Linear Dynamical Systems
Prof. Tushar Jain
Department of Electrical Engineering
Indian Institute of Technology, Mandi

Week – 08
Observer Design and Output Feedback
Lecture – 41
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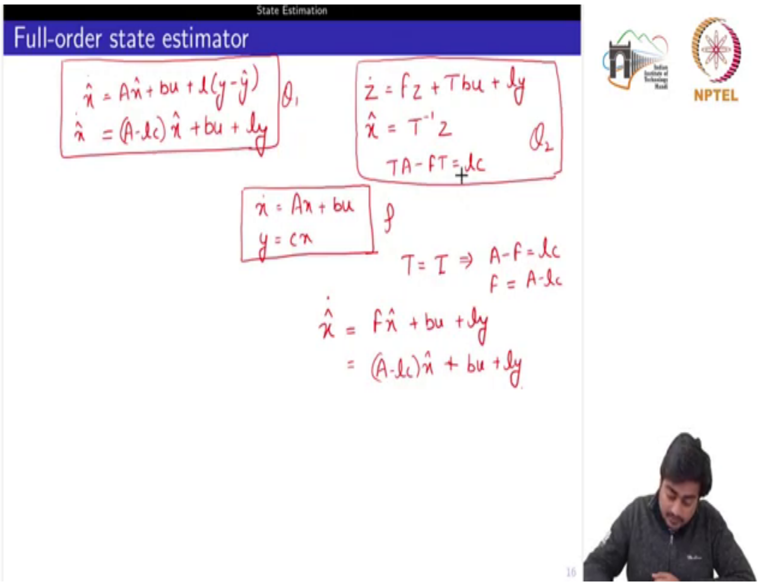

So, in the last lecture, we had seen the two approaches for designing the observer. So, the first observer we had seen is given by x hat dot is equal to A x hat, then we added one term which defines y minus y hat right. And, we also parameterise this in the form of the into this form x hat plus b u plus l y ok.

So, this was one observer, another observer we design using the Lyapunov equation method. So, that observer is given by z dot and here we had the output of this observer x hat is equal to T inverse z, where T was basically obtained after solving the Lyapunov equation T A minus F T equal l c ok. So, these two observers we had seen yesterday or in the last lecture for the given plant. Let us write the equation of the plant also, let us call it the plant and this one is the observer 1 and this one was the observer 2 ok.

So, there are couple of points which we want to highlight before we go onto the next topic within this module. So, here you would notice that say for example, if the plant is n dimensional system meaning to say that x is a n dimensional variable, then both the observer; observer 1 and observer 2, both were having the x hat of the similar dimension. So, x hat was having the n dimension similarly, in in observer 1 and similarly in observer 2 right.

So, we call such observer as the full order observer or the full order state estimator. Now in these two observer design you would also notice that if I put t is equal to the identity matrix in this observer, then I would obtain x hat is equal to z. And, then I can replace the state equation of this observer as x hat dot is equal to F x hat plus b u plus l y ok. Now, if I substitute T is equal to I into this Lyapunov equation I would get A minus F equal lc or I can replace F by A minus lc.

So, if I substitute this here. So, I would get A minus lc x hat. Excuse me, plus b u plus l y ok. So, basically both observers are now the same. It just thus synthesize procedure of the f matrix and the other associated matrix in the two observer design.

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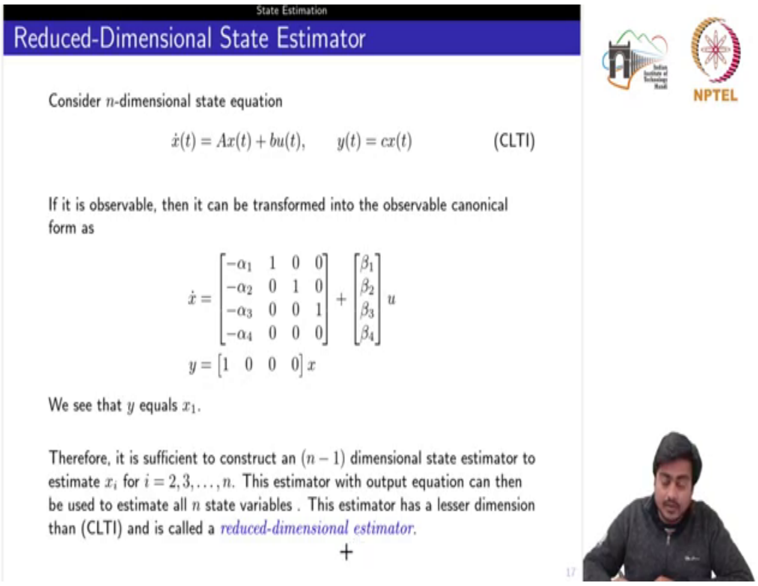

So, today we will introduce the reduced dimensional state estimator, where if x is an n dimension, x is of n dimension then, we need to design the observer whose state is lesser than the n dimension ok. So, if it is possible, then we call those observer as the reduced dimension observer or the state estimator. So, the first question arises here that whether such kind of produce design is possible or not.

So, if we go back to the definition of the observer canonical form. So, that if the system is observable then the system can be transformed into the observable canonical form as this one. So, here we have considered the one specific case where the where n is equal is equal to 4.

So, we had discuss at several places, the significance of this word canonical because if the if say for example, for a system. It could have many state space representation. Now it might be possible that in some space state space representation x is not directly appearing in to the output measured variable ok. But, if we carry out the canonical transformation then depending on the dimension of the c matrix or the output matrix; some of the states would directly be available in the output measurement ok.

So, here this is one specific example where n is equal to 4. So, once we convert it this into the canonical form we see that y actually becomes equal to x 1. So, x 1 is directly available in y ok. So, but we still need to estimate x 2, x 3 and x 4 ok. So, for only estimating those states we can design an observer. So, this is one specific example. So, therefore, it is sufficient to construct an n minus 1 dimensional state estimator to estimate x i for the rest of the state variables ok

This estimator with output equation can then be used to estimate all n state variables. So, this estimator has a lesser dimension then the dimension of the plant. So, and we call it the reduced dimension estimator ok.

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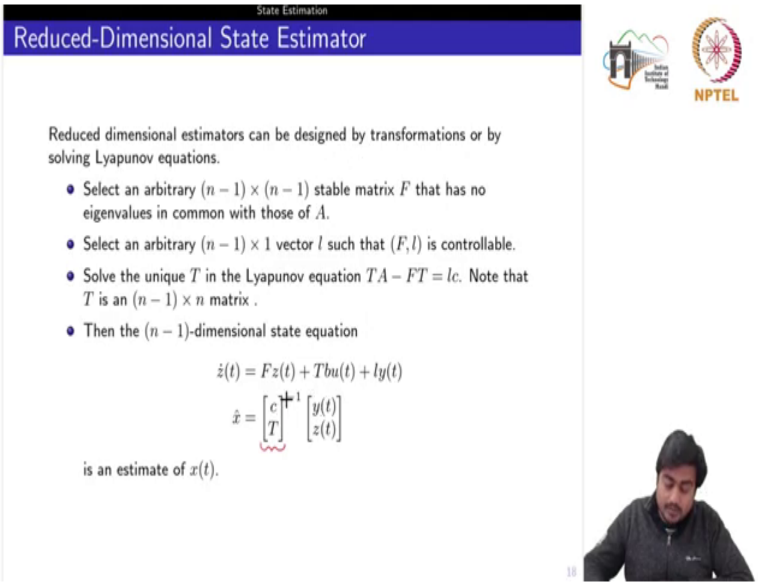

So, here we will see one design procedure which is based on the Lyapunov equation. So, we recall the steps what we had seen in the full order observer there we select an arbitrary stable matrix F of dimension n. But here we select an arbitrary matrix of dimension n minus 1 which is lesser than n.

And in addition that the eigenvalues of this F matrix are not common with those of the A matrix. The next step is to select the vector l of appropriate dimension such that this pair F comma l is controllable. Next, we solve the Lyapunov equation to compute this matrix T of dimension n minus 1 cross n and then finally, by using the state equation what we had seen earlier, we can estimate this x hat. So, here you would notice that the state equation of the observer where the state variable is defined here by z remains the same ok. It just the dimension of the associated matrices would change.

And here x hat, we see it is given by the inverse of this matrix formed by concatenating two matrices c and T and vector y and z. So, here we say that this x hat is an estimate of x ok. So, in the previous full order design for the single input, single output system we ensure the invertibility of the T matrix. But here since there is another element which is being inverted. So, we need to see whether this matrix is always invertible or not ok. First of all we will go to the justification of this procedure that whether this state estimator actually give us the estimate and then we will comment on the invertibility of this matrix.

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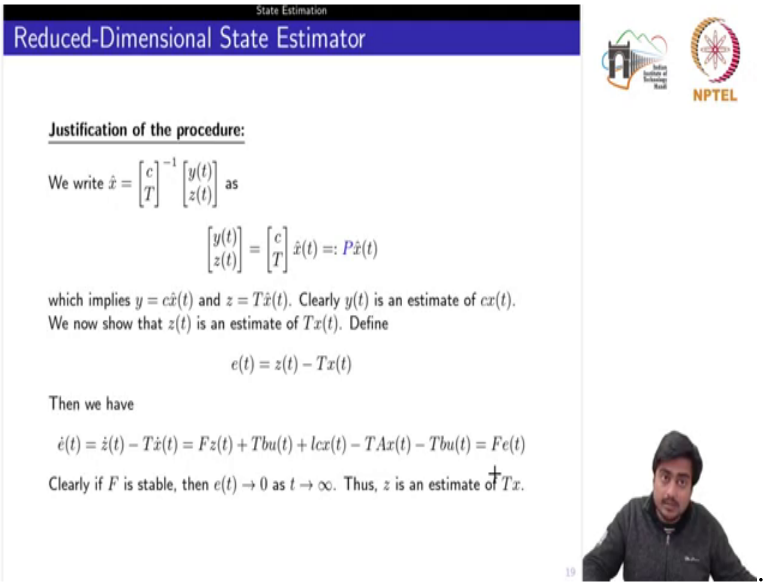

So, we write x hat which was the output equation as y t, z t is equal to. So, suppose that it is possible to take the inverse of this matrix. So, I can write this one which I can write as P times x hat; where P is that matrix of which I need to show to the invertibility ok. On which we will come on to later. So, if we see the first element and the second element I can write individually or separately as y is equal to c x hat. So, this y become equal to c x hat and this z become equal to T x hat ok. Now since here we can see clearly that y t is an estimate of c x.

So, now we need to show that z t is actually an estimate of T x ok. Then in that case we could say x become equal to x hat. So, again writing or proceeding in a similar way first we define the error vector as z minus T times x t and then taking the derivative of this error equation and substituting this z dot and x dot from the plant we get this. So, in a similar way some of the terms will cancel out and some of the term after combining would finally, give us like this relationship that e dot become equal to F e t ok.

So, we have already selected f as a stable matrix. So, it ensures that e of t would tends to 0 as t tends to infinity, thus z is an estimate of T x ok. So, we see that this reduced dimensional state estimator helps us to estimate the complete state though the dimension of the estimator is not the same as the dimension of the plant, but still we need to ensure the invertibility of the P.

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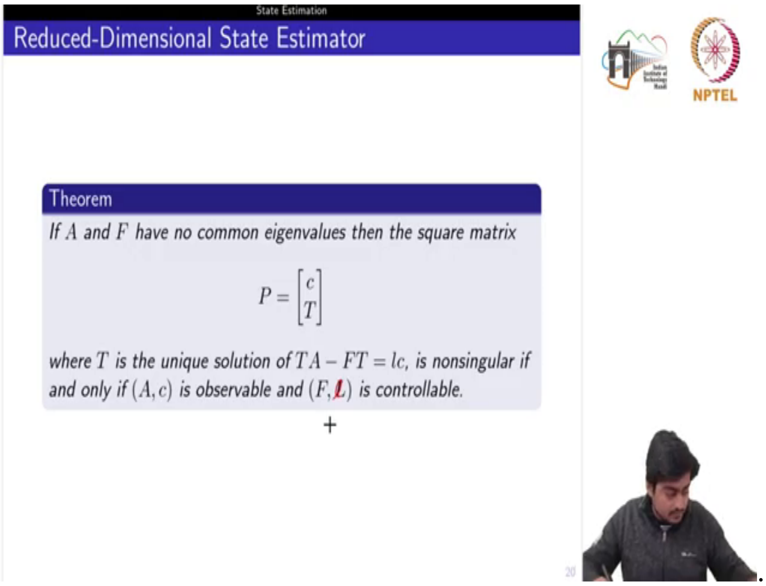

So, this result talks about the non singularity of this P matrix. So, if A and F have no common eigenvalues which we have already ensured in the design step by selecting this F matrix in such a way that the both these matrices do not share the same eigenvalues. Then the square matrix defined by this by combining the c output matrix and the T matrix. Where T is the uniques or where the t is basically obtained by solving this Lyapunov equation ok.

So, this P matrix is non-singular if and only if this pair is observable A comma c and F comma L. So, this should be small l, f comma l is controllable ok. So, we need the observability of the pair A comma c which we have already ensure or which is the basically the prime recondition for the existence of the observer. Second this pair should also be controllable if both these conditions are satisfied then it is always then whatever the P matrix we have obtained would always be non singular because this is if and only if condition ok.

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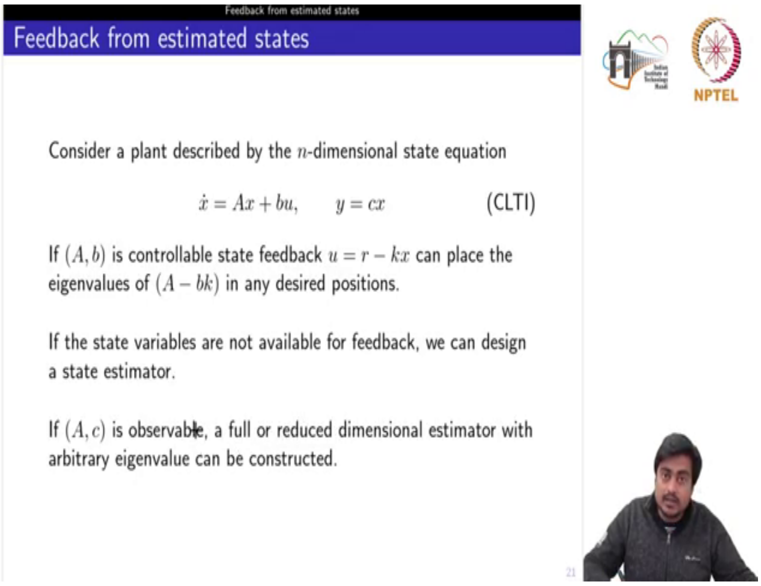

So, we shall not go into the proof of this result which we had seen at numerous occasions. So, now we will move to the next topic which speak about the feedback from the estimated states. So, this was the main issue with which we started our seventh week that the state x by using which we want to design the state feedback is not possible to measure directly. So, we have designed this observer. So, let us see that if we combined both the say the observer and the controller, then how the design or the dynamics of the overall closed loop system would behave right.

So, let us analyze this first given this plant. We know that the pair A, b is controllable then we can design the state feedback which is we had seen earlier that u is equal to r minus k x where r is the reference signal and k is the state feedback gain vector can place the eigenvalues of this

A minus b k in any desired positions ok. We had also seen that under stabilizability or if the this pair is stabilizable then still we can design the controllable ok.

So, if the state variables are not available for feedback. We can design a state estimator to estimate this x hat and then finally, so for this design of the state estimator; we need to ensure that the pair A comma c is observable. So, we can design either a full dimensional estimator or the reduced dimensional estimator with arbitrary eigenvalues ok. Now if this pair is detectable we cannot place the eigenvalue arbitrarily, but still, but still we can ensure the asymptotic stability of the error dynamics ok.

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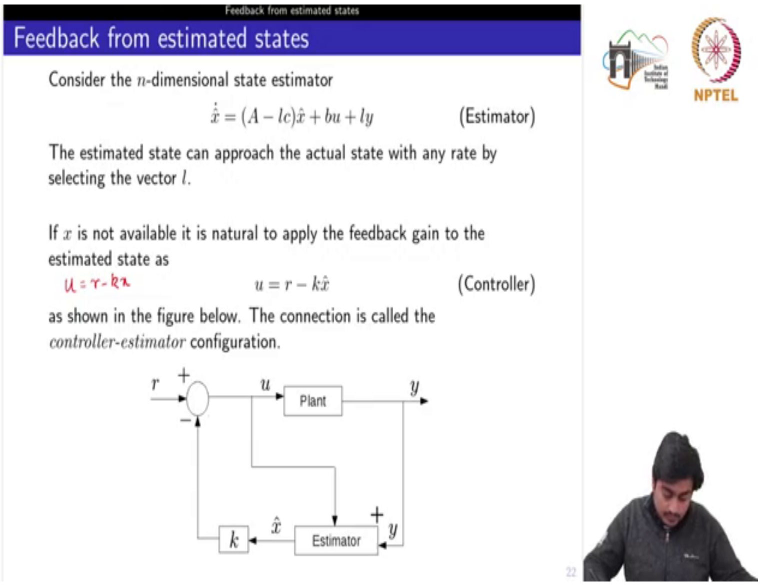

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So, let us see so, consider the n dimensional state estimator. So, here we are considering the full dimensional state estimator though the analysis will remain the same for the reduced dimensional observer. So, this equation we had used earlier, x hat dot, A minus lc x hat plus b u plus l y. Where A minus lc is a is a stable matrix and l y is a output injected term. The estimated state can approach the actual state with any rate by selecting the vector l ok. So, we have already assumed that the pair a comma c is observable.

Now, if we design the controller, if x is not available it is natural to apply the feedback gain to the estimated state as. So, earlier we have seen that the controller was u is equal to r minus k x. Now instead of using x in the controller we are using x hat which is coming from the estimator. So, we see the block diagram representation, this is the plant having input output u and y. So, taking these two signals u and y, we estimate this x hat and then this x hat we have further used to design the state feedback controller which is given by u is equal to r minus k x hat.

So, we call this configuration as a controller estimator configuration where; obviously, this part or the state feedback we have the controller and it is taking the input from the estimator.

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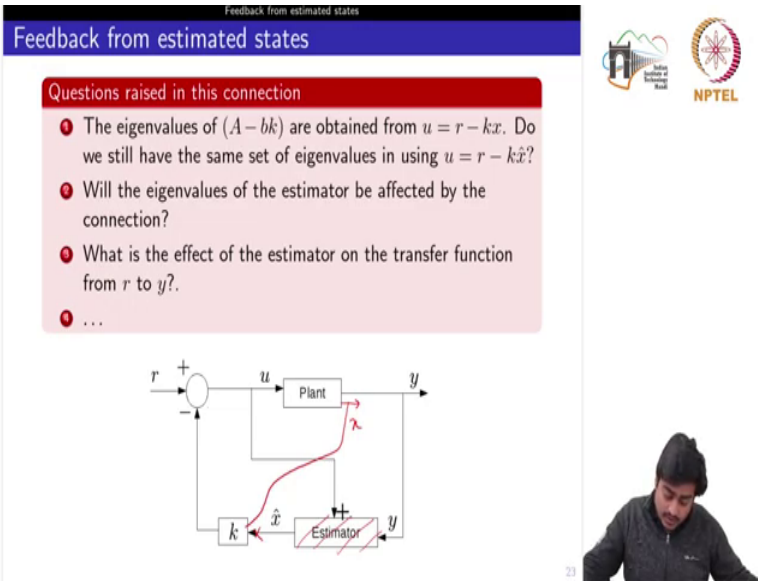

So, after seeing this connection there are some questions which arises naturally. So, the first one is that the eigen values of this A minus b k which we have obtained this matrix after applying only the state feedback are obtained from this one. So, do we still have the same set of eigenvalues in using u is equal to r minus k x hat right.

So, so, this question basically implies let us say for example, if we do not have this estimator and we have direct measurement of x and instead of using this this x hat, we are taking this one ok. So, we had seen already that the closed loop state matrix would be A minus b k.

Now if we are using this estimator whether the eigenvalues of the closed loop system would still remain the same or it would change right. So, the second question will the eigenvalues of the estimator be affected by the connection. So, it is both ways. So, whether the eigenvalues of the of the closed loop control system would be affected by the estimator or vice versa hm.

So, now, what is the effect of the estimator on the transfer function from r to y ? So, earlier we had we have derived the transfer function of the closed loop system now there is an estimator block inside it. So, there is question naturally arises whether this estimator would have any impact on the transfer function in the sense either the transfer function would change or would it going to remain the similar.

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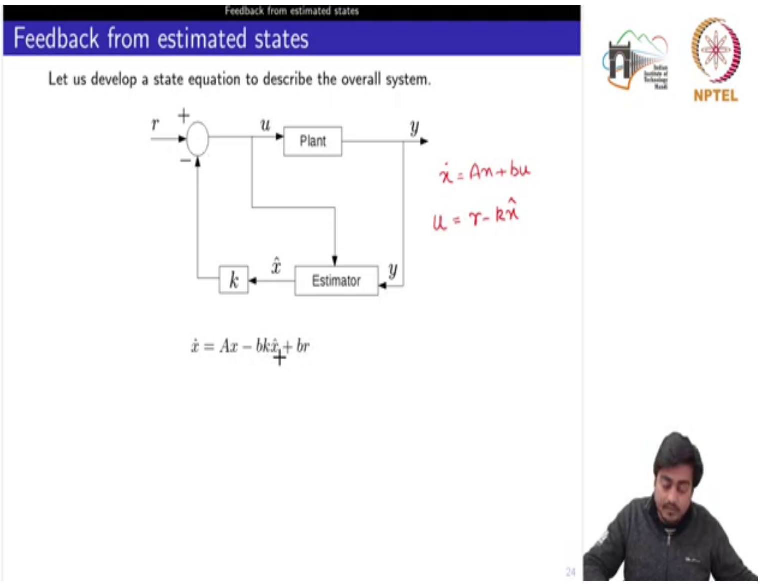

So, we will try to answer a couple of most of the questions in this connection. So, to answer these questions we will develop first of all the state equation of the overall system to further analyze that what is happening inside the closed loop. I will write here the equation of the plant also which is x dot is equal to A x plus b u ok. So, this so, and we write u is equal to r minus k x hat right. So, now, this is the plant, the actual plant and this is the controller that instead of having this x we are supplying it x hat.

Now, if I combine these two system or these two equations I would get this one; x dot is equal to A x and putting u as r minus k x hat here I would get b r minus b k x hat.

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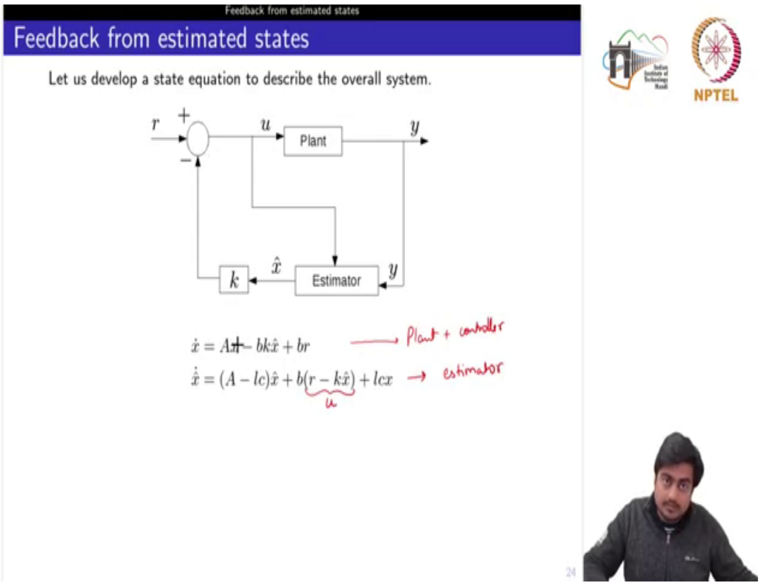

Now, seeing the second equation which is basically the estimator that this is the estimator where we have x hat is equal to A minus lc x hat plus b u. Now here we have also replaced u by r minus k x hat and plus and the last term is l y or we can write y as c x ok. So, this is you can see the plant plus controller and this one is your estimator ok.

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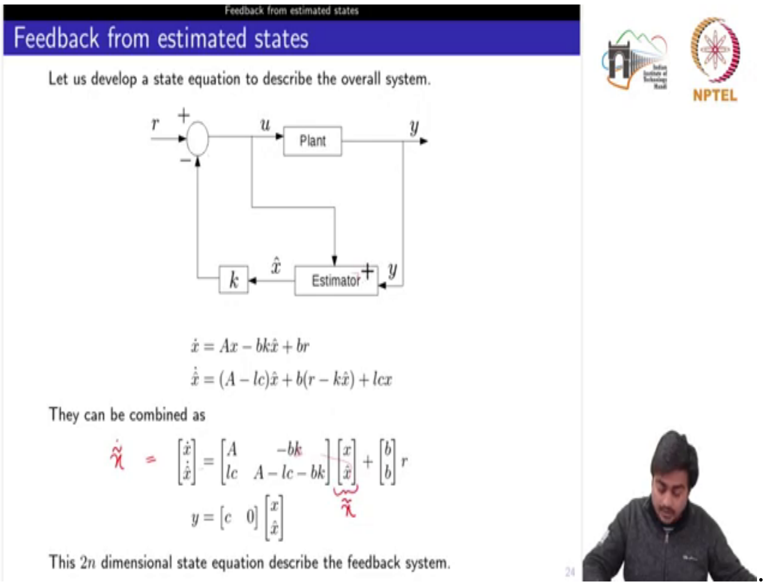

So, now we will make a new system by combining these two state equations into one as we will write this let us say, we will come onto this later. So, we can see that this x dot and x hat dot would become a new state vector. Let us it is better to define them let us call them x tilde ok. So, it would be x tilde is equal is equivalent to this one and this we have defined at as x tilde ok. So, it becomes x dot is equal to A x this term minus b k x hat minus b k x hat plus b r this is the first equation.

The second equation is x hat dot is equal to we need to find out all the terms with respect to x so, l c, l c x, then we have A minus lc minus b k time x hat ok. So, this become this element of this matrix and the last element is b times r. Similarly, the output equation will become c x hat and 0 times x hat ok. This is pretty much straight forward. So, if you see the dimension of this overall new state space system it would be 2 n. The n state variables are coming from the plant and similarly the n state variables are of the estimator.

So, the overall dimension of the state space equation in 2 n. So, this so, this system describes this overall feedback system where we have both the controller and the estimator ok.

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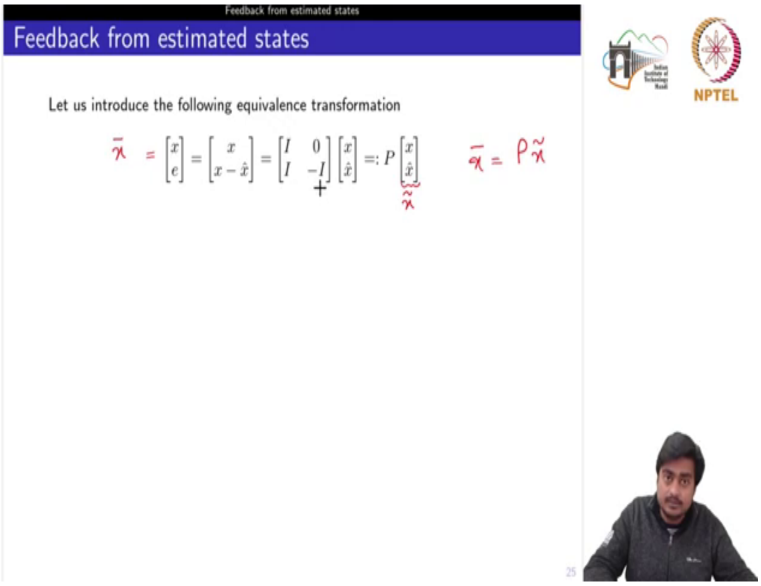

So, now we introduce the following equivalence transformation. Now we define this another state let us say x bar and this one we had x tilde. So, I make a new state vector called x bar which is basically x and e. x is the state of the plant and e is the error vector which is described by the difference between the actual state and the estimated state ok.

So, if I this is basically an equivalence transformation which we had seen in the first week. So, if we see at the outset we would get x bar is equal to P times x tilde, where x tilde is the state variable of one representation I want to transform into another representation whose state is x bar and I am using this P. Now depend now if we see the structure of the P, this P is non-singular ok.

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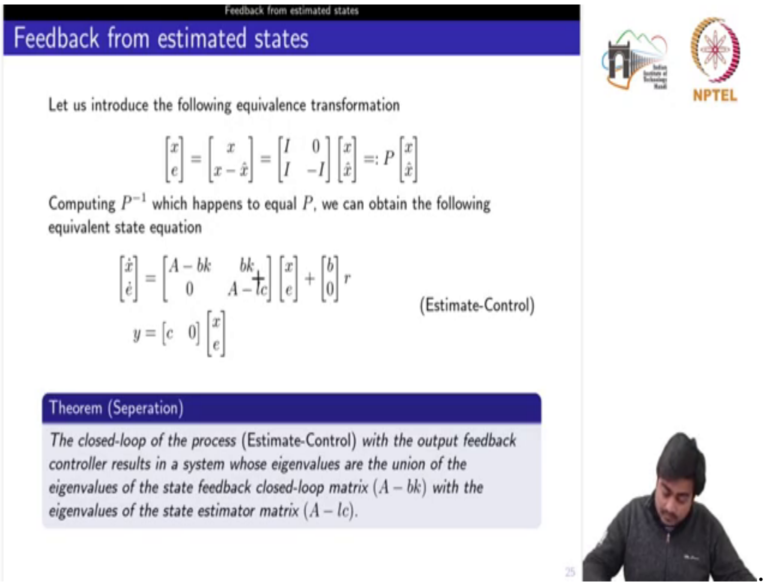

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And so, let us do the equivalent transformation. So, we can also see that p inverse also become equal to P. So, now, if we define the state space system into this new state variable I am after simplification I would get this state matrix. b matrix and the c matrix ok. So, you can do this by yourself also by using the formulas which we have introduced in the first week where we spoke about the transformation from one representation to another representation ok. So, this representation is basically equivalent to the representation we had seen in the last in the last slide which was in terms of x tilde ok.

So, now after using this transformation many things I have been highlighted. So, we can see one by one. So, that so, this is the next result the closed loop of the process basically which is this estimated estimator plus controller system with the output feedback controller results in a system whose eigenvalues are the union of the eigenvalues of the state feedback closed loop matrix A minus b k with the eigenvalues of the state matrix a minus l c ok.

So, pay attention to this state matrix. So, now, if this is a upper triangular matrix and if we compute the eigenvalues of this matrix, you would see that the overall all the eigenvalues of this state matrix would be the union of this eigen, thus eigenvalues of this matrix and the eigenvalues of this matrix. So, the eigenvalues so, so, that is why we say that this is a separation property that and it also speaks about that both the controller and the estimator can be designed separately. No one has any effect on the other ok.

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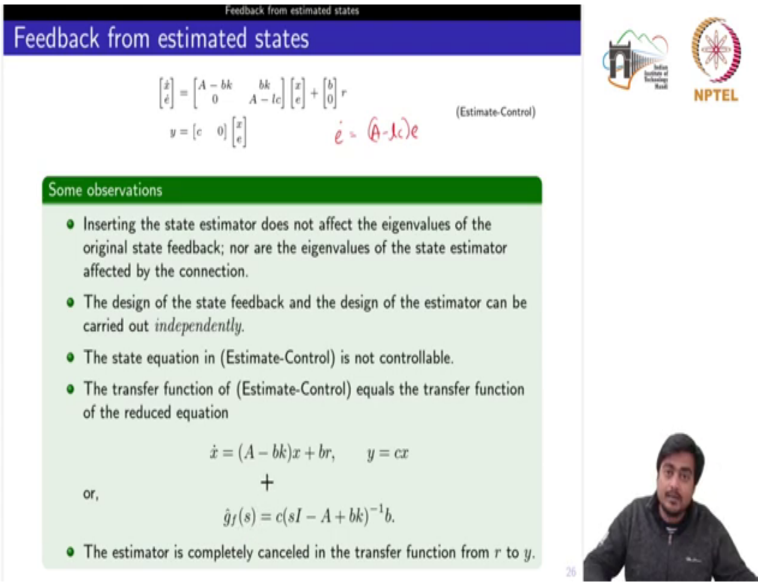

Let us see some couple of more observations. So, this was the estimate control or the closed loop dynamics we have obtained. So, inserting the state estimator does not affect the eigenvalues of the original state feedback nor are the eigenvalue of the state estimator affected by this connection. So, this we had already seen a second. So, as a consequence of this first point the design of the state feedback and the design of the estimator can be carried out now independently.

Third the which is most interesting that if you see the state equation of this overall closed loop dynamics it does not controllable right. So, pay attention to this state equation. So, once we have used the transformation we have actually end up with the decomposed form, where this if we write the dynamics of the second state which is e. So, this becomes e dot is equal to A minus l c e because we have this 0 here.

So, this e is not all controllable, but at the same time we have ensure that A minus l c is stable matrix. So, this state equation although not completely controllable, but still it is stabilizable ok. So, this is just one observation although we would not be using it in closing any feedback loop or designing an observer. Now talking about the transfer function. So, if we compute the transfer function of this one it basically equals the transfer function of the reduced equation of this one which we had already seen by g hat f which is the transfer function of the feedback system it is given by this one.

And as a consequence of this the estimator is completely cancelled in the transfer function from r to y. Then because the last equation which is e dot is equal to let us say A minus l c times e. This is a homogeneous system and this would not appear anywhere in the input output signals that is why we are getting the same transfer function what we were obtained when we had not used any estimator ok.

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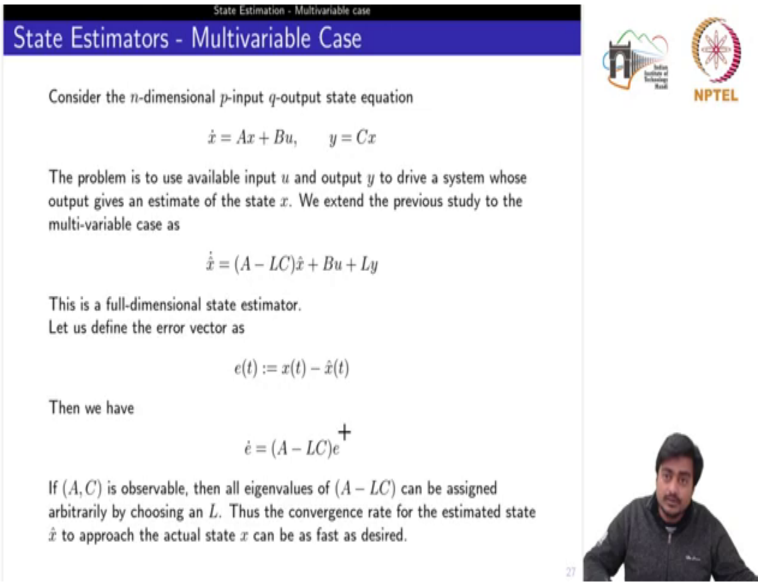

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So, this was all on the feedback from the estimated states. So, now we can see or we can quickly go through all the results under the multi variable environment that consider the n dimensional p input q output state equation where we have now capitalize the matrices B and C because they are no longer the vectors. So, the problem remain the same that the input and output are available to us and also we have the knowledge of the matrices. So, we need to compute x given the information of u y and A, B, C ok.

So, we can extend the previous study to the multi variable case as this one. So, this observer we also called it the Luenberger observer which is x hat dot is equal to A minus LC x hat plus B u plus L y. So, this is basically the full dimension observer. So, by seeing in a similar way that we can first define the error signal as a difference of the actual state and the estimated state and then taking the derivative and substituting these two state equation into the derivative of this error we would finally, end up having these error dynamics ok.

Where we know that if the pair A comma C is observable then all the eigenvalues of this matrix can be assigned arbitrarily by choosing an L ok. If this A C is detectable then although we cannot put the eigenvalues of this matrix arbitrarily, but still we can ensure the stability of this matrix. So, that the error dynamics will be approaches to 0.

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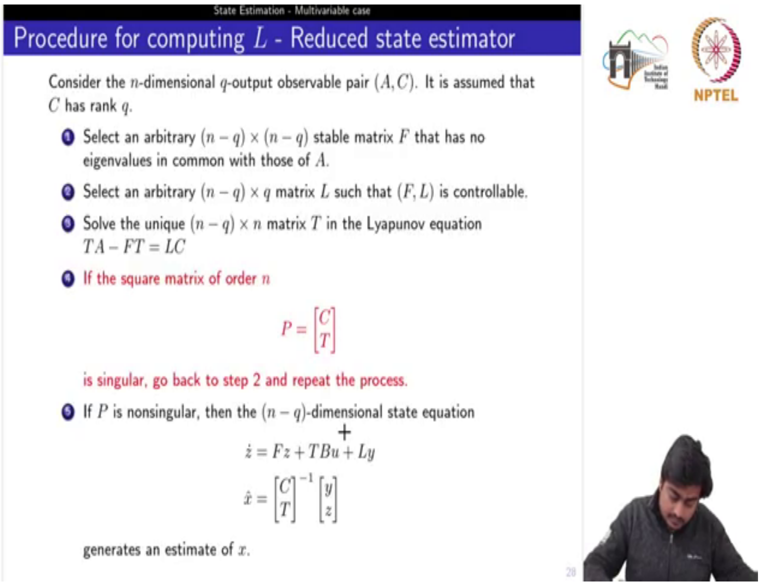

So the full order design would remain the same there is a slight change in the reduced state estimator which we can see. So, we would not be discussing the design procedure of the full order state estimator which you can do by using either of the method that is eigenvalue assignment or the Lyapunov equation method.

So, here we would see the reduced state estimator. So, we have this n dimensional q output observer pair A comma C, there we have assumed that the matrix C has rank q ok. Now the first step we select an arbitrary matrix stable matrix F of this appropriate dimension which shares no eigen values with that of the a matrix. Again, the next step we select the L matrix such that this F comma L pair is controllable then we compute those matrix t by solving this Lyapunov equation given by this ok.

So, if you recall that in the single input single output case we had formed this p matrix as the combined matrix of C and T. So, if the square matrix of order n this one is singular then we go back to step two this one and repeat the process ok. So, we need this means to say that we need to find another L matrix such that if we put that L matrix into this Lyapunov equation we would obtain another T matrix ok.

So, for some L matrix this P matrix would definitely be non-singular. Now if P is becomes non-singular by carefully selecting this L matrix or F matrix then this n minus q dimensional state equation which is given by this one where the first equation remains the same and x hat generates an estimate of actual state ok. Where this is our p matrix.

So, again the problem here we itself we see in the algorithm that P could easily be a singular matrix because in the single input single output case when we had ensured the observability of this pair and the controllability of this pair. So, it is always true that the p matrix would be non-singular. But although we have ensured these two conditions into the in the multi variable environment still we cannot ensure the non singularity of the P matrix ok. So, we will see what we can say at least of this p matrix.

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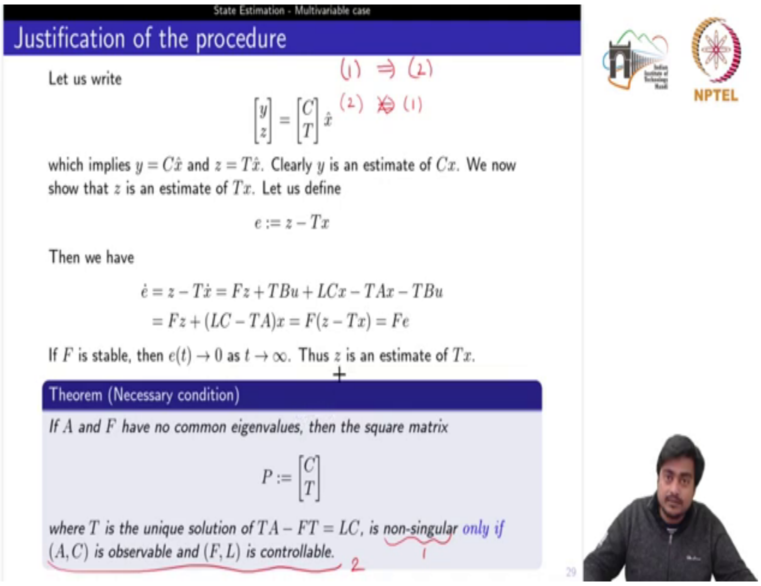

So, now if we go through the similar procedure to ensuring whether the designed estimator is basically estimate give us the estimate. So, this is the standard procedure what we can what you can follow ok.

Now, this is the main result. So, basically this is only the necessary condition that if the matrix A and F have no common eigenvalues then the square matrix P which is formed by combining C matrix and the T matrix, where t is basically obtained by solving this Lyapunov equation is non-singular only if this pair is observable ok. This this you can also understood in a way that let us take this a second statement and this take this as a first statement that the P is singular ok.

Now, it says that first the P is a non-singular, if P is non-singular then it will always the second condition would always be true ok, but if the second statement is true then it does not imply the first statement would also be true ok. This we had already seen in the algorithm also basically that although we have ensured the observability of the pair a c and the controllability of the pair F L still we can end up with having a P matrix which is a singular matrix ok. So, here in the multi variable case for the reduced state estimator we have a weaker condition which is only necessary nor the sufficient one.