Linear Dynamical Systems
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Week – 08
Observer Design and Output Feedback
Lecture - 42
Observer Design and Output Feedback

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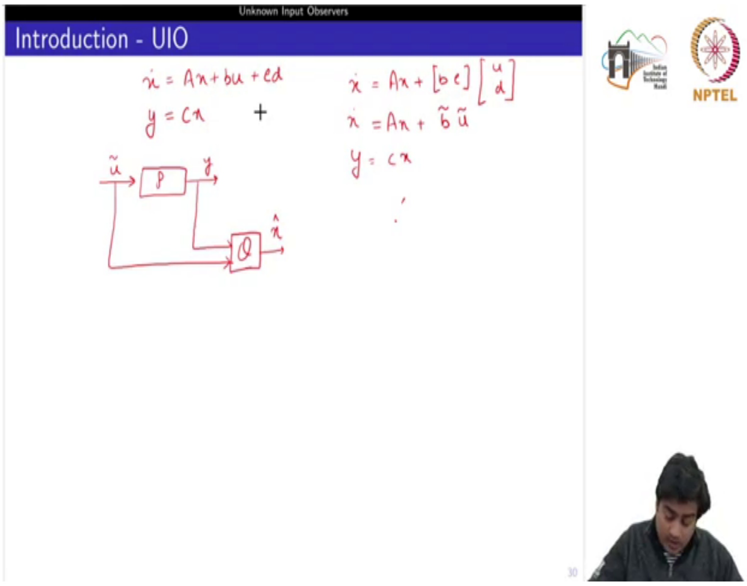

So, now we shall be taking the last topic of this week 8 where we will be focusing on the unknown input observer. So, first we shall recall the type of observer. We had discussed previously and then we shall highlight the significance of this unknown input observer. So, for example, given the LTI system let us say x dot is equal to A x plus b u, y is equal to C x, C x right.

So, say example we have this one, now in the observers what we had seen so far. We did not have any additional term let us say i ed a disturbance term with its distribution matrix e. So, we had focused on to the issue of the observability that the we discuss the observability of the pair A comma C ok.

So for this disturbance, let us say we have this plant with input and output y. So, by taking these two measurements, we have design an observer which gives us the estimate of the state right. Now if we have this disturbance term here in the plant itself, then what would happen to the observer. So, we can parameterized this one as let us say x dot is equal to A x. So, this I can write as A x plus, let us write this as b tilde and this becomes my u tilde. My output term remains as it is.

So, now instead of giving only this u. I have to supply this u tilde if I want to design an observer for this system right. So, I would change here as u tilde and y would remain as it is and this would give me the estimate of that state x right. Now here you would notice that in this u tilde, we have two input variables. One is the what we have been using so far is the control output u and d which is an external disturbance. So, if I want to design an observer which can give me the estimate of the state x then I also need to supply the information of the disturbance to my observer.

But, in many cases as we had seen during the controller designing week that many times we do not have the information of the disturbance. So, if we do not have this information of disturbance, how we can still design an observer which can give us the estimate of the state. So, this topic talks in this topic we shall see how we can solve this problem.

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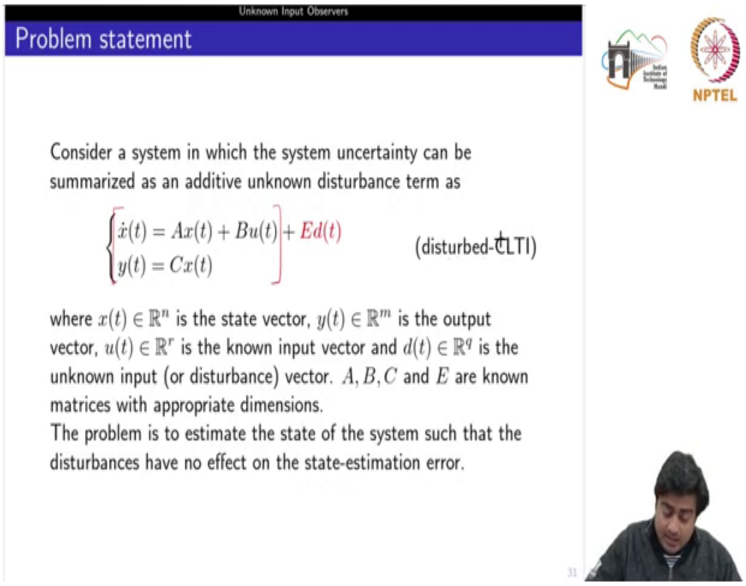

So, defining the problem statement completely. So, this part we have already seen without this additional disturbance term. So, now, we would be considering a system which we are calling as the disturbed continues time LTI system with A, B, C is the matrices and E is the distribution matrix of the disturbance ok. Where x y u are the state output and input vector and d is the unknown input or what we I call the disturbance of which we do not have the information. But we also, but we have the information of the matrices A, B, C, E which known to us having appropriate dimensions.

So, here the problem we want to solve is to have the estimate of the state of the system such that the disturbances have no effect on the state estimation error. So, this formulation, what you what we have labeled as disturbed CLTI is pretty much a generic formulation.

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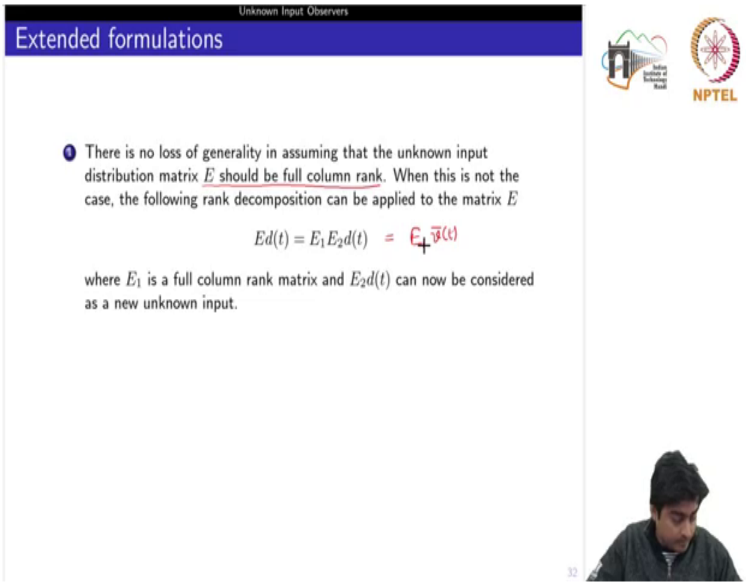

So, now one point which should keep attention that here the distribution matrix E we have assumed that it is a full column rank matrix. Now, if E is not a full column rank, then using this rank decomposition what we had discussed earlier. I can decompose this E matrix as the multiplication of two matrices E 1 and E 2; where E 1 is now full column rank and E 2 might not be. So, in my new formulation I can write this as E 1 with some variable v bar. Where v bar becomes my E 2 d t and this v bar is now my unknown vector.

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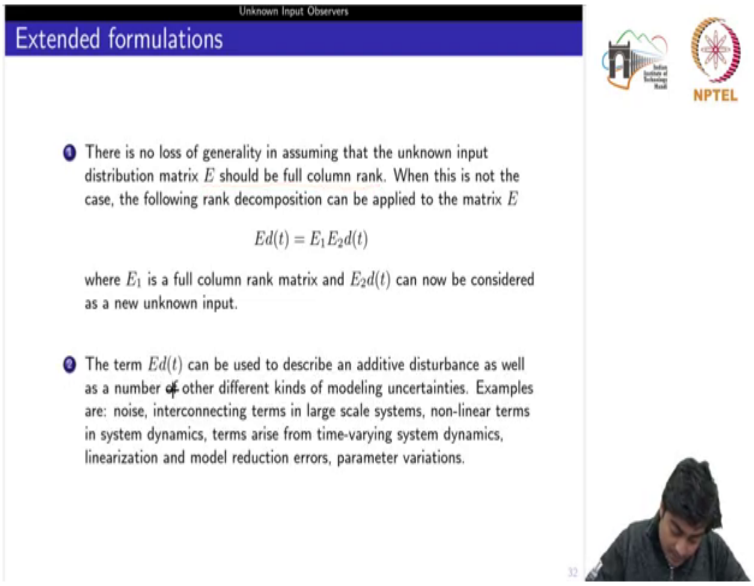

Now second which is pretty much important that the term E of E times d t can also be used to describe an additive disturbance as well as a number of other different kinds of modeling uncertainties. So, one of the examples we would take this into the when we will come on to the tutorial that how you can model some uncertainties and some disturbances which are pretty much non-linear in this L T I system in terms of u and y.

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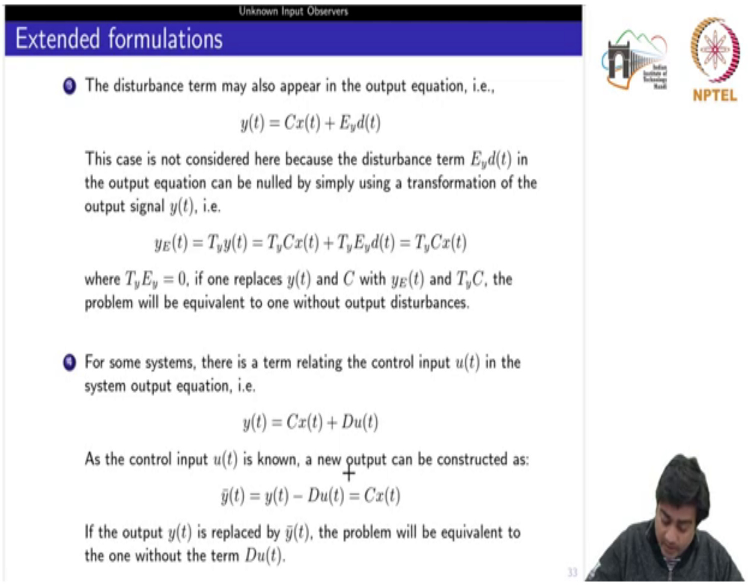

Third; so, if you recall in the output equation we have only considered that y is equal to C x t, but there might be a disturbance term in the output equation with its distribution matrices E y. So, to get rid of this issue; what we can do that we can design a transformation matrix let us say T y. In the same that if I multiply or pre multiply this whole equation by the matrix T y such that my T y E y becomes equal to 0 and at the same time my T y C is not is equal to 0.

So, I can write my or instead of taking y as an output variable I am now taking as y E which is equal to T y C x t and this would become my new C matrix let us say C tilde ok. Another generalization; now instead of having the disturbance term into the output equation we could also have the input term right. In the output equation is plus D u t in addition to what we had seen earlier. So, since this u is already measurable we can take it on the left hand side and as defined new output vector as y bar which becomes equal to y t minus D u t and finally, it would again become equal to C times x t ok.

So, all these four formulations we can use the standard formulation what we had introduced.

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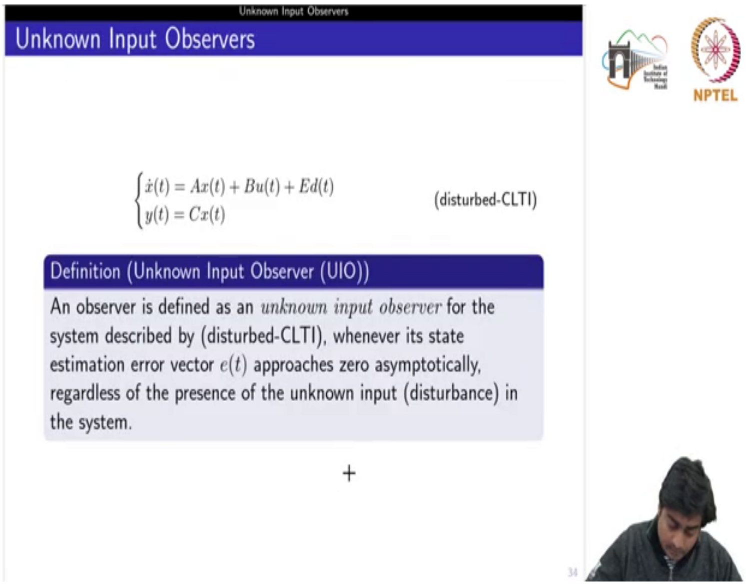

So, now coming on to the definition of the unknown input observer, consider this disturbed CLTI. So, we define an observer as an unknown input observer whenever its state estimation error e of t approaches zero as asymptotically regardless of the presence of the unknown input disturbance in the system.

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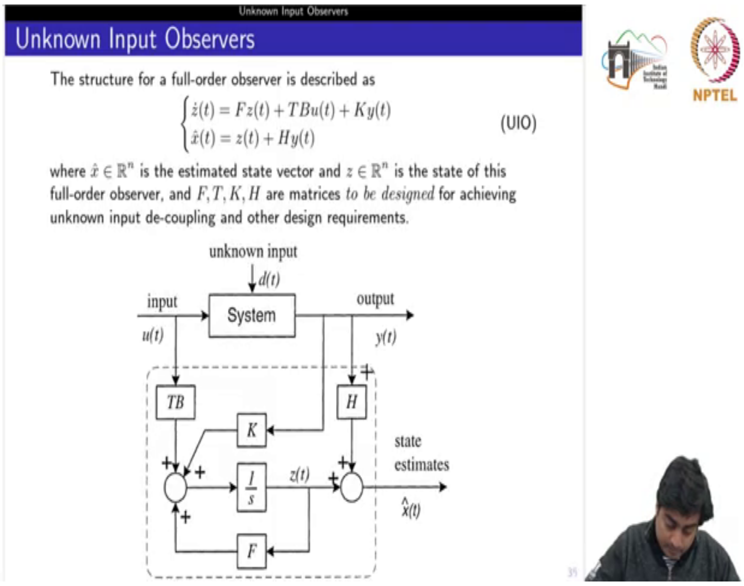

So, let us see so, the structure for a full order observer; we described as a z dot is equal to F z plus T B u plus K times y of t. Where x hat the estimation of the state is given by the summation of the z t plus H times y t ok. Where, z is the internal state of the observer and we call it the full observer because the dimension of the state of the observer and the plant was basically remains the same.

So, where here x hat is the estimated state vector z is the state of the full order observer. Here we now need to design the matrices F T K and H, because the B matrices is already know to us ok. So, these are the matrices to be designed for achieving the unknown input decoupling and other design requirements if we have any. So, this is the block diagram description of this observer.

Where you see that now the disturbance term which we are calling as the unknown input is only acting onto the system. But we are not taking any information of this input to our observer. We are only taking the information of the input or let us say the control output and the output of the plant.

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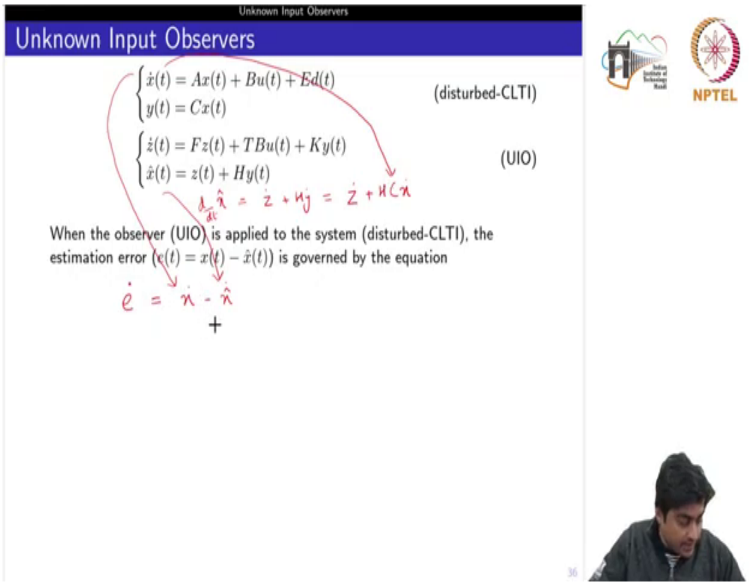

So, let us see how we can compute the matrices F T K and H. So, this was the L T I plant and this is the observer we have proposed. Now in a similar way we can see whether the state estimation error actually becomes equal to 0. So, in the earlier observer design we have defined the estimation error as the difference between the actual state and the estimated state.

So, the next step is to take the derivative of this state as x dot minus x hat dot ok. Now here I can substitute this x dot to here and after taking the derivative of this output equation of the observer, I will substitute it here. So, here in the output equation you will see that the additional term I would get is d, x hat is equal to z dot plus H y dot ok. And y is equal to C x. So, I can replace it by C x dot then again I can substitute x dot here ok. So, computing the error dynamics and simplifying.

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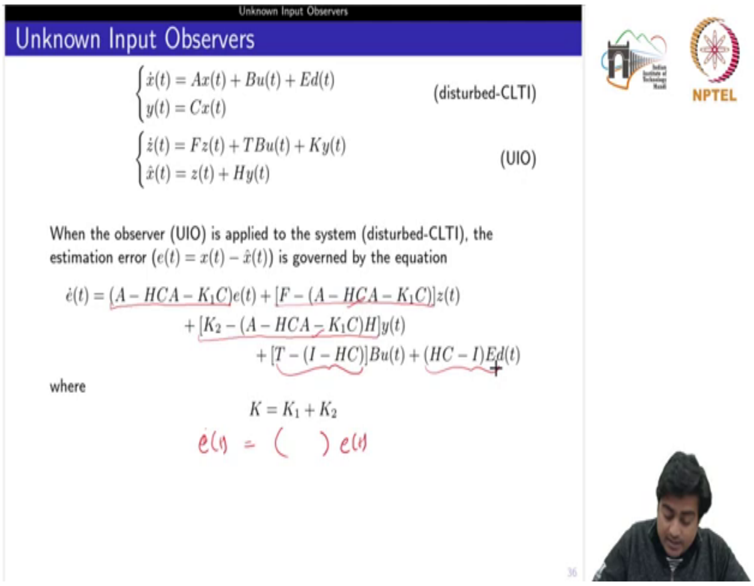

Finally, I would obtain these terms where we have this e dot is equal to some combination of this matrices times e of t, which is basically the state of the error dynamics. This matrix times z of t, which is the state of the observer. Here we have introduced K as equal to K 1 plus K 2 into this equation and finally, I can write this as some matrix times y of t some matrix time u of t and some matrix times d of t.

Now this equation is pretty much important because what do we want that the error dynamic should be equal to 0 under the presence of the external disturbance d. So, if we make all these matrices; this one, this one, this one and this one. If we make all these matrices somehow equal to 0, then we would be having only this e dot of t is equal to some matrix times e of t ok. Meaning to say that the error dynamics now become completely or let us say the error dynamics are now completely decoupled from the external disturbances.

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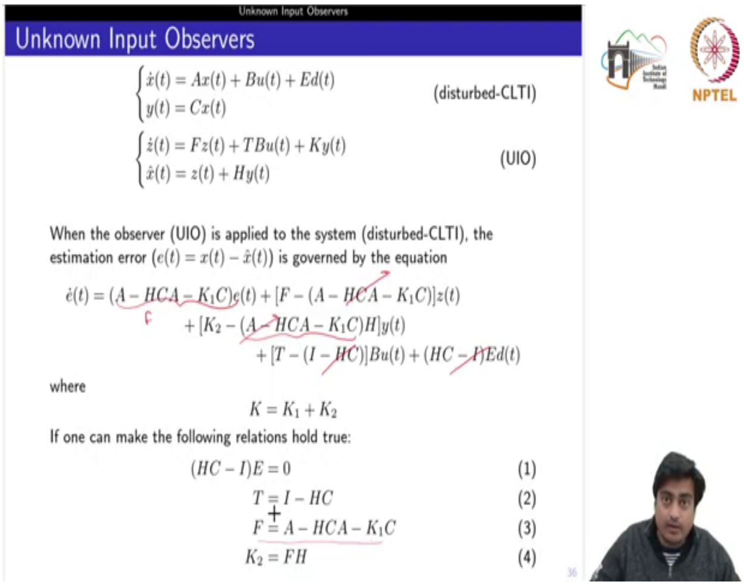

So, what do we need to solve? We need to solve these four equations which basically say the same thing that H C minus I times E should be equal to 0. Now T should become equal to I minus HC. So, that my this term would go to 0, here we are substituting the state matrix of the error dynamics is some F matrix right. Meaning to say so, if or let us say my F if my F becomes equal to this would also become equal to 0. Now see this part; this is my F. So, if my K 2 becomes equal to F into H this part would also goes to 0.

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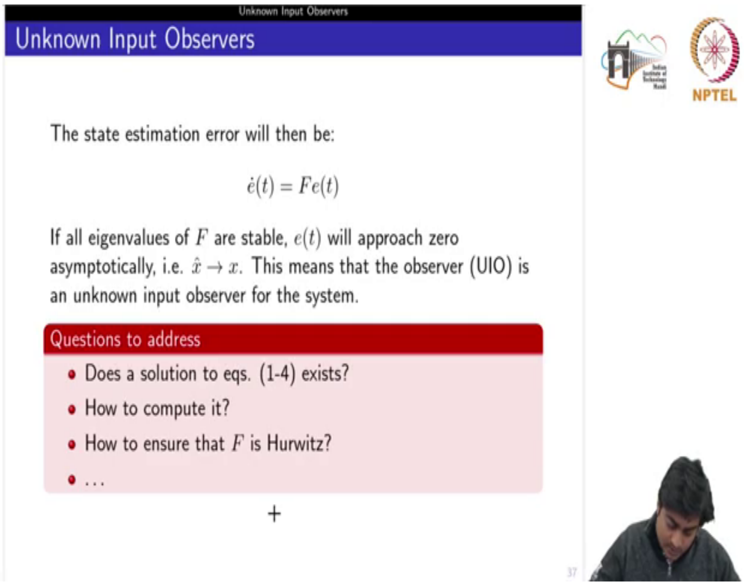

So, the remaining part is e dot is equal to F times C. Where if I ensure that my f is a Hurwitz matrix, then the error would become 0 asymptotically meaning to say my state estimate becomes equal to the actual state.

But the questions which for arises that does a solution to equations 1 to 4 exists. Meaning to say whether we could be able or we could solve these equations. As we had seen that these are the parameter matrices of the observer which we need to synthesize F T K and H. If there is a solution then how we can compute the solution of these equations and at the same time how to ensure that F is a Hurwitz matrix. Because, F is not the matrix which we are assigning or which we are defining. This basically comes from this part ok.

So, we do not know yet whether after designing my H and K or how to design this H and K such that my F becomes Hurwitz matrix right. So, we will see the answer to these questions by equations by taking equations one by one.

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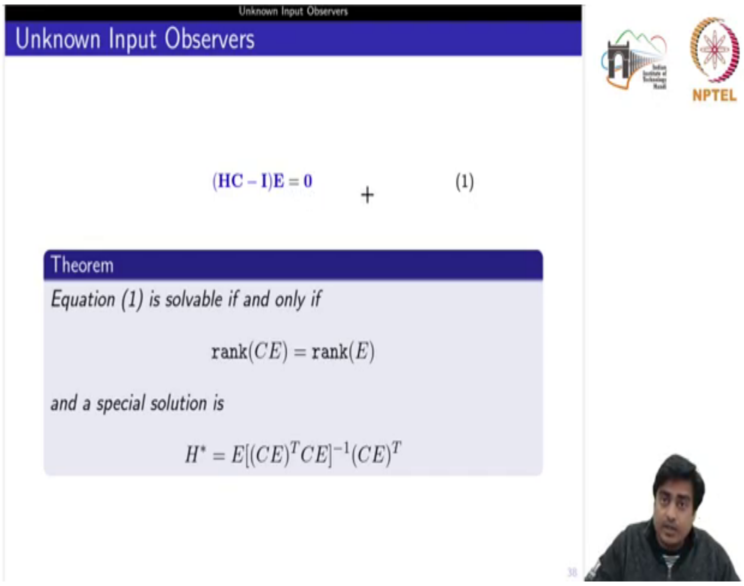

So, first of all we will take the first equation which is H C minus I where I is a identity matrix, by the way H C minus I time E is equal to 0 ok. So, first we will see whether the solution of this equation exist and what would be the solution of this equation.

So, this is the one of the important results which says that equation 1 is solvable. If and only if the rank of C E matrix becomes equal to the rank of E matrix and a special a special solution of this equation is given by as H star is equal to E times this matrix ok. So, now, we will see the proof of this theorem very quickly that how this equation becomes solvable if this rank condition is satisfied.

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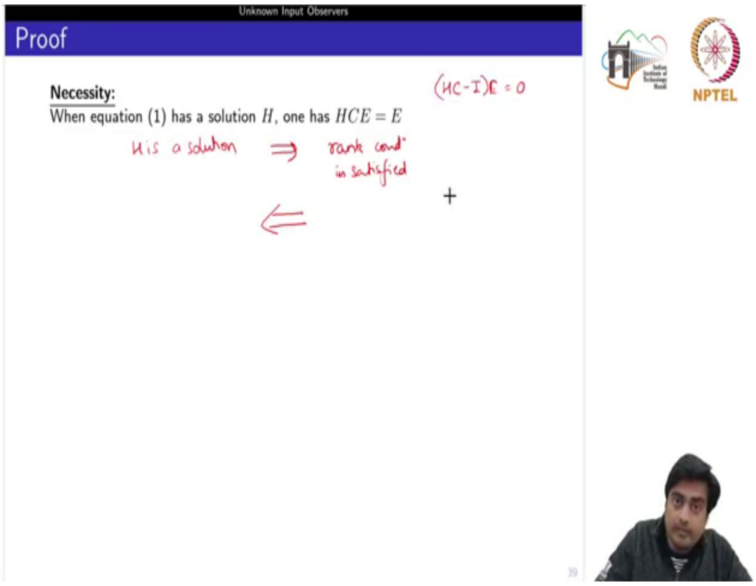

So, again there are two parts of this proof; first is the necessity and another is a sufficiency. So, we say that there is a solution H and let us say H is a solution implies that the rank condition is satisfied. So, first we will see this part of the proof; the sufficiency part that if the rank condition is satisfied implies that H is a solution ok. So, when equation 1 has a solution H. I can write H C E is equal to E, because we have this H C minus I times E equal to 0.

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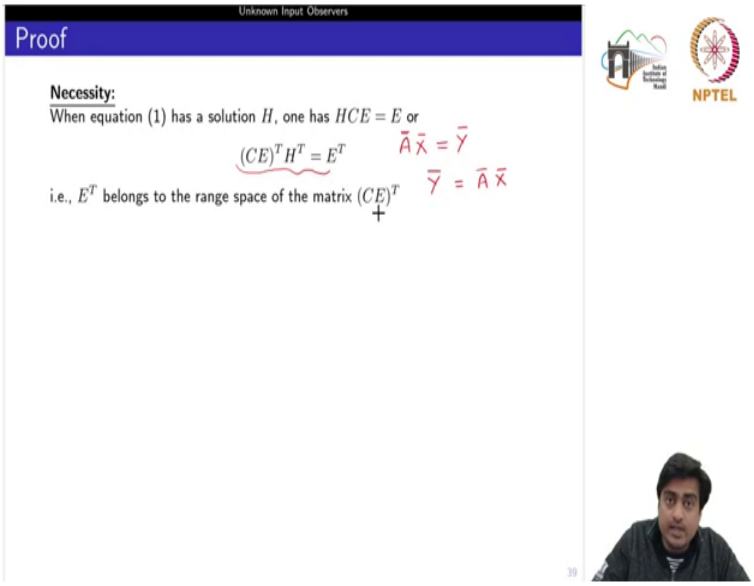

Now, this part I can also write as by taking the transpose both sides by combining the CE matrix. So, CE transpose H transpose becomes equal to E transpose. Now pay attention to this equation. So, I can write this equation as let us say C E, I call it A bar and H transpose, let us call it X bar and E transposes Y bar ok. So, we had seen this these types of equations earlier.

So, let us write this in a common notation as Y bar is equal to A bar X bar ok. Where Y bar is E transpose and X bar is H transpose and this a bar is CE transpose. So, if we pay, if you recall the controllability week where we have introduced about the range spaces and the null spaces. So, you would see there your Y bar should belong to the range space of this matrix A bar ok.

Now, this Y bar is nothing but, your e transpose and this A bar is nothing, but your CE transpose. Meaning to say that e transpose should belong to the range space of the matrix CE transpose.

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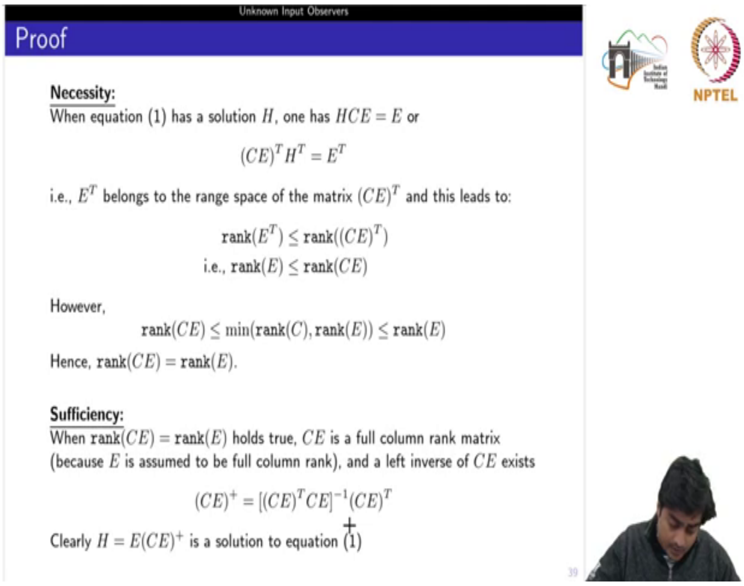

And this leads to that the rank of this E transpose matrix or the rank of Y bar should be less than or equal to the rank of this A bar right. Now if this inequality would also hold and if I take the transpose both sides that is to say the rank of E is less than equal to rank of C E. However, that the rank of this matrix CE should be less than equal to minimum of rank C and rank E ok. Which would again be less than or equal to rank of E, meaning to say for example, if the rank of E is greater than rank of C. Then the minimum of this one would be C and then rank C is again less than rank of E.

Now if rank of E is less than rank of C and taking the minimum of this rank C and rank E would yield rank E, meaning to say that it becomes equal to rank E. So, the rank of CE would never be greater than rank of E. It would always be either less than or equal to rank of E. Now see both these conditions. The first condition says; rank of E is less than equal to rank of CE and the second condition says that the rank of C E should be less than equal to rank e and this becomes possible if and only the rank of e is actually equal to rank of C right.

So, this finishes the necessity part and the sufficiency when rank of CE equal rank of E and we know that E is already a full column rank. Meaning to say that CE would also be a full column rank matrix. Now if C E is a full column rank matrix, then I can take a left inverse of this matrix as and which can be written as C E, we have used another denotation as plus. So, c raised to the power plus becomes equal to C E transpose C inverse times C E transpose. This is basically the pseudo inverse of this CE matrix.

So, finally, I would have H is equal to E pseudo inverse of C E is a solution to equation 1 ok. So, this finishes the complete proof of the theorem 1. So, now, we have the solution and we have the conditions under which this equation 1 can be solved.

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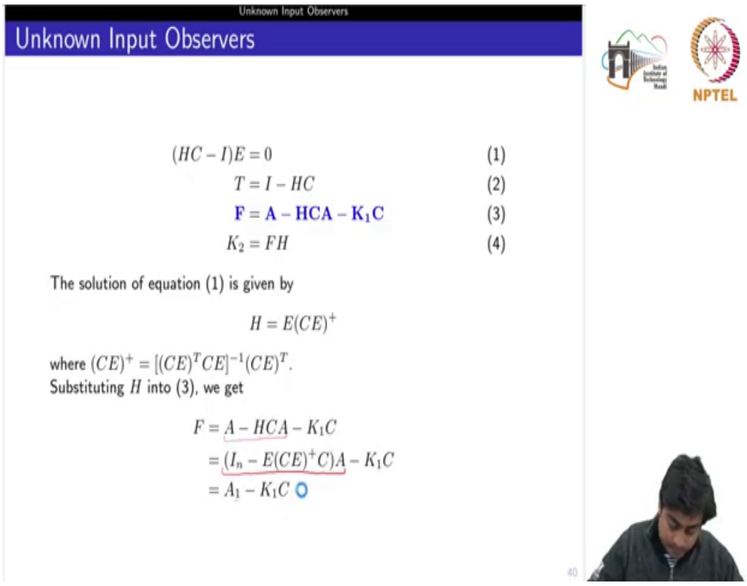

Now, if once you have computed this H, you can directly compute this matrix T. Because C is already known to us and I minus H C would give us the T matrix ok. So, this is quite trivial. Now the third equation is F is equal to A minus H C A minus K 1 C. So, now, we need to see that whatever the H we have computed from the first equation if we plug that H into this equation can we show that my F matrix would be a stable matrix for some K 1 let us say.

So, let us see. So, the solution of equation 1 is given by this one; H is equal to E, CE plus, where CE plus was defined as this. Now if we substitute this H into this third equation; that is to say A minus H C A minus K 1 C would become this one. Where I have taken a as a post multiplying factor from the first two parts of this equation that is I N minus E CE plus is H C and A minus K 1 C. Now I substitute this whole part as A 1 and so, finally, this f becomes A 1 minus K 1 C ok.

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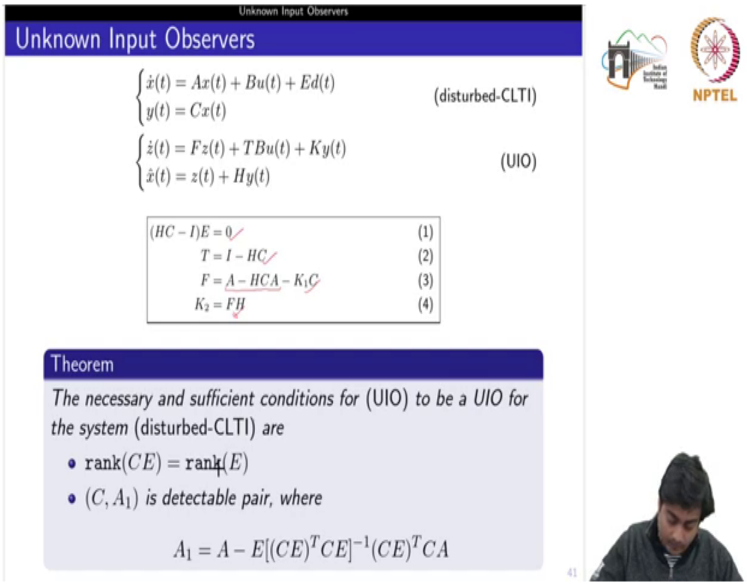

So, now this equation you would see which is pretty much familiar to you. We will summarize all the results later on I mean into this slide. So, given this plant which is having an additional disturbance term, we have designed this UIO and ensuring that the solution of this equations exist. Where the key matrix is the H matrix depending on the H matrix, the further three matrices could be computed ok. The hurwitzness of this F matrix is a separate part which we will see now.

So, this is the main result for the unknown input observer that the necessary and sufficient conditions for UIO to be an unknown input observer for this system that the rank condition is satisfied of which the importance we had seen already. And the pair C coma A 1 is detectable where A 1 is basically A minus H C A ok.

So, the significance of this first condition is that if the rank condition is satisfied we can compute H ok. So, this equation is solved if this first condition is satisfied. Second; we can compute the t matrix. Now if C or A 1 comma C pair is detectable. Where this one is A 1 if A 1 comma C pair is deductible meaning to say F would definitely be a Hurwitz matrix ok. Now so, this equation matrix we can also compute whatever F we have obtain from here and H from the first equation multiplying these two matrices would give me K 2.

So, basically we have computed all these four equations and the necessary and sufficient conditions for solving these equations basically this one ok.

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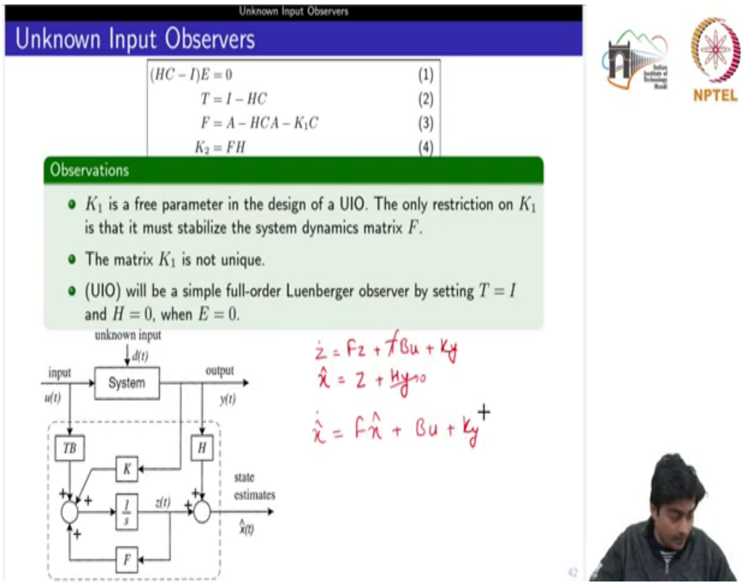

So, these four equations which are the key equations for designing the UIO. Here you would notice that we have not used K 1 and K 2 for any purpose. So, basically, K 1 is a free parameter which basically ensure the hurwitzness of this f matrix. So, this matrix K 1 possibly would not be unique. Now the interesting part of this UIO is that this UIO actually becomes a full order Luenberger observer if we substitute T is equal to I, H is equal to 0 and E is equal to or under the absence of any disturbances.

So, this Luenberger observer we had seen earlier we can see very quickly. So, this is the observer we have z dot is equal to F z, then x hat is equal to z plus H y right. So, this is what we has is a observer. So, now, if you substitute T is equal to I. So, this will go away as an identity matrix H is 0. So, this part would become equal to 0. So, the remaining part is x hat would become equal z. So, substituting z by x hat here in the first equation we would have x hat dot is equal to F x hat plus Bu plus K y.

So, this is the same observer we had designed in the starting of this week. Where if you recall instead of this K matrix, we have used an L matrix where we have also ensure the hurwitzness of the matrix F. So, we had discussed two methods; one is the eigenvalue simon method and as the Lyapunov method for designing this Ff matrix F and l matrices.

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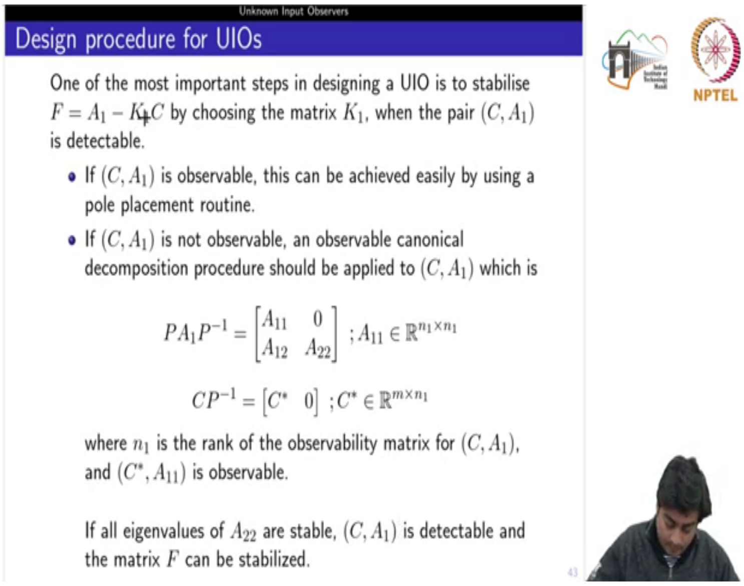

So, as the necessary insufficient condition; we need the f matrix to be a Hurwitz matrix. Now if this pair a one comma c becomes an observable pair then we can substitute any K 1 matrix meaning or we can place the eigenvalues anywhere onto the left hand side by designing this k 1 matrix.

So, we have more design freedom. Now if this pair is not observable then of course, by using the concepts we had introduced earlier on decomposition procedure we can check the stability of the unobservable part ok. Which is here A 22 matrix. So, if the eigenvalues of this A 22 matrix are stable then with the detectability or with the observability of the pair A 1 1 and C transpose C star we can say that at least A 1 and C is a detectable pair. So, that we can carry forward with the designing of this K 1 matrix.

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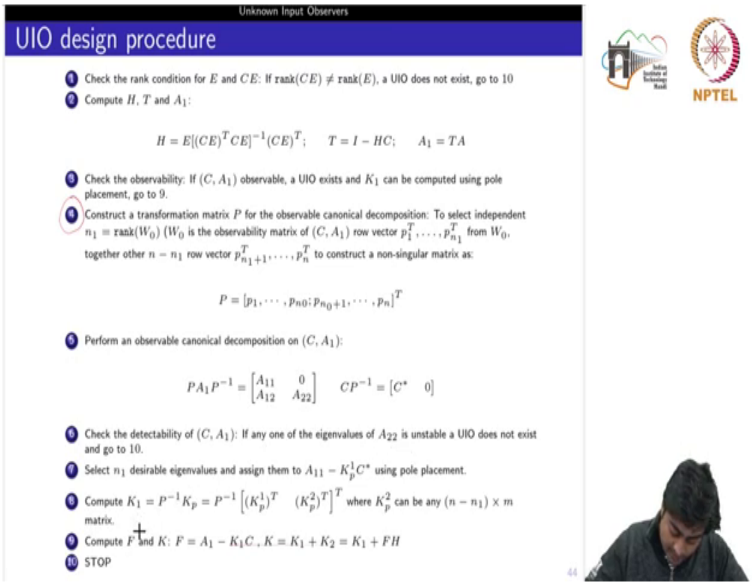

So, we see the entire procedure. So, the first check condition is the rank condition, whether the rank of E is equal to the rank of C. If this condition does not hold then the UIO does not exist because this is a necessary and sufficient condition and we will go to stop.

Now, if the same condition is satisfied. We can compute this H matrix and further we can compute the two matrices T and A 1. If this pair A 1 comma C is observable we can directly compute this K 1 using the pole placement technique or either of the method we had discussed earlier and finally, we compute this F and K matrix and then we can stop the algorithm. Now this step 4 is basically the decomposition procedure if the pair a one comma c is detectable.

So, that we can compute this transformation matrix P and using that transformation matrix I can obtain the observable part of this un observable system. So, checking the detectability, we can see if the A 22 is unstable then again the UIO does not exist and we can stop the algorithm. So, depending on how many states are observable we can assign the desired egienvalues and the rest would stay as it is. So, again we can compute this F and K matrices which would finally, yield us the unknown input observer.