So, the idea of designing the observer originated from when we want to close, when we want to design a state feedback controller and we do not have the direct measurement of all the states. So, instead of using this state x 1, we want to use the complete state. So, this is what we had seen in the previous example that somehow if we are able to design an observer for estimating both the states, then we know from the controllability property of the pair, the system can be stabilized.

So, here again we have a direct measurement of only one of the states, but not the all of the states. So, we want to design a full order state observer by placing the eigenvalues onto these values.

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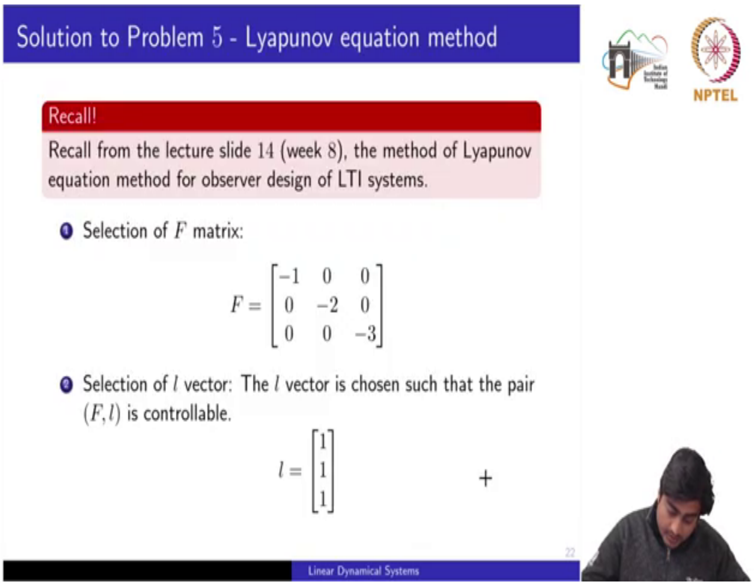

So, here we would. So, in the lectures, we had discussed two methods first is the eigenvalue assignment method which is basically the give you the Lemberger observer. Second is the based on the Lyapunov equation. So, here, we will go through the steps of computing the observer matrices by using that method and finally, we will see some simulation results.

So, the first step in this method is we want to select an arbitrary F matrix, whose eigenvalues are actually at located at the desired values. Now, the second step we select arbitrary l vector such that the pair F comma l is controllable. So, we have selected this l vector and this pair F, l is also controllable.

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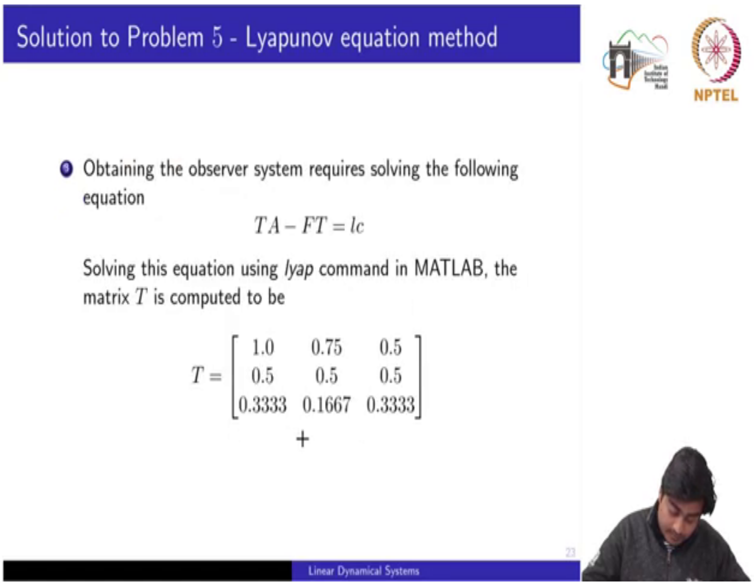

Now, the third step is computing the t matrix which is obtained by solving this Lyapunov equation and we have used the Lyap command, so this command in MATLAB to obtain this t matrix.

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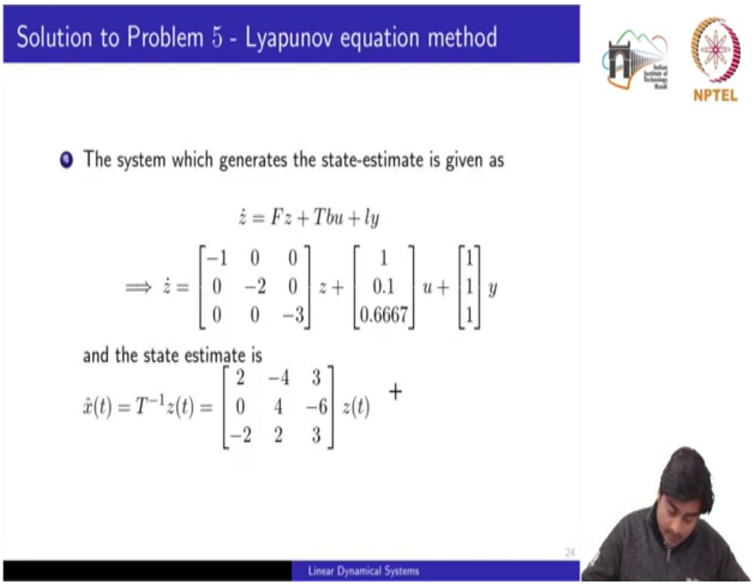

Now, the observer if you recall was given by this state equations z dot is equal to F z plus T b u plus l y and the state estimate is given by the inverse of that t matrix times z of t. So, substituting the matrix, the computed matrices by using the steps we had seen earlier. Now, we can go directly to the simulations, if I place this observer to estimate the actual sales.

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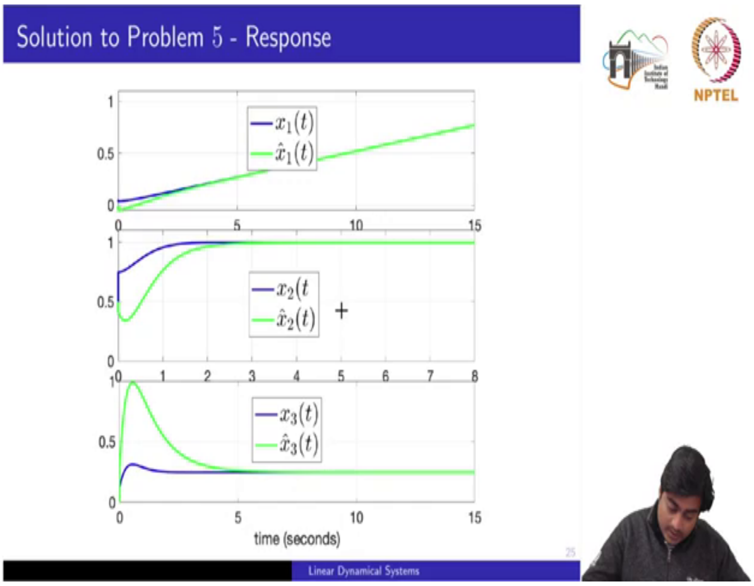

So, here we see that the blue curves are the actual states and the states which are drawn in the green colour are the estimates. So, only during the transition the transient periods the state values do not match; but when it reaches to the steady state both the states overlap; meaning to say that whatever the observer, we have designed is actually able to estimate all the states.

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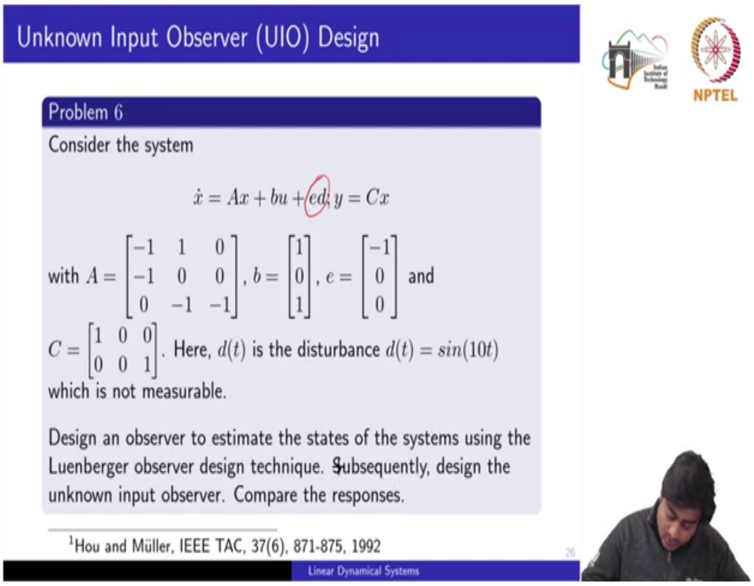

So, this is the last problem, where we want to design the unknown input observer. So, we are considering a system where now we have added a disturbance term with its distribution matrix e. So, this A, b, e, c values are given here. So, here d is the disturbance which is sinusoidal term and it is not in this variable is not directly measurable ok.

So, here so, this problem was also this numerical problem was given by one of the paper appear in the transitions some automatic control in 92. Here, we want to design two observers. So, in the first part, we will design the Luenberger observer which we have designed using the eigenvalue assignment technique and the second, we will design the unknown input observer.

So, if you recall in the Luenberger observer design technique, we require or in fact, in both the observers, we would be taking only the input and output variable as the input signals to the observer because we do not have any information of the or we are not supplying any information of the disturbance variable to the observer and finally, we will compare both the responses obtain from both observers ok.

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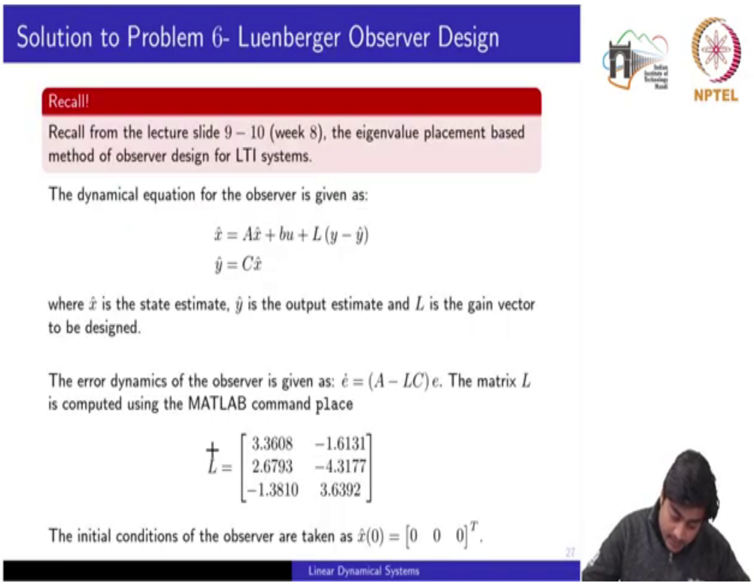

So, let us see. So, this was the observer equation which we had discussed on slide number 9 to 10, where x hat is the state estimate y hat is the output estimate and L is the gain vector which is supposed to be designed. So, if you recall that the error dynamics are given by e dot is equal to a minus L times C e; where, we can compute directly this l matrix if the observability of this pair A comma c is satisfied right.

So, we have use the MATLAB command place to place the eigenvalues onto the left hand side and we have computed this L matrix. Now, for simulations, we have taken the initial conditions at the region.

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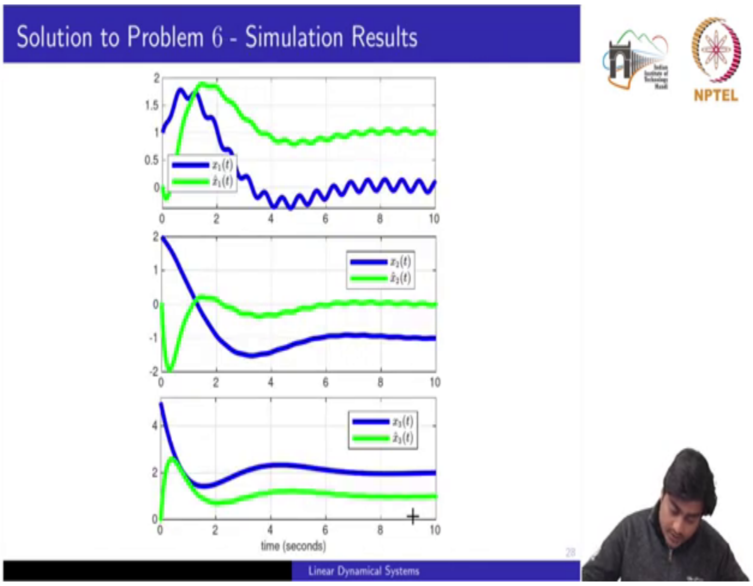

So, if we see the response of the system and the observer, we say we see that not only at the transient period, but also at the steady state period the states that state estimation error do not go to 0 and it is because that we have not supplied the information of the disturbance to the observer right.

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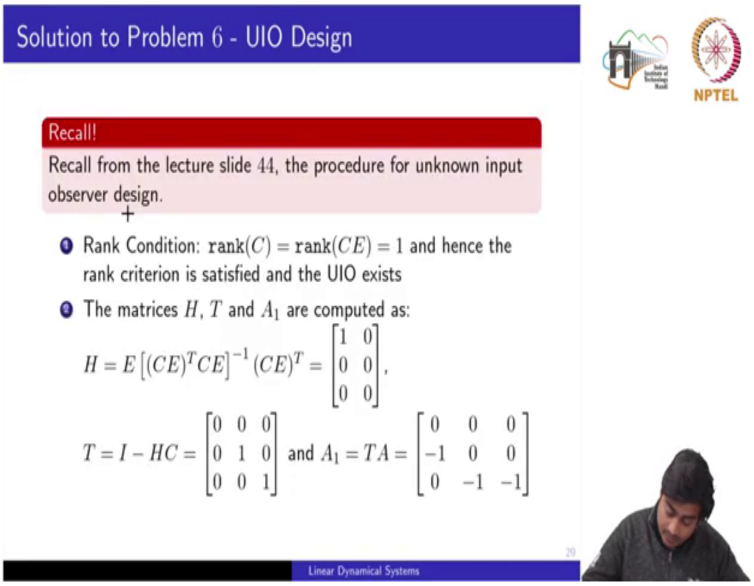

Now, we shall see that if this problem can be overcome by designing the unknown input observer. So, the detail designing steps were given on to the lecture slide number 44. So, here we will quickly go through all the steps to compute the relevant matrices and then finally, we will see the simulation results.

So, the first condition which is was basically the existence of the UIO and from there we give a spatial solution of this H matrix. So, we check the rank condition of C and CE; it is equal to 1. So, a UIO exists and this H is a special solution which we can we have computed with this matrix. So, the rest of the matrices were a straight forward to compute the T matrix and the A1 matrix, once we have computed the H matrix from the first step.

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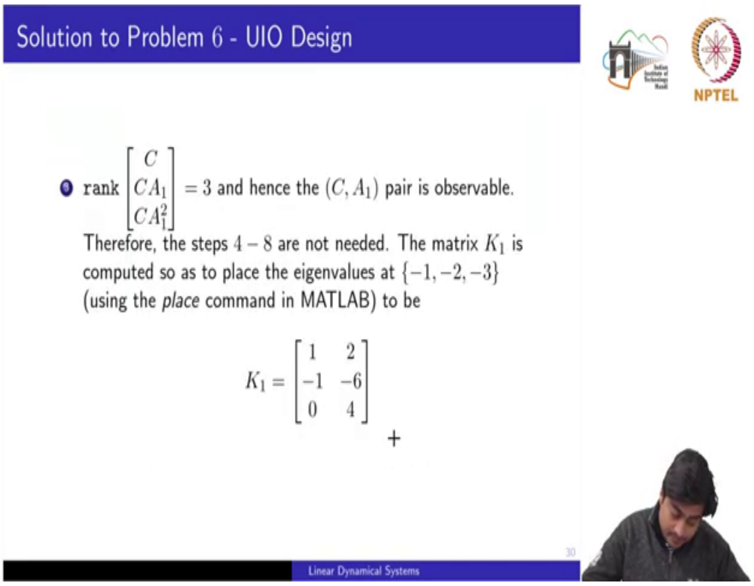

Now, the third step were in the third step, we need to investigate the observability of the pair X and A1 and A1, we had computed from this second step. So, here we see that this pair is observable. So, we do not actually need to go through the steps 4 to 8. Because in the 4 to 8 steps, we basically discuss that if this pair is detectable, then how you can compute this matrix K 1 ok. But here since the pair is observable, we can directly compute the matrix K 1 using the MATLAB command place ok. So, here we have place the eigenvalues at minus 1, minus 2 and minus 3 and we have obtained this K 1 matrix.

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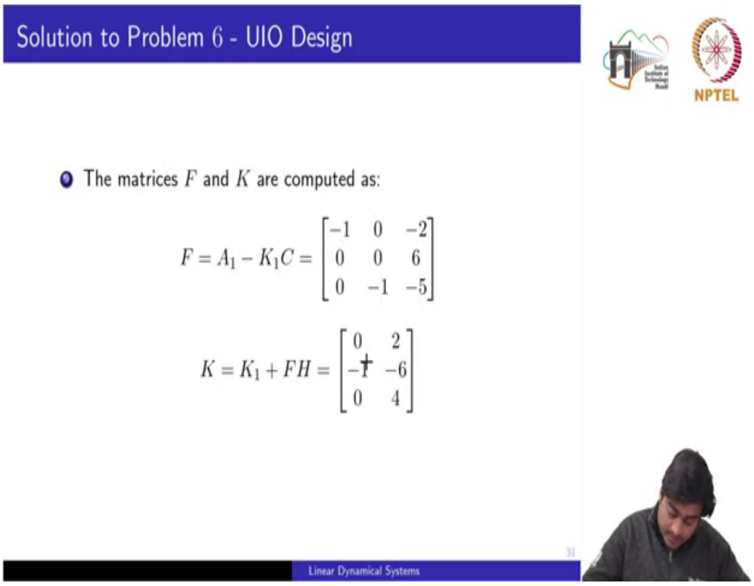

So, from the K 1 matrix, we have obtained f matrix which is supposed to be a Hermits matrix. So, you can compute the eigenvalues and from K 1 and H and F, we have finally compute the K matrix ok.

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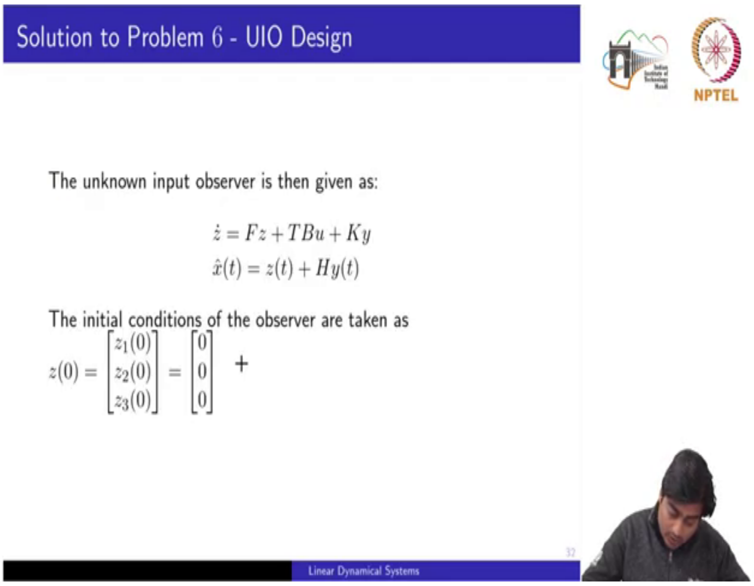

Now, putting all those matrices into the observer, F should be a stable matrix, T we have already computed, B is the output, u the input matrix and k is the an output injected term and in x hat z t plus H y t ok. Again here, we are taking the initial conditions of the observer at 0 0 0; meaning to say so the state of the observer is basically a z and the output of the observer is the x hat ok.

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Now, if we see the simulation results that although we have not supplied the information of the disturbance to the observer still the observer is able to estimate the actual value of the stage ok. So, this is the significance of the unknown input observer because we have eliminated the effect of that unknown disturbance onto the observer which was pretty much visible, when we had discussed the Lemberger observer ok.