Linear Dynamical Systems
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Week – 08
Observer Design and Output Feedback
Lecture - 40
Observer Design and Output Feedback

So, hello everyone. Today we will be starting with the week 8 of the course Linear Dynamical System in which we will talk about the Observer Design and we will see the Output Feedback control also. So, this problem of output feedback we take this or we took this as a motivation when we introduced the concepts of the observability and the detectability. And now we will see that how we can actually achieve this output feedback.

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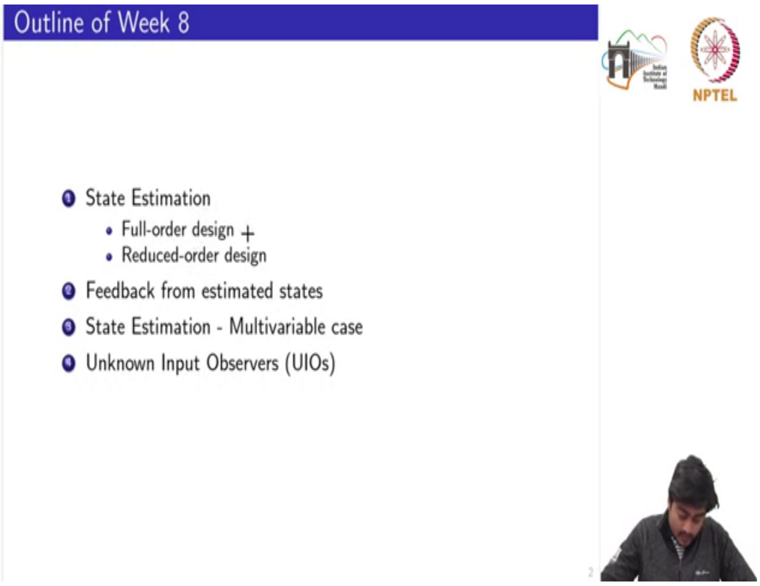

So, the outline of this week 8 is that we will be discussing about the state estimation algorithm, and in this algorithms we will discuss two types of design; first is the full order design and second is the reduced order design. Once we come on to the reduce order design we will first of we will define it first and then we will go to the synthesis procedure.

Second we will see the feedback from the estimated states and so, both these first point and the second point we will be focusing only on the single input and the single output case. In the third point we will generalize the previously discussed results for the multivariable cases. And finally, we will discuss about the unknown input observer, its definition and also the synthesis procedure for the UIO

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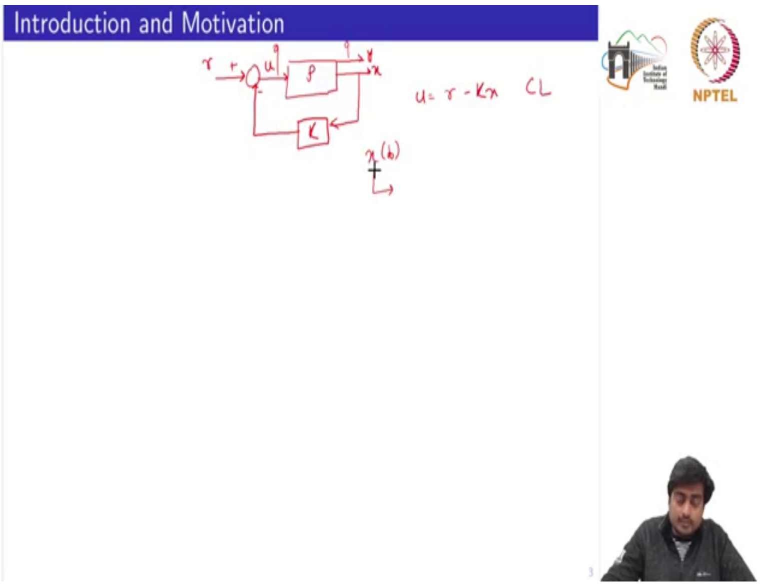

So, starting with the introduction and motivation about the observability, so if we see the lets say we have the plant govern by I will draw two output; one is the x another is the y and this is the u. So, so far we have seen the controller design procedure where we are putting the state feedback as to some reference signal x r by taking directly the measurement of the state variable ok.

So, it was u is equal to r minus k x, so this is the control signal; k of x this is the control law ok. So, in this controller design or the state feedback mechanism we see that in the control law we need the information about the state variable ok.

Now, we had said number of time that is the state variable as an internal variable and many times it might be possible that this x might not be directly available, whatever be the; we have only two external signals external signals by mean; to say which we the signals which we can measure directly. So, these two signals are the input signal and the output signal.

So, if I still want to design the state feedback controller without using the knowledge of the x, how we can do this? Ok. So, in the last week we introduced a concept of observability and the detectability, which helps us to at least ensure that we can design a observer which would give us the estimate of this state variable x ok. So, now, if you recall that we have discussed one procedure also, which is the Gramian based construction of the state signal and that state variable was actually compute in x sum time t naught ok. So, now we will see that how we can compute a continuous estimation of the state x ok.

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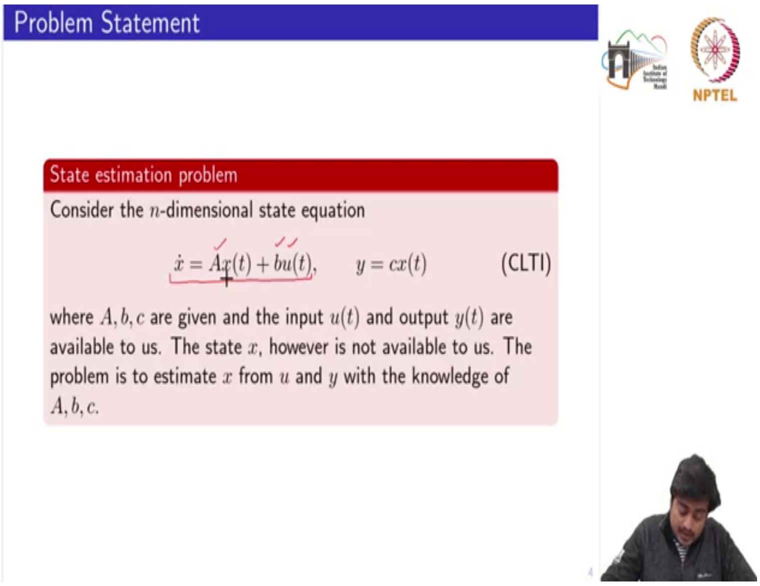

So, if we go to the precise problem statement we would consider the n dimensional state equation where x is n dimensional vector and u and y are scalar, mean; to say that it is just single input single output system. So, the information which is available to us is we know all the matrices and the vectors A, b, c is available to us, since u and y are the external signal, we also have their knowledge

So, the problem we need to solve is to estimate x from u and y with the knowledge of A, b, c and u y. So, there is a standard procedure of doing this. So, first of all pay attention to this state equation ok. So, if we see that the A is known to us, b is known to us, and u is also known to us what we do not know is only x. So, if I want to compute this x, how we can do this?

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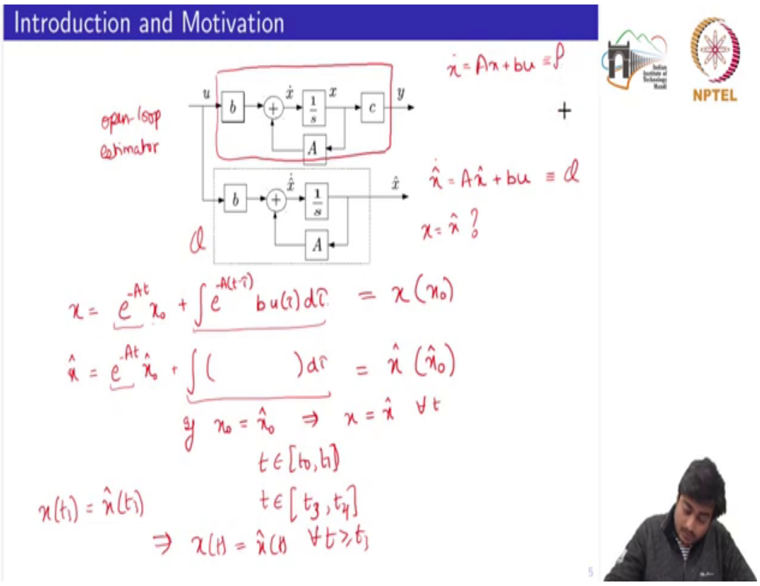

So, let us first of all write the equation of this plant; this is the internal description of the plant in terms of u and y. So, it is x dot is equal to A x plus b u ok.

So, I would design and observer let us call it o where I am taking the measurement only of the u and since b A are known to me, so I can construct this one ok. So, let us write the state equation for this one, it is x hat dot is equal to A x hat plus b u ok. So, this is my plant, this is my observer.

Now, the problem is I want x is equal to x hat right. So, if you recall that in the previous week we actually did the construction of this state signal, but that reconstruction was done at a particular time t naught ok.

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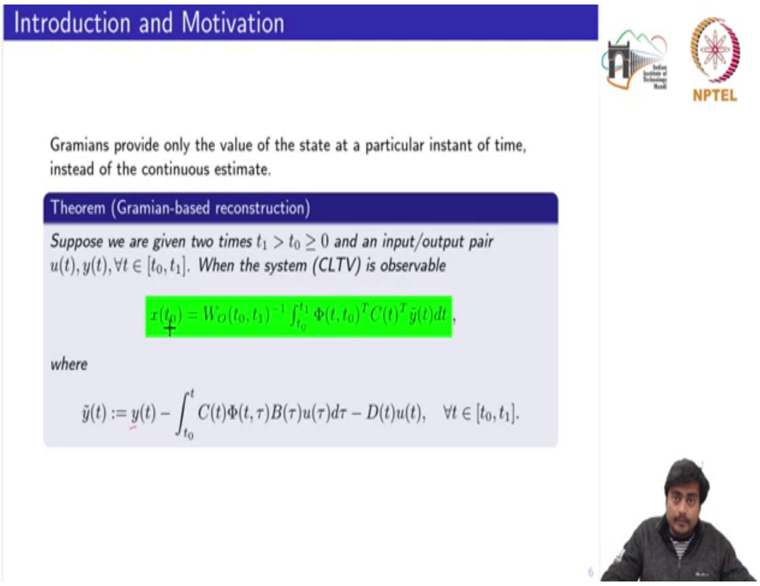


So, if you recall very quickly that the Gramians provide only the value of the state at a particular instant of time instead of the continuous estimate.

And this was the theorem we had discussed that x, that given the data during the time interval t naught comma t 1, we computed x of t naught as the inverse of the observability Gramian multiplied by this integral where we need the information of the state transition matrix which we can compute C is already known to us and y tilde it basically computed by this.

So, here if you see we can directly compute y tilde because we have the knowledge of all the matrices and the signals. So, this formula was written in the time variant case, but this you can also compute for the time in variant case right.

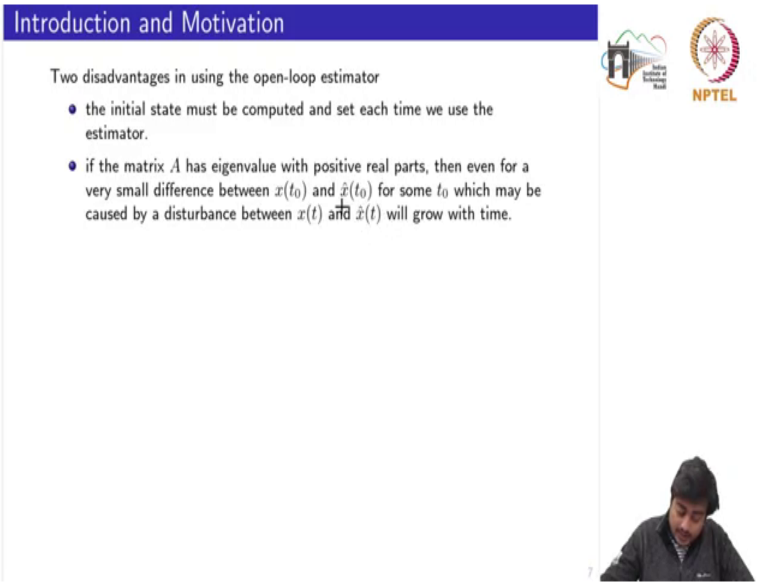
So, if we go back to this problem, what do we need? So, first of all let us see the solution of this x of the plant. So, the solution of x we know that you can compute by e to the power minus A t x naught, ok this would be the solution of this plant and the solution of the let us say the observer it is x hat, this part would remain the same as it is ok.

So, if this integral remains the same, so the value of this one would also remain the same, but the only change you see here is the value of x naught and x hat naught, because this exponential, but also remain the is basically the same. So, I can write this as x as a function of x naught ok and this is x hat as a function of x naught hat. So, I know that if x naught become equal to x hat naught, then it implies that I would definitely have x equal x hat for all time ok.

And, so now, if I use the Gramian based reconstruction of this x naught which I can do, let us say if my data is given from t naught to t 1 ok. Now, say suppose if I want to do a continuous estimate of the state what do I need to do? That every time I have to compute this x naught hat, by taking the data from t naught to t 1.

So, for example, again if I need a continuous estimate for some anther time window let us say t 1 comma t 2 or t 3 comma t 4, then again I need to compute this initial condition x hat of at t 3.

So, that I can ensure that x at t 3 is actually equal to x hat of t 3. And if this big if this condition is satisfied then I would say that x or let us say this implies that x of t becomes equal to x hat of t for all t greater than or equal to t 3. And, so every time if I use this kind of an estimator or we call; this as an open loop estimator also, every time I need to compute this Gramian to compute the value of x hat at that at that time instant ok. (Refer Slide Time: 10:59)



Now, there is another way of doing the reconstruction also say for example. So, this is the first point we will come on to that a bit later, then that some disadvantages in using that open loop estimator, that the initial state must be computed and set each time we use the estimator, otherwise it would not be possible the pure estimation. Now, let us say we have these two equation of the plant and the observer, lets rub this a bit ok.

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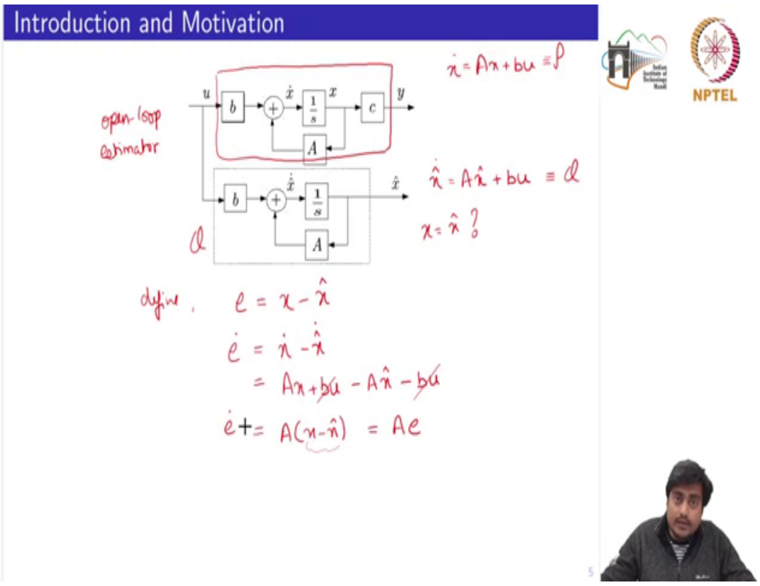

So, here I will define an error signal e is x minus x hat, we defined the signal e which we called the error between the actual state and the estimated state.

So, on taking the derivative of this error I would get x dot minus x hat dot. So, from these two equations I would substitute this x dot and x hat dot and finally, I would have A x plus b u minus A x hat minus b u. So, you see that this b u part get canceled out and the remaining part is A x minus x hat, which is nothing but A e ok. So, we have discussed this number of time that this now becomes a homogeneous system even if there is a mismatch between x and x hat at any point of time ok.

And if this A matrix if the eigenvalues of this A matrix are on the left hand side, then we can say; the error between the state and the estimation of its and its estimation would go to 0 as t times infinity ok. But the extra condition we need to put here the A should be a stable or

Hurwitz matrix, which again might not be possible all the time because if a is a. Say for example, if you are computed the x hat at some initial condition by using this Gramian and you put it here, then what would happen? We have already seen that in that case x would become equal to x hat ok.

So, this part would go to 0 again you would have e dot is equal to 0, but there could be some external disturbances or some uncertainties due to which this difference might not be 0 all the time even if you have used the Gramian base reconstruction. So, because of the non Hurwitzness of this a matrix your error will go towards to the infinity right.

So, this is the second disadvantage that if the matrix A has eigenvalue with positive real parts then even for a very small difference between x of t naught and x hat of t naught for some t naught which may be caused by a disturbance between these two state trajectories and finally, the error will grow with time.

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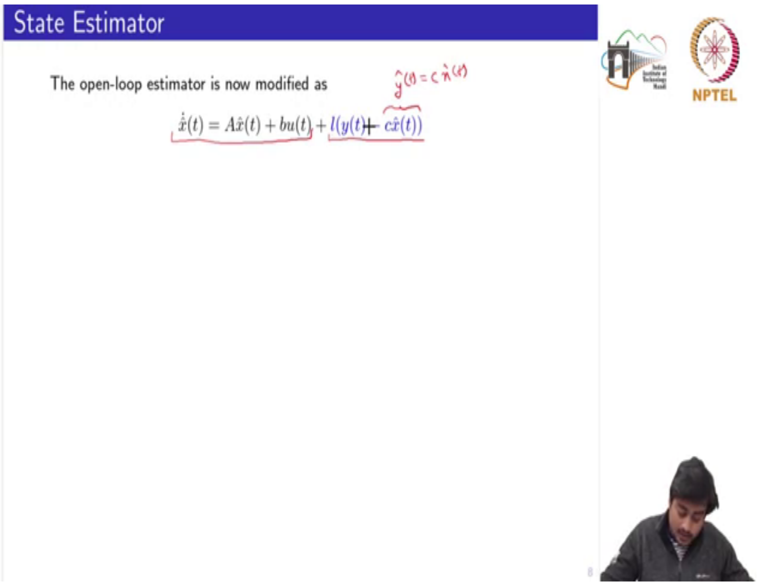

So, now we will modify that open loop estimator by introducing this term. So, this state equation we had already seen which is the open loop estimator ok. Now, in this open loop estimator I added this term where I have introduced a vector l multiply by the output signal minus, you can visualize this as y hat, which is c x of t ok, y minus y hat. So, I introduce this term and this introducing this term will bring some advantages to the estimator.

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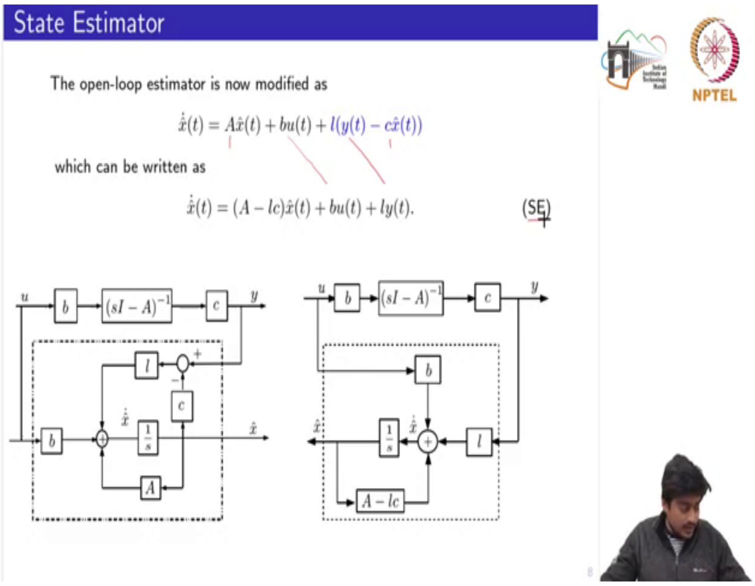

So, if I see its construction. So, this is the plant again, now I have written it in to the by using the complex variable s in the frequency domain. So, if we do not have this part l c here, so in the open loop estimator b we were using this part and only this part ok.

So, now by introducing this term into the open loop estimator basically I have introduced a feedback mechanism into the observer itself, by introducing this vector l. So, this figure is basically the implementation of this equation I can parameterize this equation to extract some nice features of this introduction.

So, after parameterizing, so what I have done? So, once you replace y hat by c x hat I can club this term and this term. So, I would have a minus l time c with a common factor x hat, this term comes as it is and similarly this come appears as it is ok. This I have done so, that I could get the modified state matrix and this state matrix would be now A minus l c right, because this u and y would now appear as a input to this system. And this is the block diagram representation of this equation which we call the state estimator ok.

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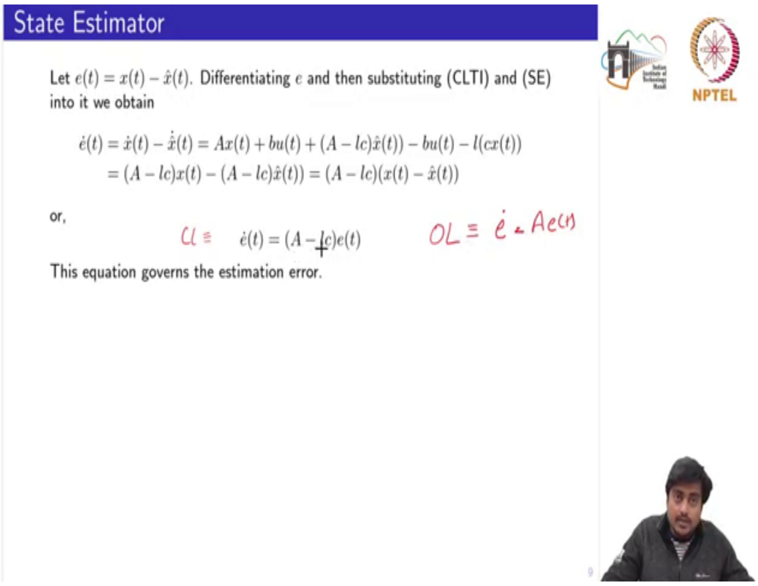

So, we will see what are the benefits of doing of using this feedback mechanism in the observer itself, again we will introduce this error signal as the difference between the actual state and the estimated state. So, using the same procedure I will differentiate e and then substitute the plant equation and the estimator equation into this and finally, obtain basically x dot minus x hat dot. So, x dot is only this part, and x hat dot is this one which is the parameterize one which we had seen in the second figure A minus lc x hat minus b u minus l c x of t ok.

So, I will start clubbing the terms that see here we have A first of all we will rub this, so that there is no confusion. So, first of all we take this term A and minus lc, taking a common factor x now again this a minus lc, we keep as it is ok. And this part bu part we will get canceled out. So, the remaining part if again taking the factor A minus lc common from this we will get A minus lc times x t minus x hat or I can represent e dot is equal to A minus lc time e t and this equation governs the estimation error ok.

So, there is a clear difference in the error equation of the open loop estimator and the closed loop estimator, in the open loop estimator let us note it by open loop, we got e dot is equal to A times e of t ok. And in the this is the closed loop estimator where this A matrix is now modified to A minus l time c

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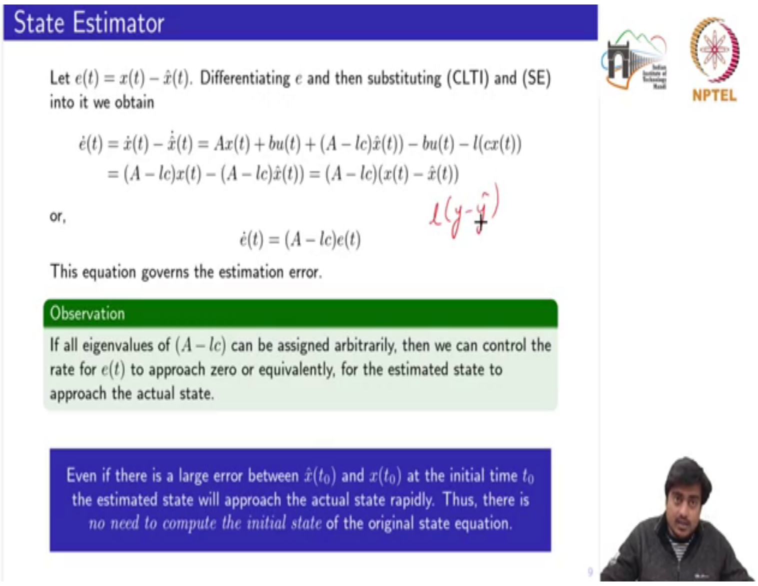

So it say that if all the eigenvalues of A minus lc can be a assigned arbitrarily, then we can control the rate for e of t to approach 0 and equivalently for the estimated state to approach the actual state right. We only need to ensure that the eigenvalues of A minus l c is on the left hand side.

Now, even if there is a large error between the initial conditions of the actual state and the estimated state at some time t naught the estimated state will approach the actual state rapidly now ok. Thus there is no need to compute the initial state of the original state equation.

So, with this introduction of the additional term; which is l times y minus y hat in the open loop estimator, we got rid of the those two disadvantages which we had seen in the open loop estimator.

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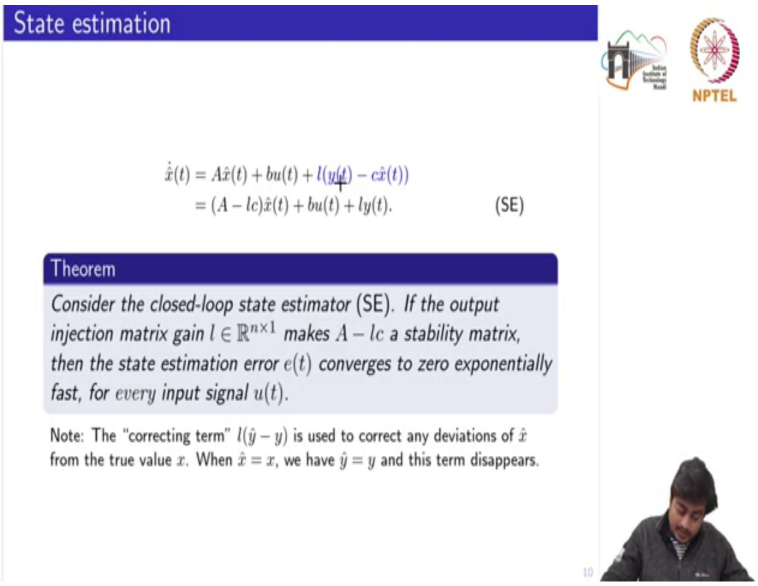

So, let us try to formulate it now. So, this was the original equation by the addition of this term and this is the parameterize one which we called it the state estimator. So, this result says that consider the closed loop state estimator this one, if the output injection matrix gain l. Now, we also call this output injection because we can see it here, that we made the output appear into the state equation itself, through this vector l.

So, we call it the output injection matrix gain l or we can call it a vector also, makes A minus l time c a stability matrix, then the state estimation error e of t converges to zero exponentially fast for every input signal u of t ok. We call it for every signal because we had seen here that the influence of the u signal on to the error signal is nullified or got get canceled that is why. So, we can put any u the state estimation error would not be affected ok, and the proof of this we had already seen that how we had approach to the error dynamics of the observer.

So, note that the correcting term l y hat minus y or y minus y hat is used to correct any deviations of x hat from the true value x. So, when x become equal to x hat we would have y become equal to y hat and this term also disappears ok. So, the only resting part is this plant and this state equation which is nothing, but the plant equation because we actually want x equal x hat.

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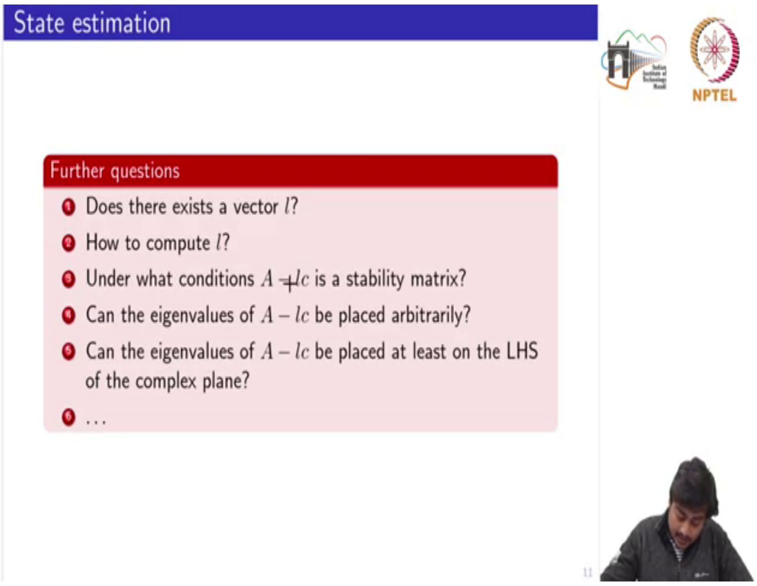

So, the though we had reduce some of the disadvantages we had seen in the open loop estimator and we had seen that if this we can ensure if there is or let us say if A minus l c is a stability matrix then the error will converse towards to zero, but there are further questions which are raises after that formulations.

So, first of all we need to answer does there exists a vector l? Right because if there does not exist a vector l you cannot introduce that term, and if you cannot introduce that term you are observer would still be an open loop observer. The second, if we have ensure the existence of the l we need to compute the l. The third point at under what conditions we can say that A minus lc is a stability matrix?

Fourth the can the eigenvalues of a minus lc be placed arbitrarily or under what conditions we can say that the eigenvalues of a minus l c be placed arbitrarily? Now, this is related to the third part the fifth one that can the eigenvalues of A minus lc be placed at least on the left hand side of that complex plane? Ok. Now, there are further questions also which we will see as we go ahead into the lecture.

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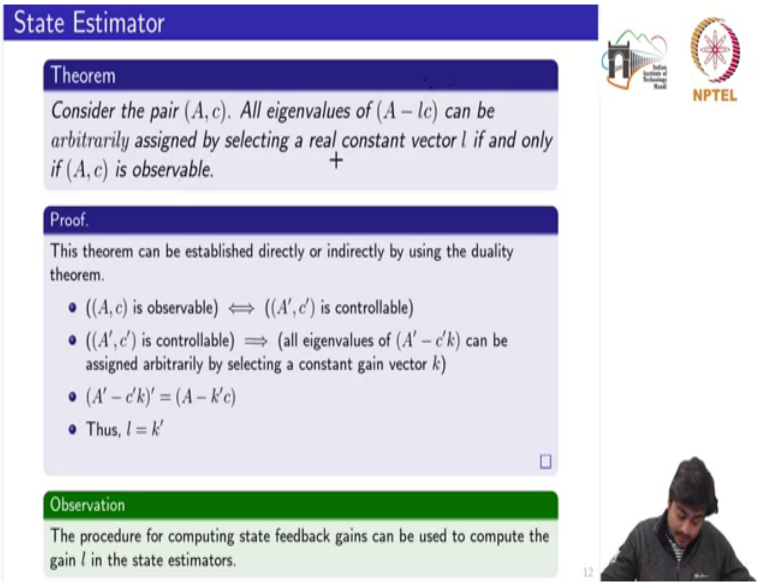

So, first of all we will see the one of the key results which ensure the existence of this vector l and it has some connection with the observability which we had discussed in the seventh week. So, consider the pair A comma c, all eigenvalues of A minus l c or this modified or the error matrix can be arbitrarily assigned by selecting a real constant vector l if and only if A comma c or this pair is observable right.

So, we can see a quick proof that this theorem can be established directly or indirectly by using the duality theorem which we have introduced last week also on the 7th week. So, it says that if this pair is observable. So, we know that the observability of this pair A comma c is equivalent to saying that the this pair A transpose comma c transpose is controllable.

Now, the controllability of this pair A transpose comma c transpose implies that all eigenvalues of A transpose minus c transpose k can be assigned arbitrarily by selecting a constant gain victor k and this is nothing, but a state feedback controller design ok. So, this result is taken from that week.

Now, if I take that transpose of this matrix which I got by introducing this vector k I will get A minus k transpose c. Now, you will notice that this A minus lc is equivalent to A minus k transpose c, if we have l is equal to k transpose right, and k we know already we have designed by using the state feedback design procedure right

So, the procedure for computing the state feedback gains can be used to compute the gain l in the state estimators right. So, with this theorem we have ensure we had guaranteed to two things; first the there exist any l and under this condition that the pair a comma c is observable this l would definitely exist. And at the same time we have shown that how you can compute the l ok, by using the procedures we had discussed during the state feedback controller design.

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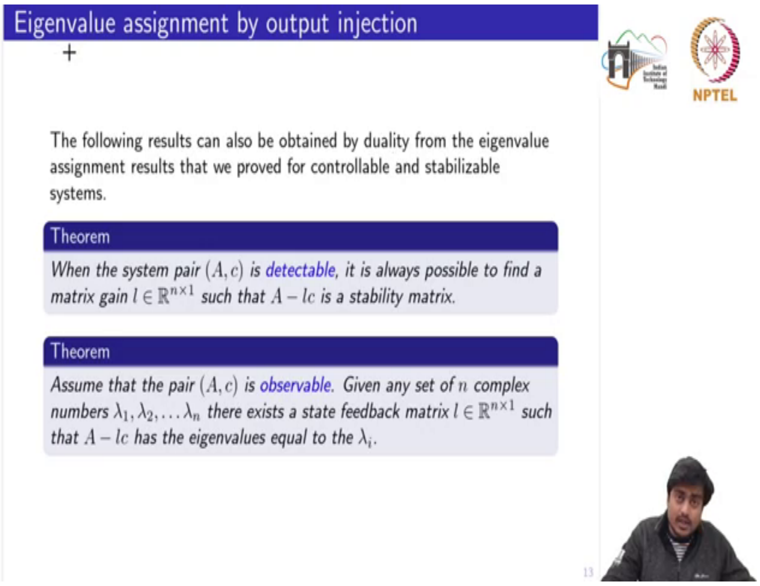

So, we can quickly see those others. So, one of the method you would see or you would recall is the eigenvalue assignment approach. So, the following results can be obtained by duality from the eigenvalue assignment, that we proved for controllable and stabilizable systems

So, first we discuss or let us say first we will see the observable case that assume that the pair A comma c is observable, we need the observability, otherwise they would not exist any l. So, assume that this pair observable given any set of n complex numbers lambda 1 to lambda n there exists a state feedback matrix l such that A minus lc has the eigenvalues equal to the lambda i right.

So, this result ensures that given any eigenvalues, we can design or there exists this l such that the eigenvalues of this matrix becomes equal to the given, these complex numbers.

Now, if we have a weaker condition then the observability which is the detectability. So, let us say if the pair A comma c is detectable it is still or always possible to find the matrix l such that A minus l c is the stability matrix ok. So, under both the cases either we either the pair is observable or the pair is detectable they would always exist any l ok. With this observability we get some additional freedom that we can place any eigenvalues to the desired location, but with the detectability we at least ensure that the this error matrix would be a stable matrix ok.

And, we had discuss the method of computing this where we have assigned the eigenvalues right, so you can see that method for computing this a vector l in a similar way what when we had designed the state feedback vector ok.

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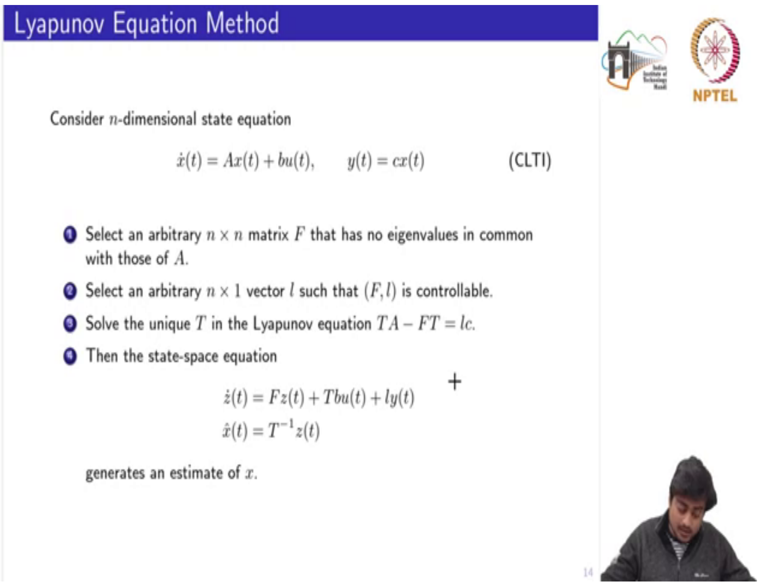

The second method if you recall is the Lyapunov Equation Method that consider this n dimensional state equation or the pair A, b, c which we have labeled this is the CLTI. So, the first step; in the first step we need to select an arbitrary n cross n matrix F that has no eigenvalues in common with those of A ok. Second I can arbitrary select a vector l ensuring that whatever the matrix F I have selected arbitrarily in the first step pair with the vector l is controllable ok.

Now, using this Lyapunov equation I will solve for the unique T which is TA minus FT is equal to l c. So, then the state space equation which is given by z dot is equal to F z plus Tbu plus ly would be the internal state of the observer now ok, and x hat would become the output of the observer. So, x hat is equal to T inverse z. Now, we know from the Lyapunov equation that whatever the T we would get it would be a non singular ok.

So, this is one state space system with the state equation and the output equation on the plant with this Lyapunov equation method by following these three steps, we can write at another state space system for the observer whose internal state is z and the output is x hat, and this x hat would be equal to x right.

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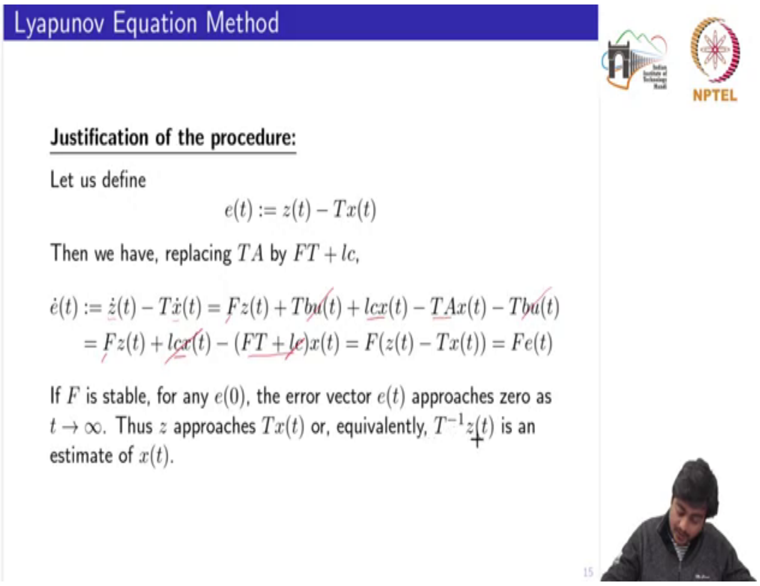

So, we can quickly see that whether the whatever the observer we got by using that Lyapunov equation method is actually a or would lead mean to the state estimation error equal to 0. So, here we will define the state estimation error is equal to z minus T times x ok where, z is the internal state of the observer and x is the original of the plant.

So, defining this and following the procedure the similar procedure that we get we first of all take the derivative to of this error. So, we will get z dot minus T x dot, then here I will replace this z dot and x dot by their respective equations. So, z dot is basically F z plus Tbu plus lc x or l y and T times x dot. So, T would be pre multiplied with this complete equation which is A x minus bu which is x dot.

So, again I will start pairing the terms, so, the first term will come as it is. Similarly this I wrote this is second term lc x. So, here you would see again that the influence of the input has been canceled out completely on to the estimation error on the error dynamics ok.

So, whatever the observer we have synthesize here again it would be come it would the state estimation error would be 0 for any u right, but still we need to show that the state estimation error goes to 0 ok. Now by using this Lyapunov equation, we have replace this TA part by FT plus lc ok, that because if you see that the Lyapunov equation is TA minus FT is equal to lc. So, this term we got in the error dynamics and I am replacing it by FT plus lc.

So this TA has been replaced by FT plus lc and times x t, so all the terms we had seen now you would see that this term lc x again got canceled, and the remaining part is F z minus T x which is nothing, but my F times e which we have defined earlier z minus t x t ok. Now, if you in the design procedure itself we have ensure that F is a stability matrix. So, if F is stable then for any e of 0 the error vector e of t approaches zero as t times infinity. Thus z approaches to T a of x t or equivalently T invers z is an estimate of x and this was the observer basically T invers z.