Linear Dynamical Systems
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Week – 01
State-space solutions and realizations
Lecture - 04
Equivalent State Equations

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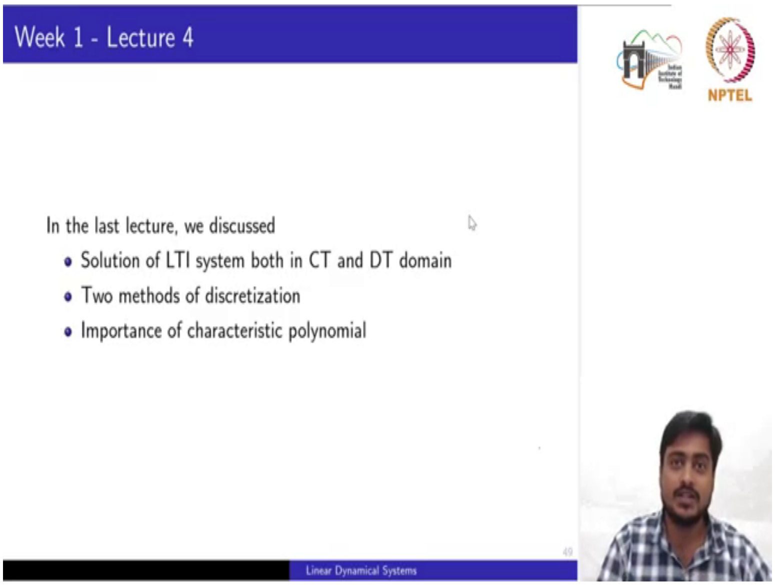

So, hello everyone. Today, we will be seeing the lecture 4 of the week 1. So, in the last lecture, we discussed about the solution of the LTI system both in the continuous time domain and in the discrete time domain. We also saw the two methods of discretization, where in the first method. We did the approximation while in the second method, we consider the input as a piecewise constant signal. Third we also discuss the importance of the characteristic polynomial and its eigen values.

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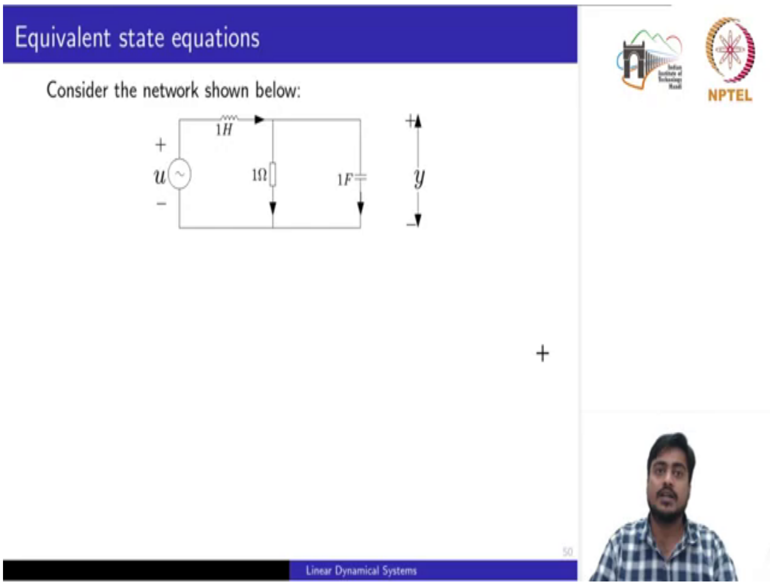

So, today, we will start with the notion of equivalent state notations. First of all, we will see the what is the problem statement. And then we will see the solution or the and the definitions of when can we say that this is two states equations are equivalent. Consider this network which is shown in the figure there we have a three elements the registers and the capacitors are connected in parallel in series with the inductor.

So, all these variables the resistance capacitor and the inductor are having the value of 1. Input u is defined by the input voltage and the output y we are collecting or we are seeing the output as the voltage across the capacitor. Now as we also saw in the first lecture of defining the state variable, similarly there are two ways or there are multiple ways or defining the state variables.

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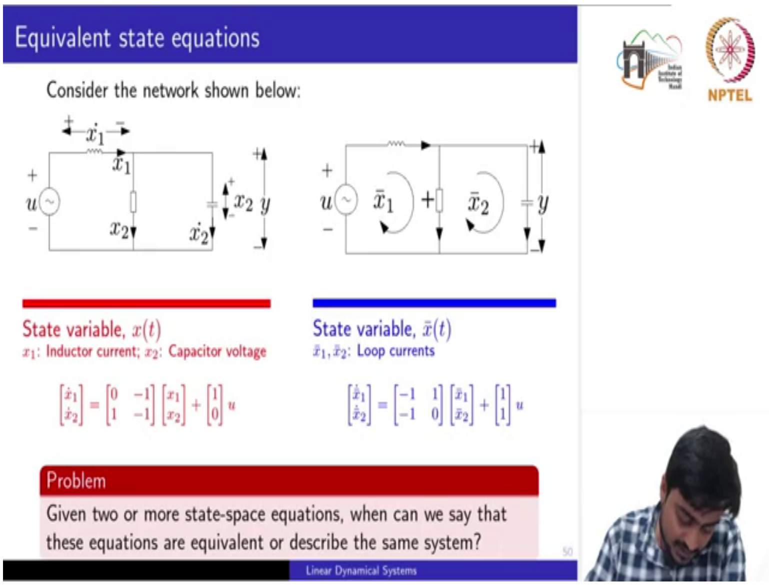

So, let us see these two set of state variables. So, the circuit remain the same. Here in this let us denote this first figure and let us denote this second figure ok. So, in the first figure, we denote; we define two state variables x 1 and x 2, where x 1 is the current flowing inside the inductor in this branch. So, naturally the voltage across this inductor is given by the derivative of x 1. Second the x 2 which is the voltage across the capacitor.

So, here you are saying that x 2 is flowing inside this branch which is the register. So, if I compute the voltage if I compute the voltage across this resistor which is equal to x 2. In the same voltage would also be across the capacitor. So, that is why we are denoting x 2 as the capacitor voltage or the current flowing inside the resistor branch ok.

In the second figure, we define the state variables x 1 bar and x 2 bar, which are basically the loop currents in this first loop and in this second loop. So, you can compute the voltages now, across the inductor the resistor and also the capacitor right. Now using these two different set of state variables, we can write two different state space representation. By the state space representation we mean the a b c d matrices. Let us denote this matrix A this matrix B similarly let us denote this A bar and B bar right.

So, if we pay attention to this first set of state variables, we know that x 1 dot is given by minus x 2 plus u. Which is basically we are solving the loop equations for both the circuits by considering the x 1 is the state variable and x 1 and x bar as a state variable. So, the here the question arises that given two or more state space equation, then can we said that these equations are equivalent or describe the same systems. Now naturally we see that given a circuit, we can define we have defined two different state space representations. So, we know at the outset that these two representation basically describe the same system.

Now we will take the converse problem in the sense given any two representations any two state space representation, how we can ensure that these two state space representation basically describe the same the dynamics of the same system ok. So, this is the problem we would address today.

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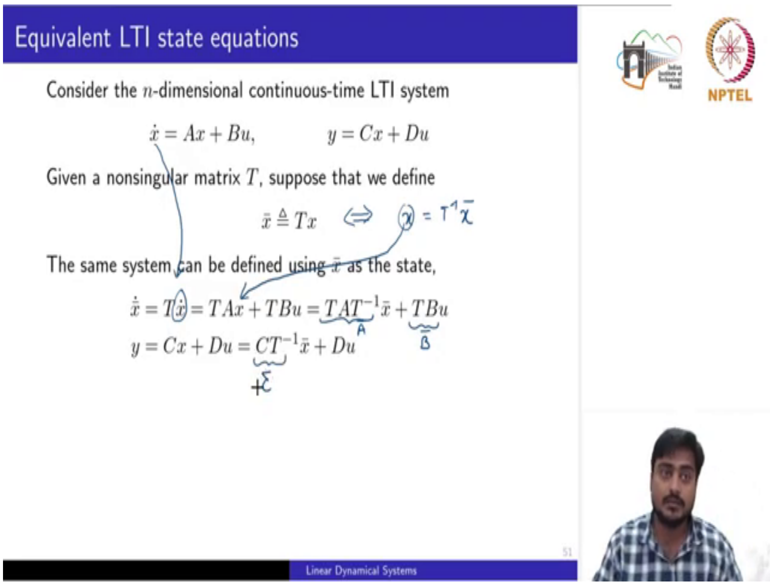

So, first we will define this notion very quickly in an informal way. Let us say we have the n dimensional continuous time LTI system. So, first we will take the discussion of the LTI systems. And then we will move towards the LTV case. So, we have the state equation given by this x dot and the output equation is given by this output equation.

Suppose we are also given a non singular matrix T which satisfy this equation. So, we define a variable x bar is equal to the multiplication of the non singular matrix T and the state variable x. So, let us take the first derivative of this part. So, we would have x bar dot is equal to T into x dot. So, here x dot we will replace by this state equation the original state equation.

So, by putting x dot from here to here we get T Ax plus T Bu ok. Now from here, we also know that since T is non singular the inverse of this matrix exists and I can present x equal to

T inverse of x bar ok. So, again substituting this x here, we would get T AT inverse x bar plus T Bu ok. Similarly we do the same we replace this x by this x. So, we get C into T inverse into x bar plus D into u ok. Now you see that this whole matrix whole multiplication I can represent by A bar this one by B bar and this one by C bar.

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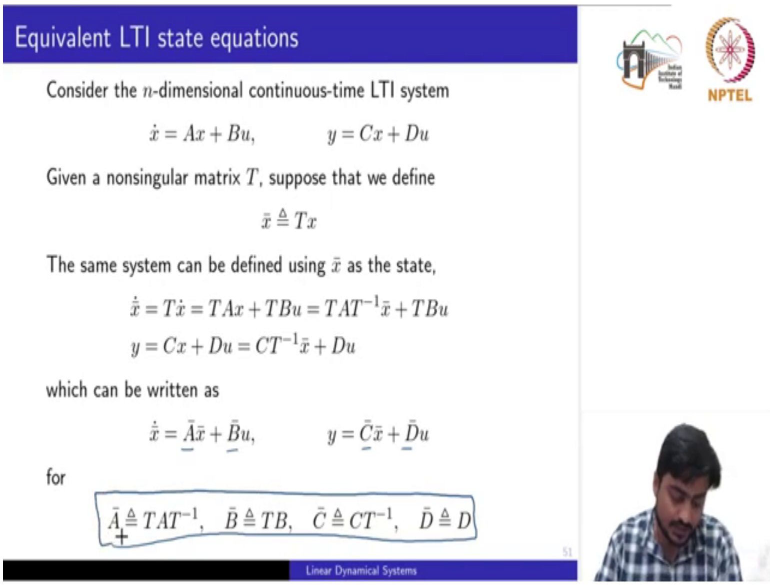

So in fact, I would obtain that x that I can represent another state space representation in terms of this A bar B bar C bar and D bar right for all these relationship. Now let us define this formally.

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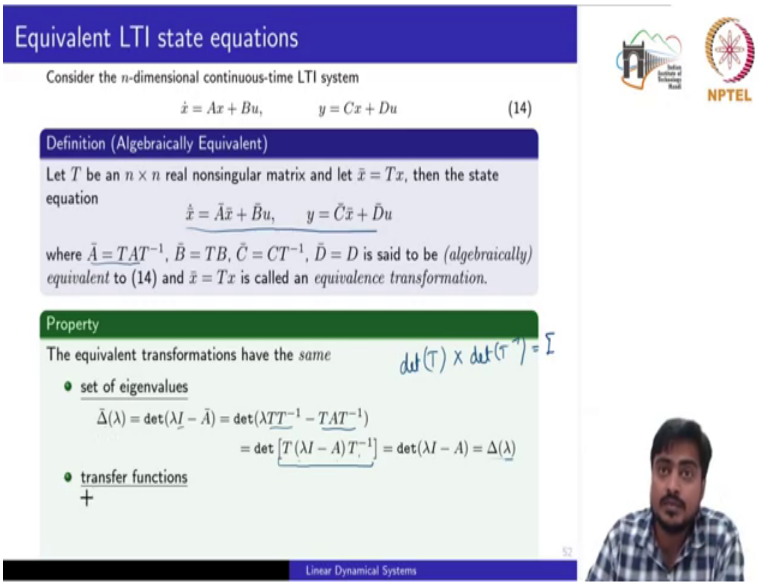

So, consider this again this n dimensional continuous time LTI systems given by the same equations. So, we define, we give first of all we give the definition of the algebraic equivalence in the sense. Let T be an n cross n real non singular matrix. And let we defined x bar is equal to T into x. Then the state equation which is defined by another state space representation A bar B bar C bar D bar, where all these A bar B bar C bar D bar satisfy hese relation.

Then we say that these two representations are algebraically equivalent. And this equation x bar is equal to T x is defined as the equivalence transformation ok. Now that the next question arises, if you have two different state space representation. Now what would happen to their eigenvalues and also the transfer function does the eigen values remain the same and the same goes with the transfer functions.

So, the equivalent transformations have the same set of eigen values and also have the same transfer functions. So, we can see quickly the derivation of the these two let us say we define the characteristics polynomial by delta bar of lambda of this representation ok which is given by the determinant of lambda I minus A bar right. Now if I replace this A bar from this equation.

So, we get this T AT inverse and I replace I by T into T inverse ok. Since the T is non singular, the inverse exists in multiplication it or pre multiplying or post multiplying with the T matrix it would give the identity matrix ok. Now I rearrange these equations finally, give me this part T lambda I minus A into T inverse ok. Now see here this whole part I can write as the multiplication of determinant of T into determinant of this part into determinant of T inverse right, but I know that the determinant of T into determinant of T inverse would be equal to the identity matrix ok.

So, finally, I would get determinant of lambda I minus A which is equivalent to the characteristics polynomial of the original state space equation. So, we if the characteristic polynomial remain the same. Then the eigen values would also remain the same.

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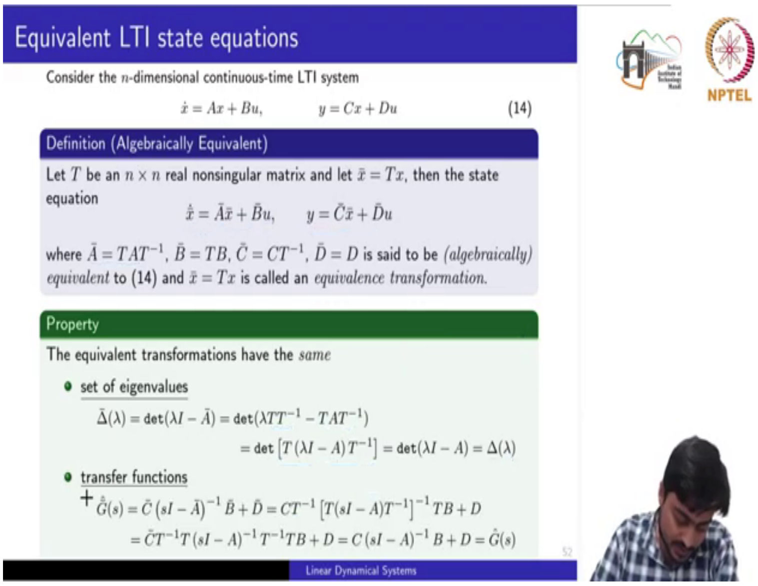

Similarly, for the transfer functions, let us denote the transfer function of the representation defined by A bar by the bars. So, this is given by C bar s I minus A bar inverse B bar plus D bar. Again if I replace all these bar matrices into using these equations, I would get this and after some simplification finally, I got this you can show. In fact, you can show it by yourself this way. So, we see that the transfer functions also remain the same, together with the eigen values.

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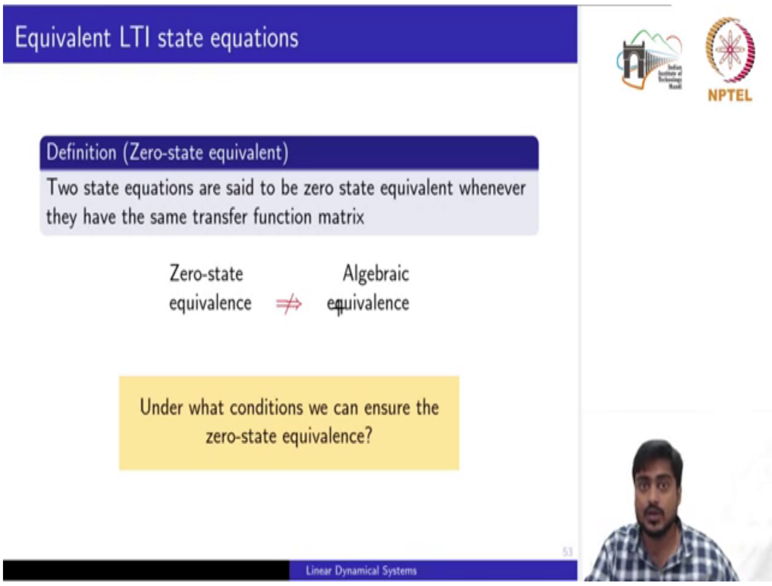

Let us introduce another definition, what we call the zero state equivalence. So, in the last couple of lectures, we have also defined what do we mean by the zero state equivalent. So, this is a natural definition, the two state equations are set to be zero state equivalent, whenever they have the same transfer function matrix.

And so, now, we have introduced two definitions, one is the algebraic equivalent another is the zero state equivalent. So, the natural question arises is there any relationship between the zero state equivalence and the algebraic equivalence meaning to say. Whether zero state equivalence implies algebraic equivalence or algebraic equivalence implies zero state equivalence or they both are equivalent right.

So, the first implication is that the algebraic equivalence always implies that the system or the two representation would have the zero state equivalenceand because it is the property of the algebraic equivalence that they would have the same transfer function ok. And we have defined a zero state equivalent in the sense that they would be having the same transfer functions.

So, this implication would always hold right, but we cannot say that this the reverse implication would work in the sense that the zero state equivalence may not implies that the system would be algebraic equivalence ok. So, under what conditions we can ensure the zero state equivalent that the two state space representations would have the same transfer functions or the zero state equivalence.

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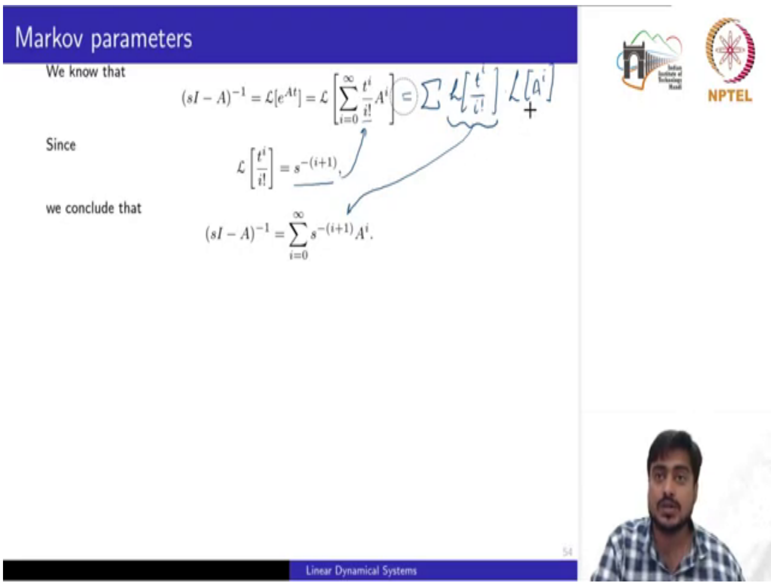

So, before answering that question first of all we will introduce the Markov parameters let us see. So, we know that the inverse of s I minus A is basically the Laplace inverse of the exponential matrix or the matrix exponential that is e to the power At and this e to the power

A t I can represent as an infinite series of this entity ok. Starting from I is equal to 0 to infinity ok.

Now I know this property that I can write this the Laplace or in fact, I can write this as the summation of Laplace inverse of t I by I factorial into Laplace of A i right using the property of the Laplace transform. So, first we will take this term ok. So, since this part can be if I take the Laplace transform of the ratio t i and i factorial it is basically given by this one ok.

So, putting this back to here, we get the inverse of the s I minus A is equal to this part. So, we just replaced this part or let us say to be precise this part by this part and since A to the power i is a constant matrix the Laplace would be the same ok.

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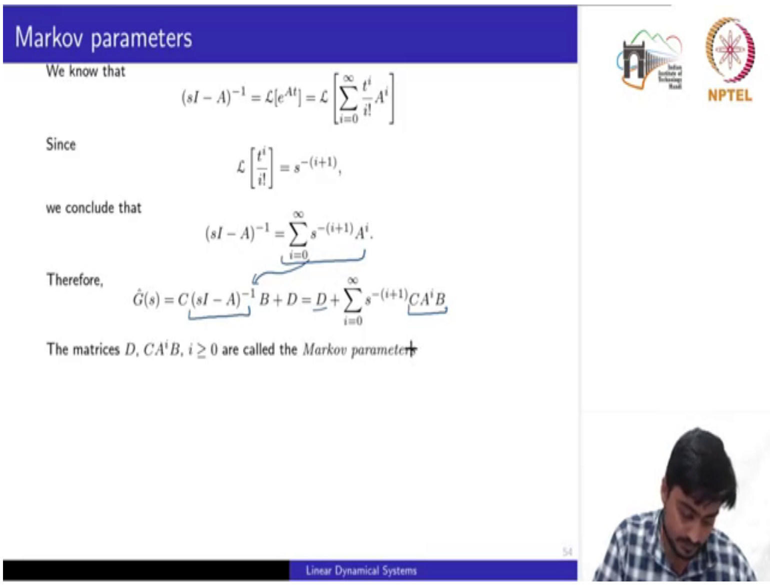

So, let us see the transfer function G hat of s which is given by C s I minus A inverse B plus D. So, which I can write by replacing this part by this part ok. So, I would have D plus summation of s to the power minus I plus 1 into C A i into B since s is a scalar I can compute ok. So, all these matrices which are denoted here as D C A i C into A to the power i into B they all are called the Markov parameters.

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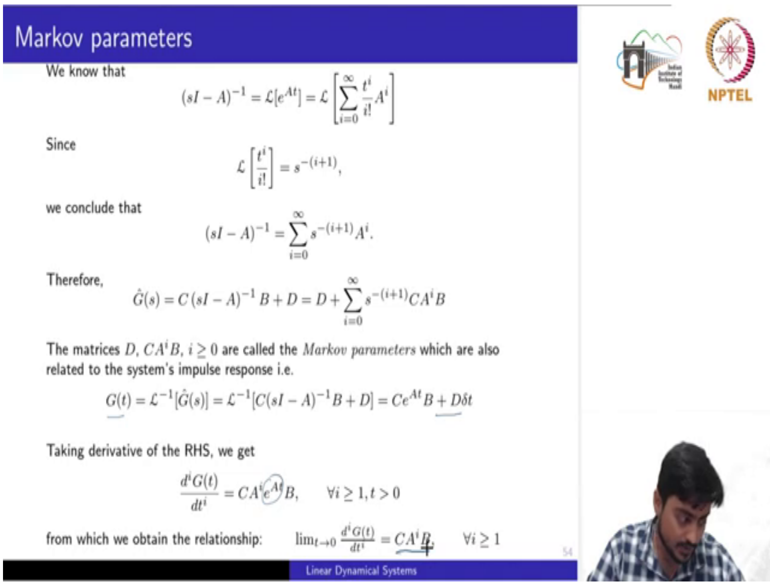

Now, these Markov parameters, we can also relate to the impulse response of the system. So, we know that G of t is basically given by the Laplace inverse of G hat. So, if I just replace G hat by this part I would get this. And this C s I minus a inverse B can be represented by C into e to the matrix exponential B plus D into delta t ok.

So, if I take the i th derivative both sides of this and this I would get C into A to the power i into Markov exponential B. So, for all I greater than equal to 1 and T greater than equal to 0 ok. From which we so, if I take the limit both sides. So, this part when t tends to 0 this part would go towards to i and I would remain this one which is also the Markov parameters ok.

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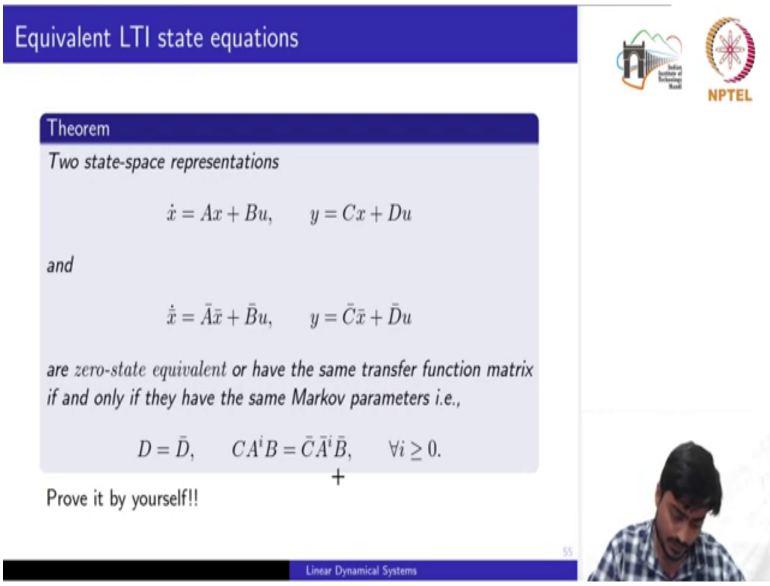

So, after seeing the definitions of the Markov parameters, we can now answer our the question what we have raised in the previous slide. That these two representations the first given by this one. Another is given by the bar matrices. They are 0 state equivalent or they have the same transfer function matrix.

If and only if they have the same Markov parameters and that is D is equal to D bar and Cinto A to the power i into B is equivalent to their bar counterparts ok. For all i greater than equal to 0 ok. So, this I think you could the proof of this result you can do it by yourself this is pretty much straightforward.

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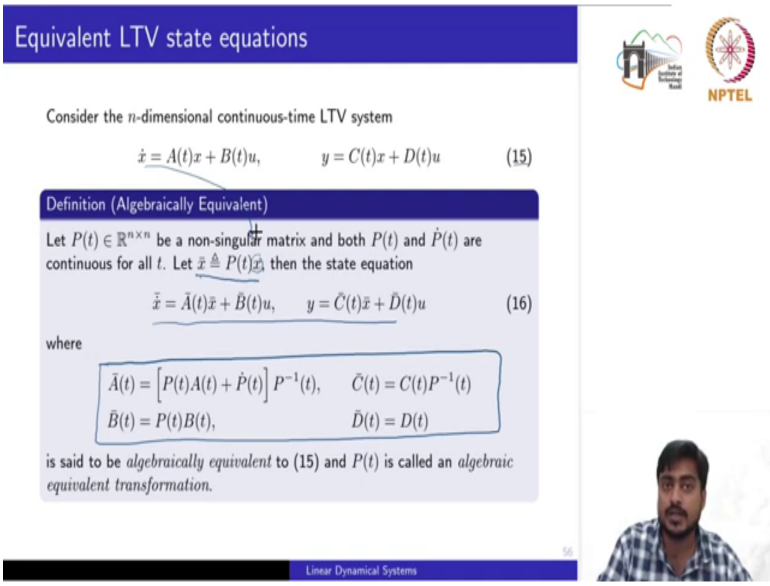

Now, moving towards to the LTV case, we if we now we know that in the LTV systems all these matrices would be time dependent matrices ok. So, similarly first of all we define the algebraic equivalence. So, here we define a non singular matrix s P of t which is also a n cross n matrix. Now this P of t also satisfy also assume to satisfy these two property that the P of t and the derivative of the P of i is continuous for all time t ok.

Now again in a similar way we define this transformation as x bar is equal P of t into x. Then we say that these equation is said to be algebraic equivalent to this one, whenever all these A bar B bar C bar D bar satisfies these equations. And P of t in that case would be said the algebraic equivalent transformation ok.

So, you can show this by yourself very quickly in a similar way what we have done for the LTI system. That if I put this part starting from this point then I will take the derivative of this part and the derivative of this I would replace by this one and similarly you would obtain these bar counterparts ok.

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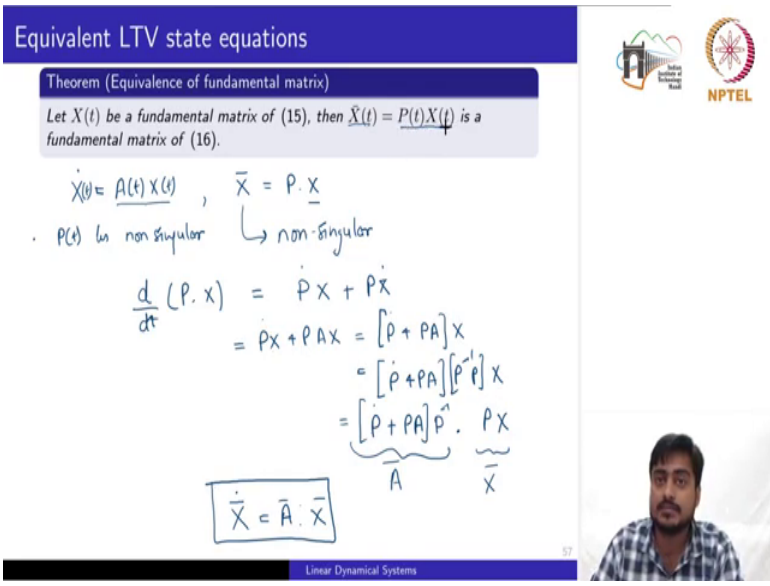

So, now what happens to the fundamental matrix because we saw that in the LTV case. The notion of the fundamental matrix played an important role. So, if the two state representations are equivalent are the fundamental matrix remains same or they would change. So, we have the next result let X of t be a fundamental matrix of 15. Now this 15 is the original state space without the bars.

So, then this X bar which is defined by the multiplication of the transformation matrix into the fundamental matrix of the original state space representation would be a fundamental matrix of the transformed system basically this part 16 ok. We can show this result very quickly in the sense. So, let us say we know from one of the properties of the fundamental matrix that the derivative of this fundamental matrix is basically equal to A of t into X of i ok.

So, we know that Xx bar is equal to P into X. So, whenever possible, we would not be using the time as the argument. So, it is implicitly clear that I m talking about the X bar of t is equal to P of t into X of t ok. So, we know first point that P of t is singular oh sorry is non singular and since the as A 1 of the properties of the fundamental matrix that X is also not singular.

So, if I multiplied two non singular matrices, the multiplication would also be non singular. It would not affect the property of the non singularity. So, we are clear that X bar would also be non singular ok. Now taking the derivative of P into X which would be given by P dot X plus P X dot. Now replacing this X dot by this equation, we would get P A into X. Again all the arguments of t are implicit, which I can write as P dot plus P A into X.

By taking X common from the both the terms. Again introducing the inverse, the multiplication of the P inverse and P which is given by the P inverse and P X which supposedly should be identity matrix ok. So, this I can separate both the terms at P X ok. And this part we know is given by A bar if we see the previous slide that A bar is defined as the multiplication of P A plus P dot multiplied by P inverse.

So, this time P dot plus P A in multiplied by P inverse is A bar. And this we have already defined here which is X bar ok. So, it would satisfy that the derivative of X bar is equal to A bar into X bar basically the same equation meaning to say that X bar is a fundamental matrix defined by the multiplication of P and X ok.

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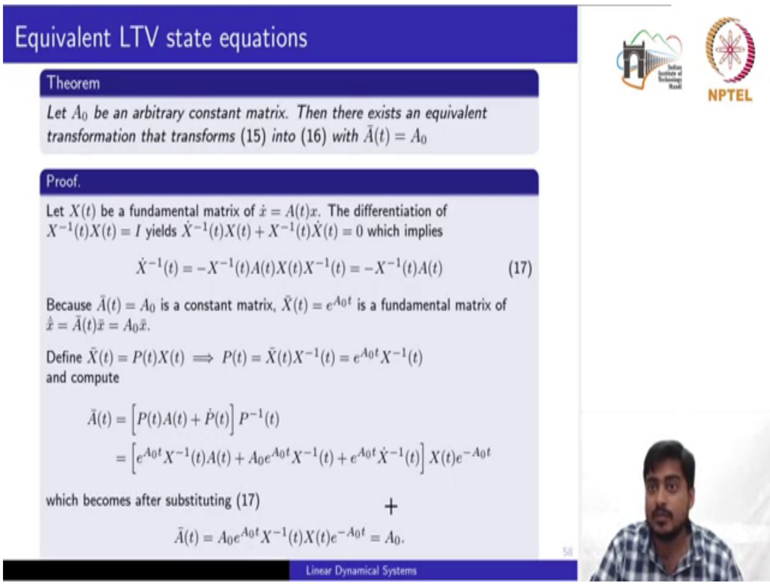

The next result says, that let A naught be an arbitrary constant matrix. Then their exists an equivalent transformation that transforms 15 into 16 with A bar is equal to A naught. This 15th number equation is the original state space equation and 16 th is the equation with the bars ok. Now there is this is one of the very important results because what happens here. That this A bar matrix which is supposedly to be a time dependent matrix I can replace that matrix with a constant matrix A naught ok. So, we will see the proof of this one.

So, first of all let X t be a fundamental matrix of X dot is equal to A t into X. Now if I take the derivative of this part I know that X is A the X matrix is a non singular matrix. So, this X inverse into X satisfy this I. If I take the derivative both sides, I would have the X inverse derivative into X plus X inverse into X dot is equal to 0 right. Which implies that I can express X inverse derivative by this part and finally, minus X inverse into A ok.

So, since because A bar is equal to A naught which is a constant matrix. If I because it is no longer a time dependent matrix. So, the fundamental matrix would be a matrix exponential and this is what we also saw, when we were first when we first computed the solution for the LTV system and then we tailor it for the LTI system right.

So, the fundamental matrix of the transfer system is basically given by the matrix exponential e to the power A naught t. And the idea here to replace this A bar t by a constant matrix A naught ok. So, again recalling the result what we had seen in the last slide, that X bar is equal to P into X. I can express P of t as the multiplication of X bar into X inverse. Where X bar I know that it is the matrix exponential. So, my P of t is basically the matrix exponential into the inverse of X. And then we compute this A bar which was given by P into A plus P dot into P inverse ok.

Now, here P of t is replaced by this part and after doing some simplification which you can look more closely. We finally, arrive that A bar t is actually equal to A naught. So, this is one of the important results in the sense, that I can replace the state matrix of the time varying system into a time invariant system; because for the LTI case we have the state matrix as the time invariant matrix which does not depend on the time.

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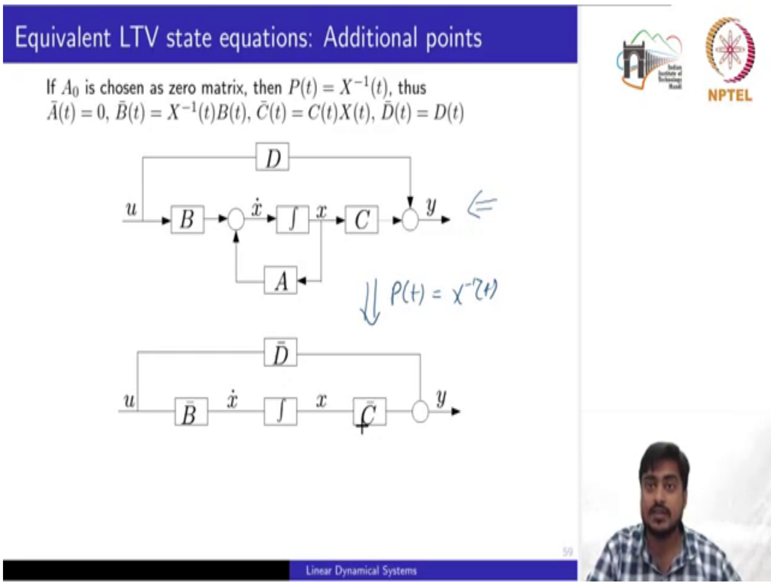

So, if A naught is chosen as a 0 matrix. Then P of t is basically given by the inverse of X. And the I can replace A bar B bar C bar D bar by these equations. Where A naught is my 0 matrix. So, if this is my original state space representation. The time varying state space representation I can transform it into this part by using this P of t is equal to X inverse of t, where there is no longer an A matrix ok.

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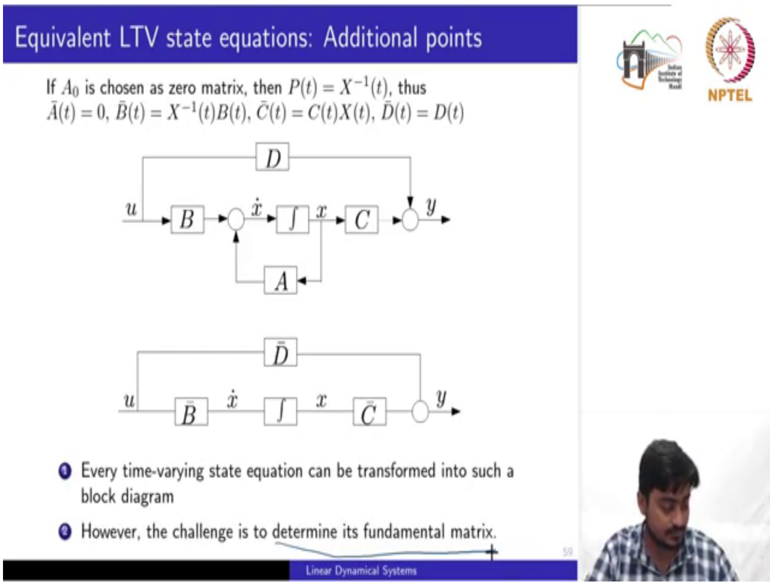

So, every time varying state equation can be transformed into such a block diagram, where you can replace the time dependent matrix into a constant matrix, but the most important thing is that the solution or basically this transformation relies on the computation of the fundamental matrix. And this we have already seen while computing the solution of the LTV system that computing the fundamental matrix is quite challenging the invariance of the impulse response matrix.

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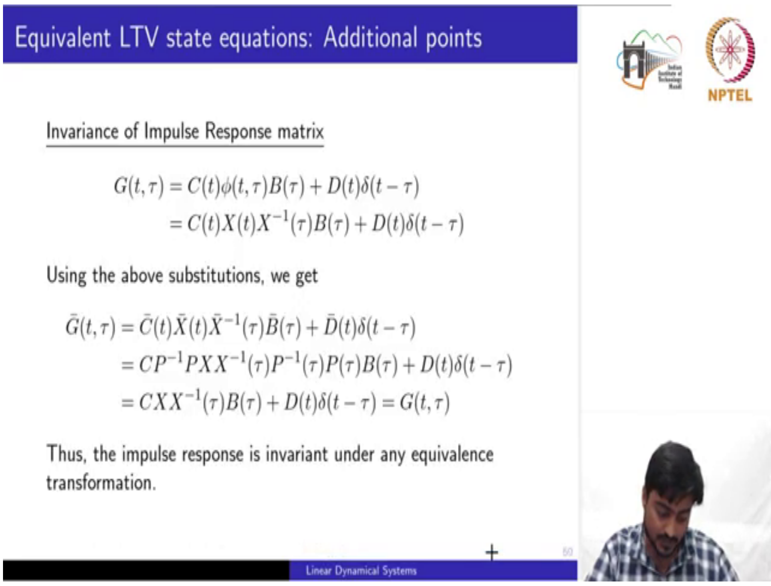

Say for example, if this G t comma tau defines the original impulse function. The impulse function of the original state space equation, where we have replaced this phi the state transition matrix by the multiplication of capital X t into inverse of capital X into tau ok.

This is what the solution of the state transition matrix. So, if I use all the if I write that impulse function or the transfer impulse function of the state space representation given by the bars and replacing all those bars by their counterparts. Finally, I would have the equivalence with the original impulse function. So, meaning to say that the impulse functions or the impulse response matrix is also in variance under this transformation.