Linear Dynamical Systems
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Week - 07
Observability and Minimal Realization
Lecture – 39
Minimal Realizations

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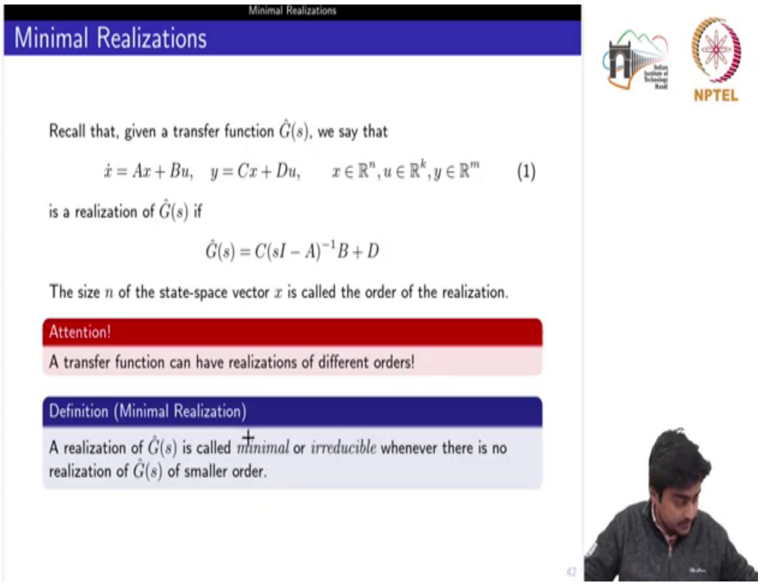

So, now we will discuss the concept of Minimal Realization. So, if you recall that the concept of realization, we had started in the week 1; where we basically gave one important result that any representation ABCD is realizable if and only the transfer function is a proper action function ok.

So, now, after introducing the concepts of controllability and observability, we can define the minimal realization because at the outset we know that for a given transfer function, there could be infinite number of realization. And by realization we set bb that the trans the given a transfer function G hat of s we say that this ABCD pair is a realization of G hat.

If G hat is equal to this one, which is supposed to be a proper rational function ok. Now in that week 1 if you recall that we have given one specific realization, in one of the canonical forms for a given transfer function and that was controllable canonical form. And now after seeing the duality between the controllability and the observability, we know that similar to what we have defined as the controllable canonical form. We can also give a realization in observable canonical form and similarly there could be infinite number of realization.

Now, on the top of that, the realization could have different orders also by different orders that we mean to say the dimension of the state vector x ok. Here we have chosen n, but it could be greater than n or could be lower than n right. So, first of all we define what do we mean by the minimal realization. That a realization on G hat s is called minimal or irreducible whenever there is no realization on G hat s of smaller order.

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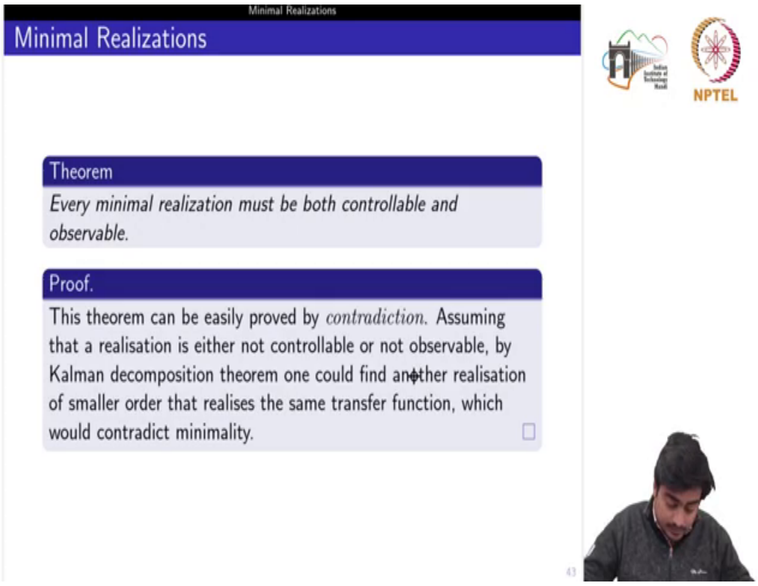

So, this is one of the key result that every minimal realization must be both controllable and observable right. We can see the proof of this theorem by contradiction.

So, suppose that our realization is either not controllable or not observable ok. So, we had seen that by Kalman decomposition theorem we can extract only the controllable part and the observable part. And we know at the same time that this extraction would lead to the dimension of the state vector which would be lesser than the original state equation ok.

So, using this decomposition theorem we could find one another realization of smaller order that realises the same transfer function right which would contradict the minimality. So, there are many other implications of this. So, for example, we know that let us say we have two realizations ok.

Now these two realization we had studied that then we could say that they are algebraically equivalent by using some similarly transformation, but we know at the same time that the transfer function would be the same. Even if the realization is of higher order, the transfer function would also remain the same because transfer function is unique ok.

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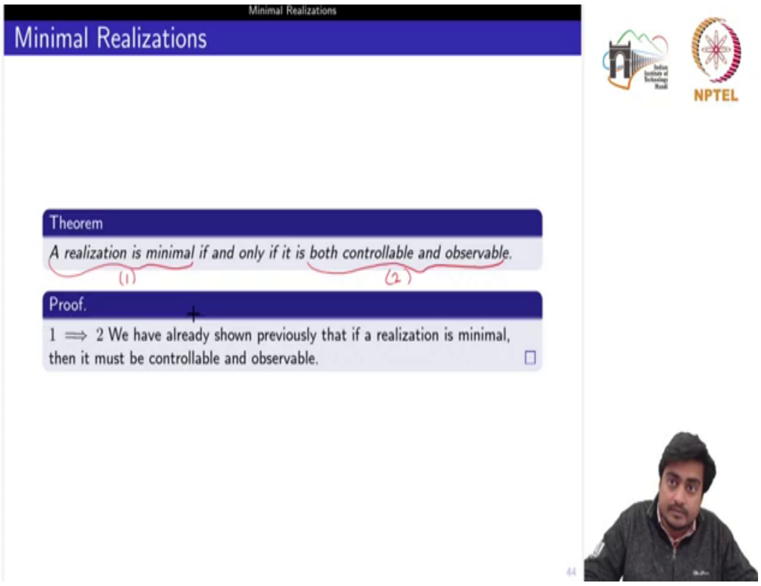

Now, the another important result is that a realization is minimal if and only if it is both controllable and observable. So, the first statement what we are saying is that the realization is minimal this is our first statement.

And the second statement is that the realization is controllable and observable. So, when we say 1 implies 2 meaning to say if the realization is minimal, then it should be controllable and observable and this what we had proved already in the previous result by contradiction. Now we need to show that if the realization is both controllable and observable, then that realization is minimal ok.

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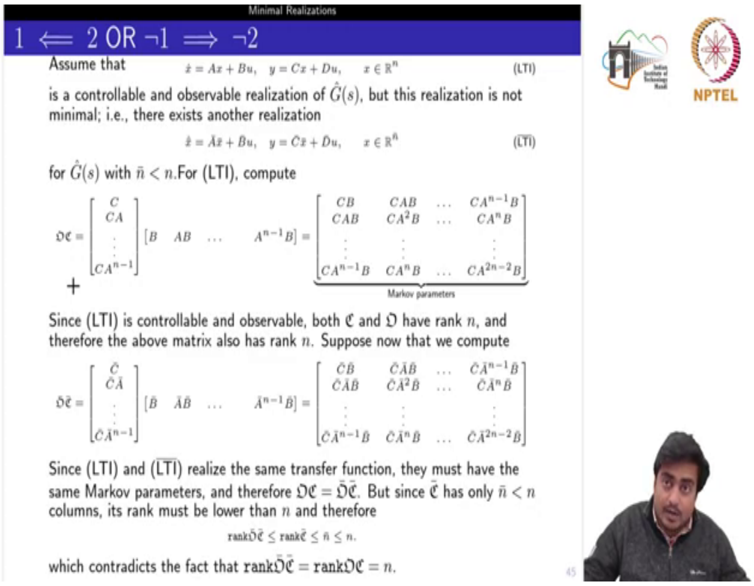

We can also prove this by contradiction meaning to say that assume that ABCD pair is both controllable and observable realization of G hat ok, but this realization is not minimal. That is to say that there exists another realization which is given by this pair A bar B bar C bar D bar in which the dimension of the state x bar is n bar now which is less than n because if this realization is not minimal.

It means that there exists another realization which we can obtain by Kalman decomposition theorem. And of which the dimension would be lesser than this one, but the transfer function would remain the same because both are the realization of the of the transfer function G hat.

So, for the system ABCD or this one we compute the multiplication of the observability matrix and the controllability matrix, first of all we compute for this pair ok. This is the observability matrix controllability matrix and the multiplication would yield this matrix which is composed of the Markov parameters.

Now the concept of this Markov parameters we have also introduced in the first week when we discuss about the realization only ok. Now since this system this LTI system is controllable and observable both matrixes would have the rank n and therefore, the above matrix would also has rank n; because this matrix basically this matrix the multiplication of the observability and the controllability matrix the rank will not change. .

Now let us compute the same multiplication of this LTI bar realization and this would be O bar C bar and again we can compute this one. Now, see one thing that since these LTI and LTI bar we realize the same transfer function which is g hat s they must have the same Markov parameters. So, this was one of the theorem we had studied in the first week.

So, they must have the same Markov parameters and therefore, both these matrixes would definitely be equal. Now if these two matrices are equal then the rank of these two matrixes would also be equal ok. Now let us see, but since C bar which we know already here has only n bar which was less than n columns it ranks must be lower than n; because the dimension of this x bar is n bar. So, if this pair is controllable, then its maximum the rank of the controllability matrix would be maximum n bar which would definitely be less than n right.

So, therefore, we would have the rank of this multiplicated matrixes would be less than equal to the rank of this controllability matrixes of this LTI bar which could maximum would have n bar and n bar we already know is less than n which contradicts the fact that the rank of both these matrixes is equal to n ok. So, this is a contradiction here. So, we so, this result basically speaks that whatever the realization is. Now this realization would be minimal or you cannot have another realization of a lesser order of a minimal realization metrics which is both controllable and observable.

And there are many more important outcomes of this minimal realization which we will discuss now and these questions basically originate from where we discuss in the previous weeks, where we had stopped did not answer them completely.

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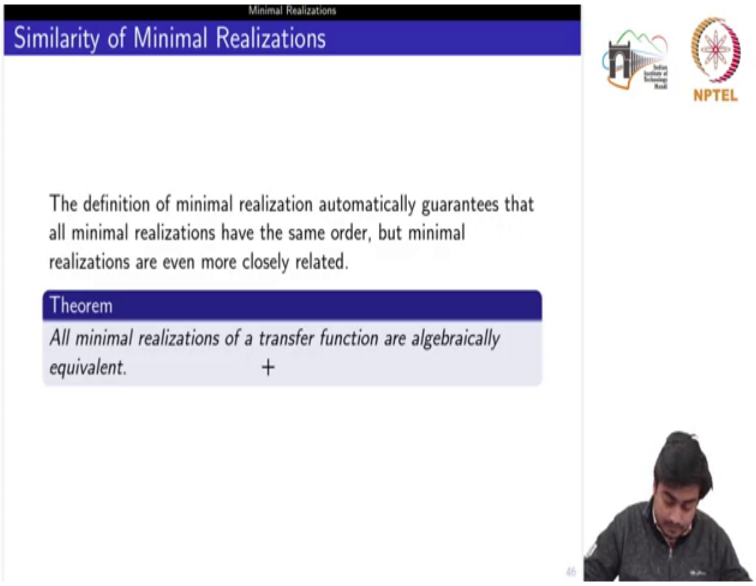

So, the definition of minimal realization automatically guarantees that all minimal realization would have the same order right, but minimal realizations are even more closely related in the sense that all minimal realizations of a transfer function are algebraically equivalent ok. So, this you can achieve by using some simulated transformation matrix, where the another realization would have the same dimension, but the matrixes might be different.

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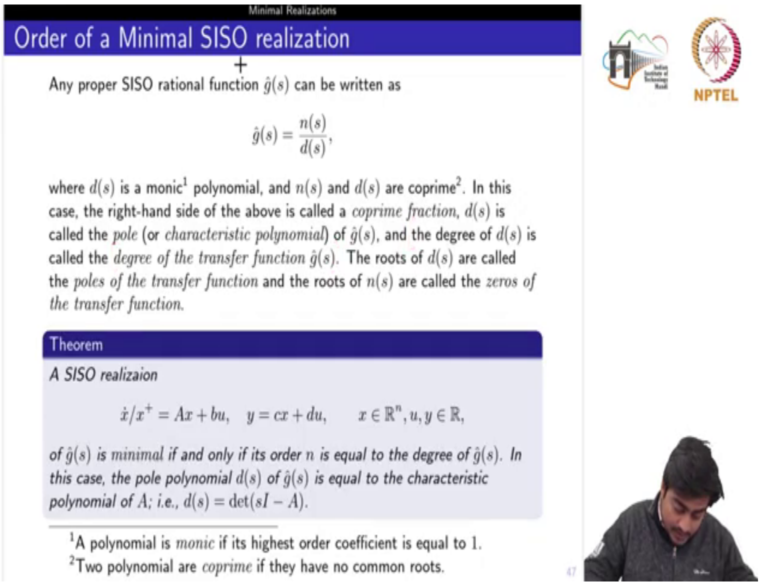

So, talking about the order of a or setting up some connection between the SISO realization and a minimum SISO or a SISO transfer function so. So we will recall first of all some basic definitions, which we had also discuss in the previous weeks. So, any proper SISO relation function g hat can be written as the ratio of these two polynomials the numerator polynomial and the denominator polynomial. Where d s d of s is a monic polynomial and a polynomial is monic if its higher order coefficient is equal to 1 and both these polynomials are coprime.

Now, if these two polynomials do not have any common factors, then these two polynomials are called the coprime polynomials and in that case the right hand side of the above is called the coprime fraction. Now ds is called the pole polynomial of g hat of s we had seen already and also describes the characteristic polynomials and the degree of ds is called that degree of the transfer function. The roots of ds are called the poles of the transfer function and the route of ns are called the zeros of the transfer function which we know already.

So, the main important aspect to look here is the degree of ds or the degree of that transfer function ok. So, we will in the next result we will set up or we will connect the degree of the transfer function g hat with the minimal realization. So, it says a SISO realization which is given by this ABCD pair of g hat s is minimal.

If and only if its order n is the order of this realization is given by the dimensional of the x as is equal to the degree of g hat of s. Now the degree of g hat s is given by their degree of ds that determine a polynomial. In this case the pole polynomial d of s is equal to the characteristic polynomial of a that is d of s is equal to the determinant of si minus a ok.

So, you can connect the degree of the denominator polynomial of a transfer function with the order of the minimal realization in the of the stage space presentation.

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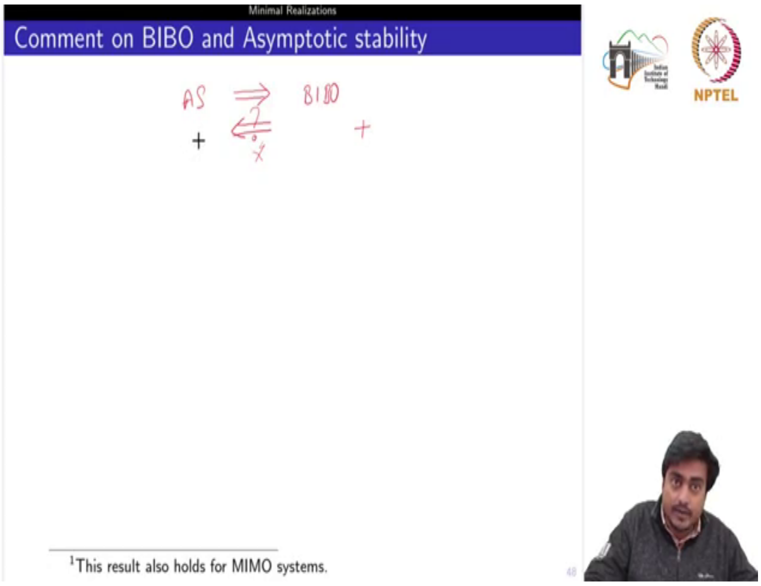

Now, comment on the BIBO and asymptotic stability. So, when we were during the stability week we discussed both these stability concepts the Bounded Input Bounded Output stability and also the asymptotic stability. So, bounded input bounded output in this BIBO stability we did not introduce or use the concept of the state, we only use the input and output signals and the boundedness on those signals.

Now this asymptotic stability we have specifically defined in terms of the a norm of the state trajectories or for transfer function by the poles of the transfer function right. Now one question we put up at that time that let us say we have asymptotic stable system and at the other end we have BIBO stable system right.

So, at that time we had ensured that if the system is asymptotic stable, then it could definitely be bounded BIBO stable. But the question we did not answer at that time that whether this implication would hold or not and we say that in general this implication does not hold. So, now, we will see one condition under which we can say that this reverse implication also holds meaning to say that if the system is BIBO stable plus some additional conditions are satisfied on the system then we could say that that system would be an asymptotic stable system.

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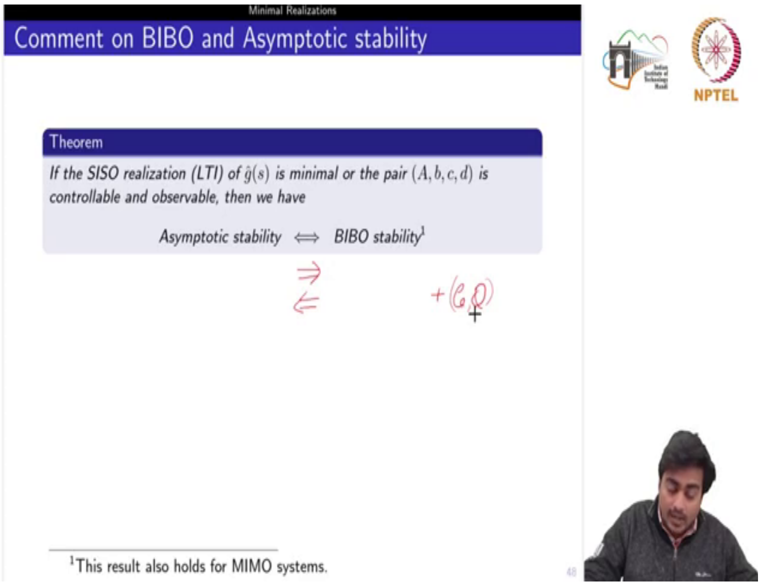

So, if the SISO realization the single input single output realization of g hat of s is minimal or the pair A, b, c, d is controllable and observable then we have this implication both ways. And this result also holds for the multi input multi output system right. So, the additional condition what we were talking about that if the system is BIBO stable we know that if the system is asymptotic stable, this implication holds. Now if the system is BIBO stable plus controllability and observability is also satisfied by the pair then this reverse implication would also hold.

And this implies that that realization is minimal that there does not exist any degree of the polynomial d of s less than the original. Now if their exists, now in that case there would be some cancellation between the numerator and the denominator polynomials.

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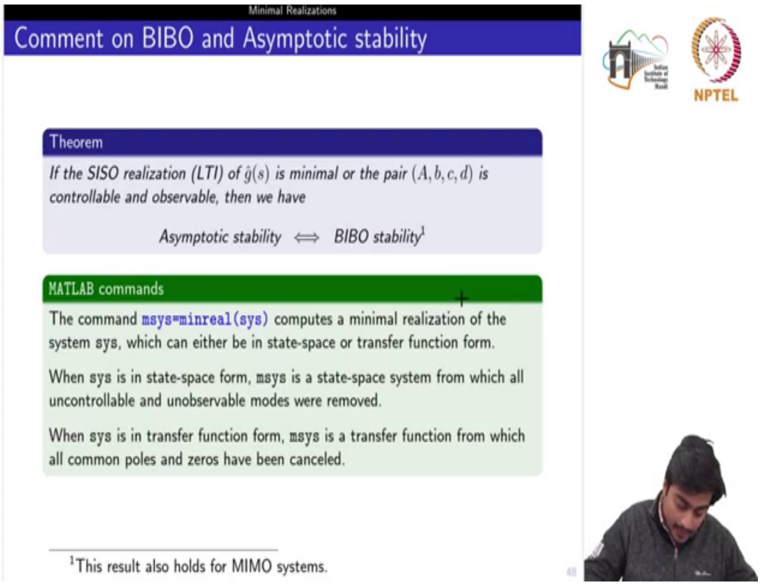

So, you can also compute the minimal realization by the MATLAB command. So, this command is given by this minreal we had this system sys defined either in the state space form or in that transfer function form. So, if this sys is in the state space form then msys would contain the realization of the minimal order in which all the uncontrollable and unobservable modes are removed.

Similarly in the if this sys is in the transfer function form then msys would contain or it would yield a transfer function from which all common poles and zeros have been canceled ok. So, you can use this minreal command in the MATLAB and this also holds for the MIMO system.