

Linear Dynamical Systems
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Lecture – 43
Tutorial on week 7 and 8

So, now, we will see the combined Tutorial of the week 7 and 8. So, in the week 7, we discussed about the observability and in the week 8, we discussed about the state estimate design. So, since both the concepts are pretty much related. So, we will take the we will go with the combined tutorial.

(Refer Slide Time: 00:33)

Outline

- 1 Observability under state-feedback } w7
- 2 Duality for time-varying systems }
- 3 Minimal Realisations }
- 4 Stabilisation and Output feedback }
- 5 Observer Design for armature controlled DC servomotor } w8
- 6 Design of UIO }

Linear Dynamical Systems

So, these topics would be covered for solving the numericals. So, the first three problems are basically from the week 7 and the next three problems are basically you can relate with the week 8. So, we in from the week 7, we will focus onto the observability under the state

feedback. So, one of the concepts we had studied during the controllability week, that once you have design a state feedback controller, then the controllability is invariant. But we cannot ensure the observability under the state feedback.

So, we will see one of the problem on that aspect. Second, we shall see the duality for time-varying systems. For the LTI system, we had seen a detailed proof that the about the duality; but we have left this question, unanswered at that time that if there exist any duality for the time-varying case similar to the LTI case.

The third problem is about the minimal realisation, where we will try to correlate the concepts of controllability and observability with the infinite with the many realizations. So, the fourth, fifth and sixth problems is are basically about the output feedback, observer design and the unknown input observer design.

(Refer Slide Time: 02:01)

Observability under State-feedback

Problem 1
Consider the state equation



$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u; y = [1 \ 2] x$$

Check the observability of the system. Consider the state-feedback of the form $u = r - [k_1 \ k_2] x$. Is the system observable under this state feedback for all k_1 and k_2 ?

¹Example 8.1, Chen

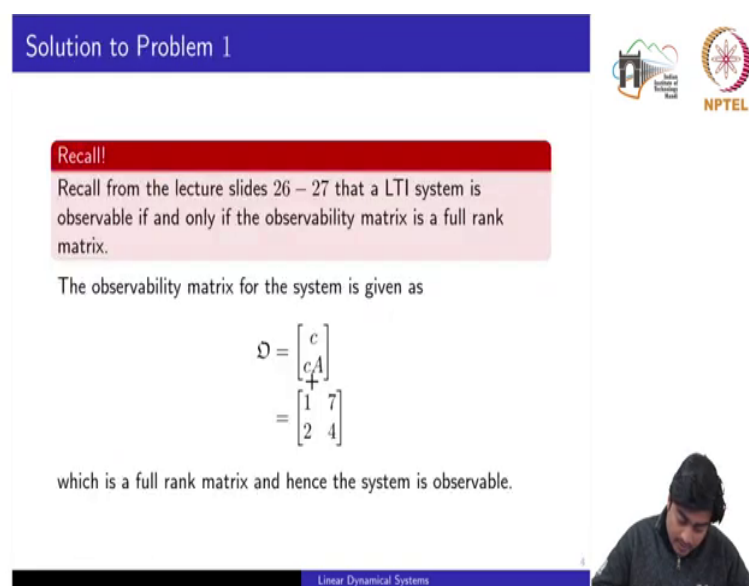
Linear Dynamical Systems

3



And the first problem, we are considering the state equation \dot{x} is equal to $Ax + Bu$ and y is equal to Cx , where A , B , C matrices are given here. So, this problem was taken from the example 8.1 of the one of the references mentioned on the course page. So, first we need to investigate about observability of the system. Second, we shall design the state feedback of this form which is $u = -Kx + r$; where, K is a vector composed of K_1 and K_2 . So, we need to investigate whether the system is absorbable under this state feedback for all K_1 and K_2 ok

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Solution to Problem 1

Recall!
Recall from the lecture slides 26 – 27 that a LTI system is observable if and only if the observability matrix is a full rank matrix.

The observability matrix for the system is given as

$$\mathcal{O} = \begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix}$$

which is a full rank matrix and hence the system is observable.

Linear Dynamical Systems

So, the first part is pretty much straight forward that you compute the observability matrix given the A and C matrices and finally, check the rank of that controllability, observability matrix. So, here it is a full rank matrix.

(Refer Slide Time: 03:08)

Solution to Problem 1

Now with $u = r - [k_1 \ k_2]x$, the closed-loop system matrix is given as

$$A_c = A - b[k_1 \ k_2] = \begin{bmatrix} 1 & 2 \\ 3 - k_1 & 1 - k_2 \end{bmatrix}$$

The observability matrix for the pair (A_c, c) is computed to be

$$\mathcal{O}_c = \begin{bmatrix} c \\ cA_c \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 - 2k_1 & 4 - 2k_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$$

which has the rank one if $k_1 = 3$ and $k_2 = 1$. Thus, the state feedback may not preserve observability.

Attention!

This example illustrates that the state feedback may not preserve observability.

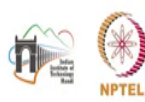


So, the system is observable, but the interesting part is to see whether the observability is invariant under this state feedback. So, let us put this controller into the state feedback system in that case the closed loop system matrix could be given by we denoted here by A subscript c is equal to A minus b k . So, substituting the corresponding values of A b and taking k_1 , k_2 as the parameters. So, we computed the observability of the closed loop system in terms of k_1 and k_2 .

Now, if you put k_1 is equal to 3 here and k_2 is equal to 1 here. So, from here we would get by putting k_1 is equal to 3, we will get 1 and putting k_2 is equal to 1, we will get 2. So, this is straight forward to see that the rank of this observability matrix is not 2, but 1; meaning to say that the system is not observable under the state feedback.

(Refer Slide Time: 04:20)

Duality for time-varying systems




Problem 2

Show that $(A(t), B(t))$ is controllable at t_0 if and only if $(-A^T(t), B^T(t))$ is observable at t_0 .

$$Q_c(A, B) \equiv Q_o(A^T, B^T)$$

¹Exercise 6.22, Chen

Linear Dynamical Systems



The second problem discusses about the duality. So, given the pair A and B which is time varying. So, we need to show that the controllability of this pair is equivalent to the observability of the pair which is minus A transpose and B transpose ok. Now, if you recall the results for the LTI system, if A and B are time variant then let us say the controllability of the pair A comma B is equivalent to the observability of the pair A transpose B transpose ok.

But here, we have a slight change here, we are having the negative sign. So, we need to investigate whether we can have some duality in for the time-varying case.

(Refer Slide Time: 05:18)

Solution to Problem 2

Consider the systems

$$\dot{x} = A(t)x + B(t)u, \quad \dot{x} = -A^T(t)x$$

$$y = B^T(t)x$$

$$X(t), \phi(t, t_0); \quad X_1(t), \phi_1(t, t_0)$$

Recall!


- Recall from lecture slide 16 (week 3) that the pair $(A(t), B(t))$ is controllable at $t = t_0$ if and only if the controllability Gramian


$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau) d\tau$$

 is of full rank for all t , where ϕ is the state transition matrix of the homogeneous system $\dot{x} = A(t)x$.
- Recall from the lecture slide 15 (week 7) that the pair $(-A^T(t), B^T(t))$ is observable if and only if the observability Gramian

$$W_O(t_0, t_1) = \int_{t_0}^{t_1} \phi_1(\tau, t_0)^T B(\tau) B^T(\tau) \phi_1(\tau, t_0) d\tau$$

 is of full rank for all t .





Linear Dynamical Systems

So, to solve this problem, we considered two systems. The first system is \dot{x} is equal to A x plus B t u . Another system is where we have replaced the A matrix by minus A transpose and the output matrix, we are taking as B transpose right. We have deliberately write these state space equation into these forms because we will investigate the controllability of this system and for investigating the controllability, we only need the A B pair.

Now, for observability, we only need the pair the state matrix and the output matrix. So, we have not consider any input term here. Corresponding to these two systems, we are defining two important parameters which are required to solve these time-varying systems. First is the fundamental matrix which we have denoted by x for the system 1 and the fundamental matrix denoted by x_1 for the system 2.

Similarly, this ϕ is the state transition matrix for the system 1 and ϕ_1 is the state transition matrix for the system 2. These are the two important parameters which we have discussed in the first week, when we had seen the solution of the time-varying systems. Now, we will recall first of all how we have defined the controllability and the observability. In terms of the Gramians. So, these concepts are basically related to the week 3 and week 7, where we have shown that the controllability of the pair A comma B is equivalent to the full rank of this Gramian which we have defined as the controllability Gramian.

So, here we are taking the directly the straight transition matrices this ϕ and the B transpose remain as it is of the input distribution matrix. For the observability, since these are the state and the output matrices, we have directly written the Gramian in terms of these pair which is supposed to be full rank, if we want to ensure the observability of this pair ok.

Now, the logical idea behind the proof is if we are able to show that W_c is actually equal to W_o and if we ensure that this is of full rank, it is necessary that this would also be a full rank ok. So, this W_c we have defined for the system 1 and W_o we have defined for the system 2. So, we only need to show the equivalence of this W_c and W_o .

Now, we pay attention to the on to the right hand side, the terms which are appearing in the at the middle of this integrant remains the same ok. The difference lies only in the state transition matrices. So, the next idea is if we are able to show the some kind of relationship between this ϕ or the state transition matrix of the system 1 and the and that of the system 2, then we can directly show the W_c is actually equal to W_o ok.

Now, with a clear look we only need to show that $\phi(t_1, \tau)$ should be equal to $\phi_1(\tau, t_1)$ transpose, similarly here ok. So, now, to show the equivalence between this W_c and W_o , we need to show the equivalence of the state transition matrix and the solution of the state transition matrix basically hinges upon the fundamental matrix ok.

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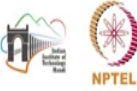
Solution to Problem 2

Recall!

Recall from the lecture slide 25 – 27 (week 1), the concept and properties of a fundamental matrix of a homogenous system $\dot{x} = A(t)x$.

Let $X(t)$ be a fundamental matrix of the system $\dot{x} = A(t)x$; then $\frac{d}{dt}X(t) = A(t)X(t)$. , $X^{-1}(t)X(t) = I$

Consider

$$\frac{d}{dt}(X^{-1}(t)X(t)) = \frac{d}{dt}I = 0$$
$$\Rightarrow \left(\frac{d}{dt}X^{-1}(t)\right)X(t) + X^{-1}(t)\frac{d}{dt}X(t) = 0$$
$$\Rightarrow \frac{d}{dt}X^{-1}(t) = -X^{-1}(t)\left(\frac{d}{dt}X(t)\right)X^{-1}(t) = -X^{-1}(t)A(t)$$


Linear Dynamical Systems

So, let us see if there is any relationship between the fundamental matrices of both the systems and then, we will come directly onto the state transition matrix. So, let $X(t)$ be the fundamental matrix of the system. We only need to consider the homogeneous system, then we know that the fundamental matrix actually satisfy this equation. So, we can write the derivative of capital X is equal to A times capital X .

Another property which we had also seen during week 1 that $X^{-1}(t)$ and $X(t)$ is actually equal to the identity and X is a non singular matrix. So, taking the derivatives both sides of this equation which is this one, we can or and further simplifying, you will end up to this relationship where it says the derivative of the inverse of the fundamental matrix is actually equal to minus inverse of X and A right.

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Solution to Problem 2

Thus,

$$\frac{d}{dt}X^{-1}(t) = -X^{-1}(t)A(t) \quad (1)$$



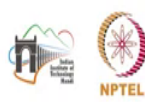
Let $X_1(t)$ be a fundamental matrix of $\dot{x} = -A^T(t)x$; then,

$$\frac{d}{dt}X_1(t) = -A^T(t)X_1(t)$$
$$\Rightarrow \frac{d}{dt}X_1^T(t) = -X_1^T(t)A(t) \text{ (transposing the matrices on both sides)}$$

Thus,

$$\frac{d}{dt}X_1^T(t) = -X_1^T(t)A(t) \quad (2)$$

Comparing (1) and (2), we have

$$\left(X_1^T(t)\right)^{-1} = \left(X^{-1}(t)\right)^{-1} \text{ and } \left(X_1^T(t)\right)^{-1} = X(t) \quad (3)$$


Linear Dynamical Systems

So, this is a key equation and we will put it as equation 1. Now, consider the second system whose fundamental matrix was X_1 and the state matrix was minus A transpose. Now, similarly what we had seen in the previous slide that X_1 again would satisfy this state equation and we would obtain directly this equation as equation number 2.

Now, if you compare both these equations 1 and 2, you would see that the A matrix is at the same place and now, X_1 transpose actually become equal to X inverse ok. Now, under this condition both these system would be the same. Similarly, if I take the inverse of the X_1 transpose, it would become equal to X . We just need to take the inverse both side ok. So, we get this equation. So, both the systems are related by the some relationship between their fundamental matrix.

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Solution to Problem 2

$$X_1^T(t) = X^{-1}(t) \text{ and } (X_1^T(t))^{-1} = X(t) \quad (3)$$


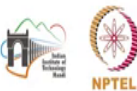
The state transition matrix $\phi(t, \tau)$ for the system $\dot{x} = A(t)x$ is given as

$$\phi(t, \tau) = X(t)X^{-1}(\tau)$$

The state transition matrix $\phi_1(t, \tau)$ for the system $\dot{x} = -A^T(t)x$ is given as

$$\begin{aligned} \phi_1(t, \tau) &= X_1(t)X_1^{-1}(\tau) \\ \Rightarrow \phi_1^T(t, \tau) &= (X_1^T(\tau))^{-1} X_1^T(t) = X(\tau)X^{-1}(t) \text{ (using (3))} \\ \Rightarrow \phi_1^T(t, \tau) &= \phi(\tau, t) \end{aligned}$$

Thus,

$$\phi_1^T(t, \tau) = \phi(\tau, t) \quad (4)$$



Linear Dynamical Systems 10

So, now let us see the state transition matrix which we had seen in the week 1 as well is given by the multiplication of the matrix x and its inverse at some time τ ok. This is how we define this $\phi(t, \tau)$. Now, for this system we can write the ϕ_1 as again this one. Now, here X_1 , we have replaced from the equation 3, what we have obtained in the previous slide.

Further simplifying it in terms of the only X , we obtain that the transpose of the state transition matrix of the second system is actually equal to the state transition matrix of the first system with the time reversal ok. $\phi_1^T(t, \tau) = \phi(\tau, t)$. So, this is again key equation which we will going to use it later to obtain the result. So, let us proceed again with this equation 4.

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Solution to Problem 2



$$\phi_1^T(t, \tau) = \phi(\tau, t) \quad (4)$$

where $\phi(t, \tau)$ is the state transition matrix of $\dot{x} = A(t)x$ and $\phi_1(t, \tau)$ is the state transition matrix of $\dot{x} = -A^T(t)x$

The pair $(A(t), B(t))$ is controllable at t_0 if and only if

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_1, \tau) B(\tau) B^T(\tau) \phi^T(t_1, \tau) d\tau$$


is non-singular. Using $\phi(t_1, \tau) = \phi(t_1, t_0) \phi(t_0, \tau)$, we write W_C as:

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \underbrace{\phi(t_1, t_0)}_{\eta} \underbrace{\phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau)}_{\eta^+} d\tau \phi^T(t_1, t_0)$$

Because $\phi(t, t_0)$ is non-singular for all t , we conclude that $(A(t), B(t))$ is controllable if and only if the matrix:

$$W_C \triangleq \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau) d\tau \quad (5)$$

is non-singular.



Linear Dynamical Systems

So, we had seen that the pair A, B is controllable, if this matrix is a full rank. So, again we are using one of the properties of the state transition matrix that if I take some time t naught in between t 1 and tau, then I can write this phi t 1 comma tau as the multiplication of 2 phi's as t 1 comma t naught and t naught comma tau right. Now, substituting this into this W c and simplifying we can take a pre multiplying factor phi t 1 comma t naught as the common part and the post multiplying factor as phi transpose t 1 comma t naught.

Now, from the property of the state transition matrix that it should be non-singular. So, for W c to have a full rank this term should be positive definite or should be in fact, definite ok; should only be definite. Definite in the sense that this matrix should be non singular which we have defined A as eta, this common part eta ok.

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Solution to Problem 2



The pair $(-A^T(t), B^T(t))$ is observable if and only if

$$W_O = \int_{t_0}^{t_1} \phi_1^T(\tau, t_0) B(\tau) B^T(\tau) \phi_1(\tau, t_0) d\tau$$

is non-singular where $\phi_1(t, \tau)$ is the state transition matrix of $\dot{x} = -A^T(t)x$.
Using $\phi_1^T(\tau, t_0) = \phi(t_0, \tau)$, we write W_O as

$$W_C(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau) d\tau$$

which is identical to η .

$$W_C \triangleq \int_{t_0}^{t_1} \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau) d\tau \quad (5)$$

This establishes that $(A(t), B(t))$ is controllable if and only if $(-A^T(t), B(t))$ is observable.



Now, coming onto the observability Gramian of this pair and using the relationship which we had obtained in the equation 4, I can replace all these phi 1 by this phi with the time reversal of course. So, directly by replacing the phi 1 by all phi, we obtain this W which is nothing but equal to eta actually. So, here we had shown that the observability matrix of the second system is actually equal to the controllability Gramian of the first system ok. So, this establish the duality which you would realize that which is slightly different from the LTIs.

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The slide is titled "Minimal Realisations" in a blue header. It contains a "Problem 3" section with a transfer function $H(s) = \frac{s+1}{s^2+2}$ and three multiple-choice options: an unobservable realization, an uncontrollable realization, and a minimal realization. Below this is an "Attention!" section in a green box stating that controllable canonical form realizations can be obtained using the `canon` command in MATLAB. The slide also features logos for "The National Institute of Education" and "NPTEL" in the top right, and a small video inset of a man in the bottom right. At the bottom, it includes a reference to "Exercise 5.14, Antsaklis" and the page number "13".

Minimal Realisations

Problem 3

For the transfer function $H(s) = \frac{s+1}{s^2+2}$, find

- an unobservable realization
- an uncontrollable realization
- a minimal realization

Attention!

The controllable canonical form realisations for this problem shall be obtained using the `canon` command in MATLAB.

¹Exercise 5.14, Antsaklis

Linear Dynamical Systems 13

The third problem is about the minimal realization. So, we are given some transfer function which is a single input single output transfer function which is s plus 1 over s square plus 2. So, here we need to find three realizations; first one should be unobservable, but controllable; second uncontrollable, but observable and third a minimal realization which is equivalent to saying that the that realization is both controllable and observable.

So, here we will going to use directly the this MATLAB command to obtain the realization which you can also compute by going through the procedure defined in the slides.

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Solution to Problem 3

Recall!



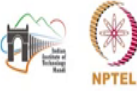
Recall from the lecture slides 21 (week 3), and lecture slide 28 (week 7) that a LTI system $\dot{x} = Ax + Bu, y = Cx$ is controllable if and only if the controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

has full rank. The system is observable if and only if the observability matrix

$$\mathcal{D} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix}$$

has full rank.



Linear Dynamical Systems 38

So, the controllability or the full rank of the controllability matrix and the observability matrix ensures that the system is controllable and observable both.

(Refer Slide Time: 16:18)

Solution to Problem 2

Let

$$H(s) = \frac{s+1}{s^2+2} = \frac{(s+1)(s-1)}{(s^2+2)(s-1)} = \frac{s^2-1}{s^3-s^2+2s-2}$$

Then, constructing the controllable canonical realisation, we obtain $\dot{x} = Ax + Bu, y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, c = [-1 \quad 0 \quad 1]$$

This realisation is controllable, because the controllability matrix



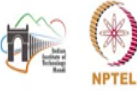
$$\text{rank} \mathcal{C} = \text{rank} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 3$$

but unobservable realisation because

$$\text{rank} \mathcal{O} = \text{rank} \begin{bmatrix} -1 & 0 & 1 \\ 2 & -3 & 1 \\ 2 & 0 & -2 \end{bmatrix} = 2$$

15

Linear Dynamical Systems



So, to obtain the unobservable realization what we have done here. So, in this transfer function we have added a one polynomial factor both in the numerator and in the denominator ok.

So, opening this polynomial, we would obtain this one. Now, for this transfer function, we obtain directly the controllable canonical realization in form of this A, b, c matrices and if you compute the controllability of this matrix it is a full rank, but it is unobservable because the rank of the observability matrix of A of the pair A comma c is only 2 ok.

So, this realization is an unobservable realization. Why? One insight of this one is that because we have added a common vector both in the numerator and the polynomial which we are

seeing only the external signals that is to say input and output signal won't be the transfer function would not be able to capture it.

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Solution to Problem 3

• The dual to the system in part (a), is $\dot{x} = A_1x + b_1u; y = c_1x$ where $A_1 = A^T$, $b_1 = c^T$ and $c_1 = d^T$ i.e.


$$A_1 = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } c_1 = [0 \ 0 \ 1]$$


This realization is observable because

$$\text{rank} \mathcal{O}_1 = \text{rank} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -3 \end{bmatrix} = 3$$

but uncontrollable because

$$\text{rank} \mathcal{C} = \text{rank} \begin{bmatrix} -1 & 2 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -2 \end{bmatrix} = 2$$





Linear Dynamical Systems

The second part, we could solve straight forwardly by taking the dual form of the system A that is replacing the A matrix by its A transpose, b matrix by the transpose of the output matrix of the first system of the first equation and the output matrix by the transpose of the input matrix of the first system. So, these are the matrices what we have obtained. Now, if we compute the observability or the rank of the observability, then it shows that the that this realization is observable. But again it is not controllable and this is what implicitly the concept of duality says.

(Refer Slide Time: 18:20)

Solution to Problem 3

Recall!

Recall from the lecture slide 44 (week 7) that a realisation is minimal if and only if it is both controllable and observable.

- The transfer function $H(s) = \frac{s+1}{s^2+2}$ is irreducible in the sense that there are no common factors between the numerator and denominator polynomials.

Consider the (controllable canonical) realisation of this transfer function $\dot{x} = Ax + bu, y = cx$ with $A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $c = [1 \quad 1]$. The controllability matrix $\mathcal{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the observability matrix $\mathcal{O} = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$ both have rank 2. Thus, this realization is both controllable and observable and hence minimal.

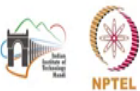


Now, lastly is the realization which is supposed to be minimal and we had seen detailed proof that the that realization should be both controllable and observable. Now, one of the inherent meaning of this is that both the numerator polynomial and the denominator polynomial are co-prime, that is there are no common factors in those polynomials ok.

So, now directly writing the or computing the A, b, c matrices, we obtain these ones for which you can compute the rank of the controllability matrix and the observability matrix which is supposed to be 2. So, this realization A, b, c is the minimal realization and this minimal realization basically in terms of the dimension of the vectors the state vector.

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Stabilization and Output feedback



Problem 4
An undamped linear pendulum is modeled by the system


$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u\end{aligned}$$

Check whether the system can be stabilised by static output feedback when

- $y_1 = c_1x = [1 \ 0]x = x_1,$
- $y_2 = c_2x = [0 \ 1]x = x_2.$

²Example 5.3, Terrell

Linear Dynamical Systems 18





In the next problem, we want to design a feedback controller, but here there is a slight difference. So, let us see first. So, given this simple system which is a second order system with 2 states and 1 input which is acting on only on to the second state dynamics. So, here we want to check whether the system can be stabilized by aesthetic output feedback under these two scenarios. So, here we are considering a two different scenarios in which our y_1 is x_1 . So, we are taking x_1 as a direct measurement and in the second case, we are only measuring x_2 into the output.

So, if you recall the concepts of the controllability, we had discussed earlier I can directly obtain the A, b, c matrices or these two equations. Now, if you check the controllability of the pair A comma b , it would be controllable ok. It would be controllable. Now, what we want to see that whatever the concept, we had studied about the controllability is there some relationship between the controller state feedback or the output feedback controller design.

Because do not forget that the concept of controllability, we had discussed is only about the state. So, basically, we should refer it as the state controllability ok.

(Refer Slide Time: 21:07)

Solution to Problem 4

The system can be written as $\dot{x} = Ax + bu$ where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$


With the static output feedback $u = ky_1 = kc_1x = k \begin{bmatrix} 1 & 0 \end{bmatrix} x = kx_1$, the closed loop system becomes

$$\dot{x} = (A + bkc_1)x = (A + bk \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})x = \begin{bmatrix} 0 & 1 \\ k-1 & 0 \end{bmatrix}x$$

The eigenvalues are the roots of $\lambda^2 = k - 1$. +

- If $k < 1$, then we have a conjugate pair of imaginary eigenvalues.
- If $k = 1$ then there is a double zero eigenvalue.
- If $k > 1$, then the eigenvalues are positive.

Therefore, the system cannot be stabilised by the output feedback $u = kx_1$.



Linear Dynamical Systems

So, let us see. So, I can. So, the A, b matrix for the system is given by these matrices and we want to apply this feedback which is given by u is equal to k y 1 and y 1 is nothing but c 1 x and c 1 is 1,0; where, I am taking actually the state 1 the first state. So, if I see the actual controller is this now u is equal to k x 1 meaning to say I am feeding only the first state and want to see whether the system can be stabilized by feeding only one state or only the first state.

So, if we check the state matrix of the closed loop system, I will obtain this as A plus b k c 1 and substituting the value of A, b k and c 1, basically we want to analyse for k. So, after substituting all these matrices, we obtain the state matrix as this one. So, we can analyse that

for what values of k the system can be analysed by computing the eigenvalues of this matrix. So, which basically aims to solve this quadratic equations for the lambdas.

Now, from this equation, you would see if k becomes equal to 1 right; k becomes equal to one then the eigenvalues would be a conjugate pair of imaginary eigenvalues, meaning to say that the system would be oscillatory. Now, if k is a becomes equal to 1, then we would have a double integrator meaning to say that the system is again unstable and if k is greater than 1, then all the eigenvalues would be positive that is on the right hand side.

So, for all the values of k the this or using this control law as u is equal to $k \times 1$ would never give you a stable system that is so, that the system cannot be stabilized by this output feedback.


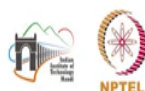
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Solution to Problem 4

If the feedback is of the form
 $u = ky_2 = kc_2x = k \begin{bmatrix} 0 & 1 \end{bmatrix} x = kx_2$, then the closed-loop system matrix is given as

$$A + bkc_2 = A + bk \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix}$$

The eigenvalues of this matrix are $\frac{1}{2} \left(k \pm \sqrt{(k-2)(k+2)} \right)$. Any choice of $k < 0$ will make the closed-loop system matrix Hurwitz.



Linear Dynamical Systems

Now, consider the second case, where instead of feeding back the state 1 or the first state, we want to feedback the second state ok. So, let us see. So, now, the control law would become u is equal to $k x_2$ finally. Because the c matrix or the c vector has been changed to $0, 1$. So, here we should have x ok. So, for this case the closed loop matrix would be given by this one; $A + b k c^T$ and in terms of k , this would be a parameterize a state matrix. Now, computing the eigenvalues, we obtain this one and you would see that any choice of negative values of k , these eigenvalues would be onto the left hand side ok.

So, this means that if I feedback both the states, if I feedback both the states, so I know from the controllability property of the pair A comma b , the system can be stabilized ok. But now if I see that instead of feeding back both the states, if I feedback the first state the system can never be stabilized. But by feeding the second state, we see that the system can be stabilized for the negative values of k right. So, the concept of the state controllability what we had discuss is basically different from the what we are seeing in this numerical.

(Refer Slide Time: 25:12)

Observer Design for armature controlled DC servomotor

Problem 5

The state space description for a armature controlled DC servomotor assumes the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u$$


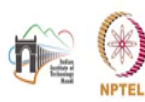
$y = x_1$

with the definition of states and input are

x_1	Shaft Position
x_2	Angular Velocity
x_3	Armature Current
u	Armature Voltage

Design a full-order state observer with eigenvalues at -1 , -2 and -3 .

Linear Dynamical Systems



In the problem 5, we have some armature controlled DC servomotor, whose a, b, c matrices are given by these values ok. So, here your c would be a vector containing only first non zero element, the other two elements would be 0 of the c matrix. So, the definitions of the station inputs are given here. We want to design a full order state observer with eigenvalues place placed at minus 1, minus 2 and minus 3 ok.

So, the idea of designing the observer originated from when we want to close, when we want to design a state feedback controller and we do not have the direct measurement of all the states. So, instead of using this state x_1 , we want to use the complete state. So, this is what we had seen in the previous example that somehow if we are able to design an observer for estimating both the states, then we know from the controllability property of the pair, the system can be stabilized.

So, here again we have a direct measurement of only one of the states, but not the all of the states. So, we want to design a full order state observer by placing the eigenvalues onto these values.

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Solution to Problem 5 - Lyapunov equation method

Recall!

Recall from the lecture slide 14 (week 8), the method of Lyapunov equation method for observer design of LTI systems.


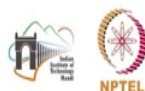
1 Selection of F matrix:

$$F = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

2 Selection of l vector: The l vector is chosen such that the pair (F, l) is controllable.

$$l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} +$$

Linear Dynamical Systems



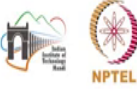
So, here we would. So, in the lectures, we had discussed two methods first is the eigenvalue assignment method which is basically the give you the Lemberger observer. Second is the based on the Lyapunov equation. So, here, we will go through the steps of computing the observer matrices by using that method and finally, we will see some simulation results.

So, the first step in this method is we want to select an arbitrary F matrix, whose eigenvalues are actually at located at the desired values. Now, the second step we select arbitrary l vector

such that the pair F comma l is controllable. So, we have selected this l vector and this pair F , l is also controllable.

(Refer Slide Time: 27:35)

Solution to Problem 5 - Lyapunov equation method



4 Obtaining the observer system requires solving the following equation

$$TA - FT = lc$$


Solving this equation using *lyap* command in MATLAB, the matrix T is computed to be

$$T = \begin{bmatrix} 1.0 & 0.75 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.3333 & 0.1667 & 0.3333 \end{bmatrix}$$

+

23

Linear Dynamical Systems



Now, the third step is computing the t matrix which is obtained by solving this Lyapunov equation and we have used the Lyap command, so this command in MATLAB to obtain this t matrix.

(Refer Slide Time: 27:54)

Solution to Problem 5 - Lyapunov equation method


The system which generates the state-estimate is given as

$$\dot{z} = Fz + Tbu + ly$$
$$\Rightarrow \dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0.1 \\ 0.6667 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} y$$

and the state estimate is

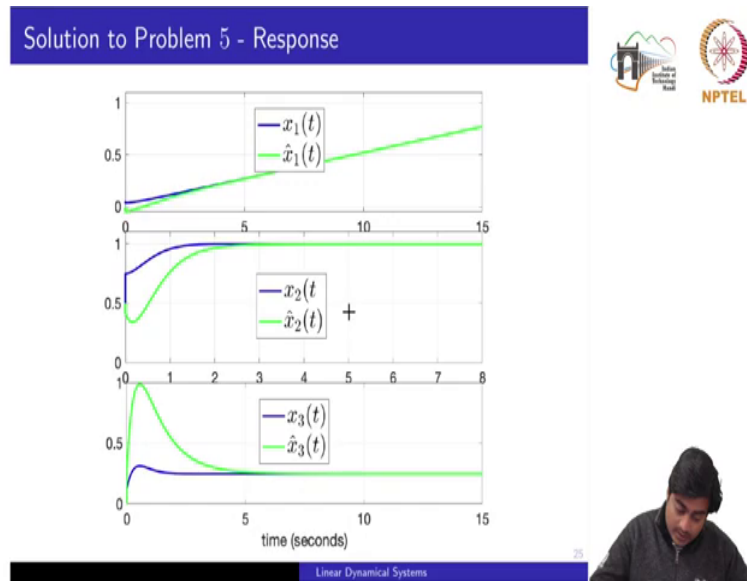
$$\hat{x}(t) = T^{-1}z(t) = \begin{bmatrix} 2 & -4 & 3 \\ 0 & 4 & -6 \\ -2 & 2 & 3 \end{bmatrix} z(t) +$$

Linear Dynamical Systems



Now, the observer if you recall was given by this state equations \dot{z} is equal to Fz plus Tbu plus ly and the state estimate is given by the inverse of that T matrix times z of t . So, substituting the matrix, the computed matrices by using the steps we had seen earlier. Now, we can go directly to the simulations, if I place this observer to estimate the actual sales.

(Refer Slide Time: 28:27)



So, here we see that the blue curves are the actual states and the states which are drawn in the green colour are the estimates. So, only during the transition the transient periods the state values do not match; but when it reaches to the steady state both the states overlap; meaning to say that whatever the observer, we have designed is actually able to estimate all the states.

(Refer Slide Time: 29:00)

Unknown Input Observer (UIO) Design

Problem 6

Consider the system




$$\dot{x} = Ax + bu + ed, y = Cx$$

with $A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $e = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Here, $d(t)$ is the disturbance $d(t) = \sin(10t)$ which is not measurable.

Design an observer to estimate the states of the systems using the Luenberger observer design technique. Subsequently, design the unknown input observer. Compare the responses.

¹Hou and Müller, IEEE TAC, 37(6), 871-875, 1992

Linear Dynamical Systems 26



So, this is the last problem, where we want to design the unknown input observer. So, we are considering a system where now we have added a disturbance term with its distribution matrix e . So, this A , b , e , c values are given here. So, here d is the disturbance which is sinusoidal term and it is not in this variable is not directly measurable ok.

So, here so, this problem was also this numerical problem was given by one of the paper appear in the transitions some automatic control in 92. Here, we want to design two observers. So, in the first part, we will design the Luenberger observer which we have designed using the eigenvalue assignment technique and the second, we will design the unknown input observer.


So, if you recall in the Luenberger observer design technique, we require or in fact, in both the observers, we would be taking only the input and output variable as the input signals to the

observer because we do not have any information of the or we are not supplying any information of the disturbance variable to the observer and finally, we will compare both the responses obtain from both observers ok.

(Refer Slide Time: 30:30)

Solution to Problem 6- Luenberger Observer Design

Recall!



Recall from the lecture slide 9 – 10 (week 8), the eigenvalue placement based method of observer design for LTI systems.

The dynamical equation for the observer is given as:


$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}\end{aligned}$$

where \hat{x} is the state estimate, \hat{y} is the output estimate and L is the gain vector to be designed.

The error dynamics of the observer is given as: $\dot{e} = (A - LC)e$. The matrix L is computed using the MATLAB command `place`

$$L = \begin{bmatrix} 3.3608 & -1.6131 \\ 2.6793 & -4.3177 \\ -1.3810 & 3.6392 \end{bmatrix}$$

The initial conditions of the observer are taken as $\hat{x}(0) = [0 \ 0 \ 0]^T$.

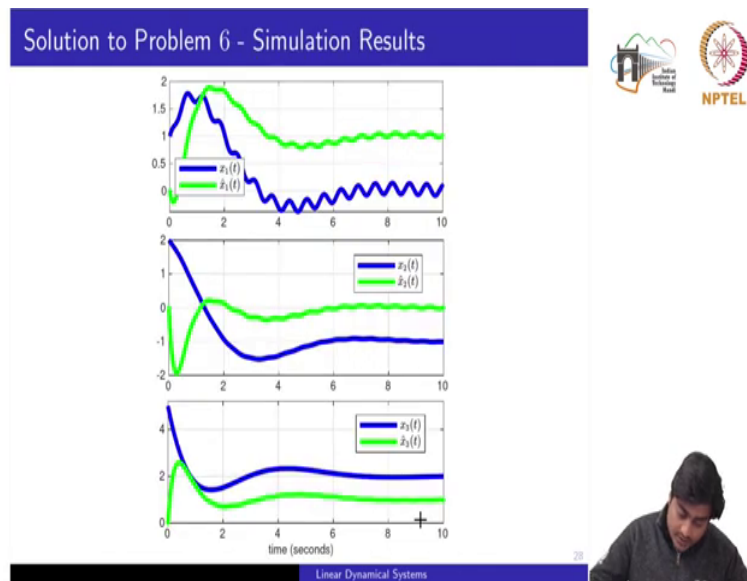


Linear Dynamical Systems

So, let us see. So, this was the observer equation which we had discussed on slide number 9 to 10, where \hat{x} is the state estimate \hat{y} is the output estimate and L is the gain vector which is supposed to be designed. So, if you recall that the error dynamics are given by \dot{e} is equal to a minus L times C e ; where, we can compute directly this L matrix if the observability of this pair A comma c is satisfied right.

So, we have use the MATLAB command `place` to place the eigenvalues onto the left hand side and we have computed this L matrix. Now, for simulations, we have taken the initial conditions at the region.


(Refer Slide Time: 31:26)



So, if we see the response of the system and the observer, we say we see that not only at the transient period, but also at the steady state period the states that state estimation error do not go to 0 and it is because that we have not supplied the information of the disturbance to the observer right.

(Refer Slide Time: 31:55)

Solution to Problem 6 - UIO Design



Recall!
Recall from the lecture slide 44, the procedure for unknown input observer design.


+

- Rank Condition: $\text{rank}(C) = \text{rank}(CE) = 1$ and hence the rank criterion is satisfied and the UIO exists
- The matrices H , T and A_1 are computed as:

$$H = E[(CE)^T CE]^{-1} (CE)^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$T = I - HC = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_1 = TA = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

29

Linear Dynamical Systems




Now, we shall see that if this problem can be overcome by designing the unknown input observer. So, the detail designing steps were given on to the lecture slide number 44. So, here we will quickly go through all the steps to compute the relevant matrices and then finally, we will see the simulation results.

So, the first condition which is was basically the existence of the UIO and from there we give a special solution of this H matrix. So, we check the rank condition of C and CE; it is equal to 1. So, a UIO exists and this H is a special solution which we can we have computed with this matrix. So, the rest of the matrices were a straight forward to compute the T matrix and the A1 matrix, once we have computed the H matrix from the first step.

(Refer Slide Time: 33:00)

Solution to Problem 6 - UIO Design




• $\text{rank} \begin{bmatrix} C \\ CA_1 \\ CA_1^2 \end{bmatrix} = 3$ and hence the (C, A_1) pair is observable.

Therefore, the steps 4 – 8 are not needed. The matrix K_1 is computed so as to place the eigenvalues at $\{-1, -2, -3\}$ (using the *place* command in MATLAB) to be

$$K_1 = \begin{bmatrix} 1 & 2 \\ -1 & -6 \\ 0 & 4 \end{bmatrix} +$$

30

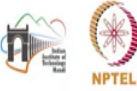
Linear Dynamical Systems



Now, the third step were in the third step, we need to investigate the observability of the pair X and A_1 and A_1 , we had computed from this second step. So, here we see that this pair is observable. So, we do not actually need to go through the steps 4 to 8. Because in the 4 to 8 steps, we basically discuss that if this pair is detectable, then how you can compute this matrix K_1 ok. But here since the pair is observable, we can directly compute the matrix K_1 using the MATLAB command *place* ok. So, here we have place the eigenvalues at minus 1, minus 2 and minus 3 and we have obtained this K_1 matrix.

(Refer Slide Time: 33:49)

Solution to Problem 6 - UIO Design




The matrices F and K are computed as:

$$F = A_1 - K_1 C = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 6 \\ 0 & -1 & -5 \end{bmatrix}$$
$$K = K_1 + FH = \begin{bmatrix} 0 & 2 \\ -1 & -6 \\ 0 & 4 \end{bmatrix}$$

31

Linear Dynamical Systems



So, from the K_1 matrix, we have obtained F matrix which is supposed to be a Hermitian matrix. So, you can compute the eigenvalues and from K_1 and H and F , we have finally compute the K matrix ok.

(Refer Slide Time: 34:05)

Solution to Problem 6 - UIO Design



The unknown input observer is then given as:

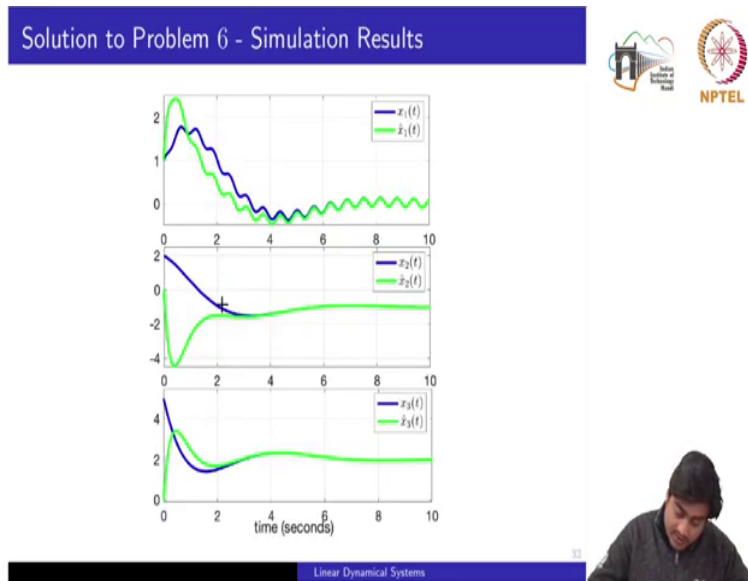
$$\begin{aligned}\dot{z} &= Fz + TBu + Ky \\ \hat{x}(t) &= z(t) + Hy(t)\end{aligned}$$

The initial conditions of the observer are taken as

$$z(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} +$$

Now, putting all those matrices into the observer, F should be a stable matrix, T we have already computed, B is the input matrix, u the input and k is the an output injected term and in $\hat{x} = z + Hy$ ok. Again here, we are taking the initial conditions of the observer at 0 0 0 ; meaning to say so the state of the observer is basically a z and the output of the observer is the \hat{x} ok.

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Now, if we see the simulation results that although we have not supplied the information of the disturbance to the observer still the observer is able to estimate the actual value of the state ok. So, this is the significance of the unknown input observer because we have eliminated the effect of that unknown disturbance onto the observer which was pretty much visible, when we had discussed the Lemberger observer ok.