Linear Dynamical Systems

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Week – 07
Observability and Minimal Realization
Lecture – 38
Decompositions and Detectability

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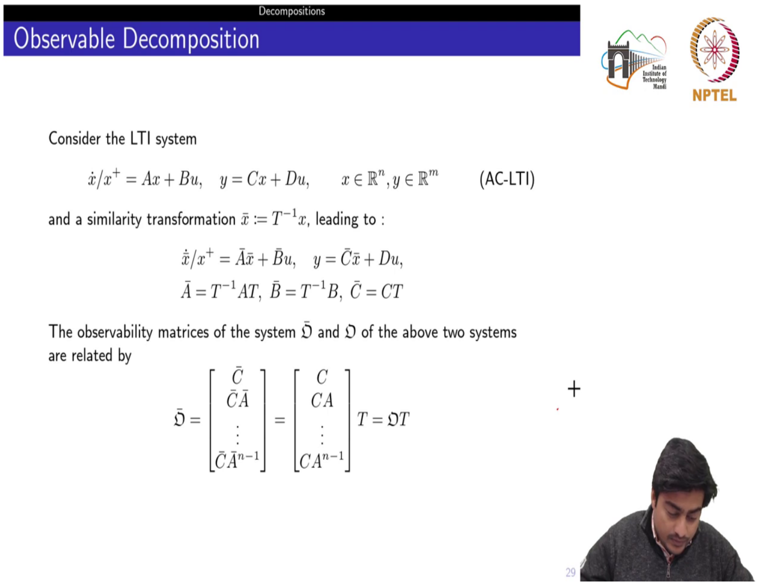

So, far we had discussed about the results of how to determine the observability and basically it requires the satisfying some rank conditions on the matrices. So, now, if the rank conditions are not satisfied say for example, for the continuous time LTI system that if the rank of the observability matrix is n, then the system is completely observable.

Now, if the rank of that observability matrix is let us say less than n, then it means that I cannot observe all the states, but is it; so, the question what I want to answer that whether is it still possible to observe some of the state instead of observing the complete state. So, this concept would remain almost similar to what we had discussed about the stabilizability, where we do not have the, we do not have the control over control signals or what the state trajectories, but still we can steer some of the state trajectories, ok.

So, before that first we will go into the decompositions that what are the different decompositions techniques to basically extract or differentiate between the observable part and the unobservable part, ok. And then we will define the detectability which is a weaker condition then the observability.

So, consider this LTI system with this pair A, B, C, D and consider this simulated transformation which is given by x bar is equal to T inverse times x, ok. So, this we had seen also in the first week on the simulator transformation, where A bar is defined as T inverse AT, B bar as T equals B and C bar as C times D, ok. So, the observability matrix of both the systems of this one and this one are related by this one, ok.

So, if you recall that the controllability is invariant under any simulator transformation; similarly it holds for the observability, ok. So, O bar is actually related to O bar times T, where since T is a simulated transformation matrix, so it is of full rank, right. So, if my this AC LTI system is observable, meaning to say that the rank of this observability matrix is n. So, post multiplying with a non-singular matrix would not change the rank of this O bar, ok. So, it is invariant under any simulated transformation, right.

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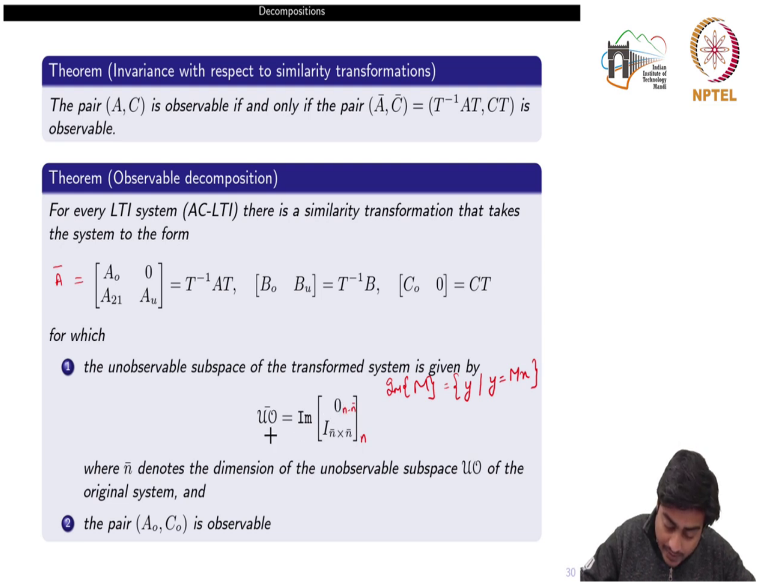

So, the pair A C is observable if and only if the pair A bar C bar is observable, where A bar and C bar are related to matrices A and C, with this simulator transformation matrix. So, this is equivalent to saying that it is the observability is invariant under any simulated transformation, ok.

So, this is an important result which speaks about the observable decomposition. So, for every LTI system there is a similarity transformation that takes the system to the form to this one. So, this is basically my new A bar. So, we had seen earlier that A bar is defined as T inverse AT now here we are giving a shape to this A bar matrix which is A o 0, A 21 and A u is equal to T inverse AT. Again, I am defining my B matrix, the transform B matrix and similarly that transformed C matrix related to the simulated transform matrices, ok.

For which the unobservable subspace of that transform transform system is now given by UO bar, ok, which is equivalent to saying the image of the 0 and the identity matrix of the dimension n bar, right. So, this result you can directly obtain from the result what we are obtain for the unobservable subspace and its equivalence with the Kernel of the observability coming.

So, what it says? That UO bar is equal to image of this one, ok. So, the total dimension of this one is n, right. It cannot be more than n. And in that n there is a number n bar which belongs to the unobservable subspace, and this is what it say that where n bar denotes the dimension of the unobservable subspace of the original system. Meaning to say among those n states I cannot observe n bar, but still I can observe n minus n bar because the remaining pair A o and C o which is of the dimension n minus n bar is observable, ok.

So, try to visualize the significance of this equation which says, so let us call it n minus n bar, ok. So, it means that the n we know that let us say we have any matrix M, ok. The image of any matrix M, we have defined as all y's such that we would have y is equal to Mx, right. Now, here it says the image of the 0 matrix, now the image of the 0 matrix meaning to say M itself is 0. So, why would definitely be 0 irrespective of whatever x is. And it says and recall the definition of the observability that the unobservable subspace should be 0. Meaning to say it does not or it should contain a 0 vector. So, this is what equation speaks. And the n bar, there are some n bar ah vectors which belongs to the unobservable subspace, ok.

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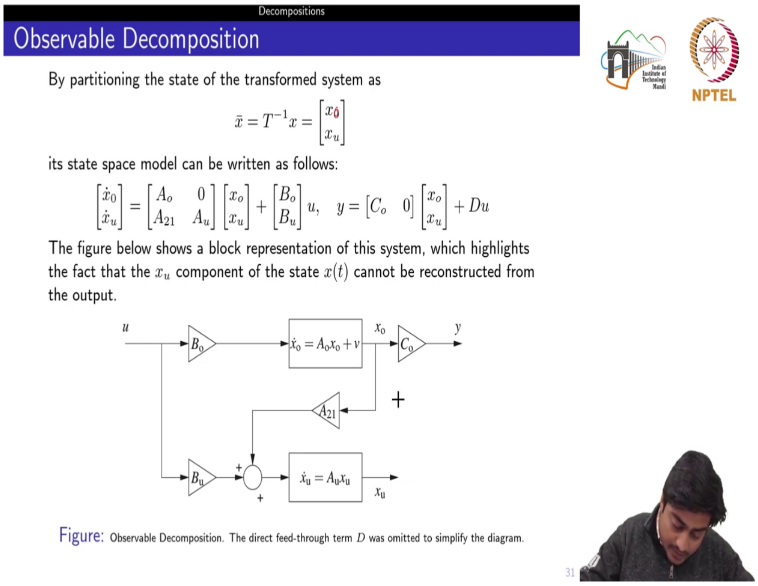

So, let us see more detailed visualization of this decomposition. So, by partitioning the state of the transform system as x bar is equal to T inverse x. So, here we are partitioning it as observable. So, this is x o instead of x 0, this is observable states and these are unobservable states.

So, if I write the transform system by taking this A bar matrix, B bar matrix and C bar matrix and the D matrix would remain the same. So, this is how the block representation I can draw, ok. So, where you would see that x naught has a path from u and y somehow, ok, but x u which is completely unobservable has some path from u, but it is not taking in any information from y, right. So, if it does not taking any information from u and y. So, I cannot observe this state particularly, ok. This is what this block structures says.

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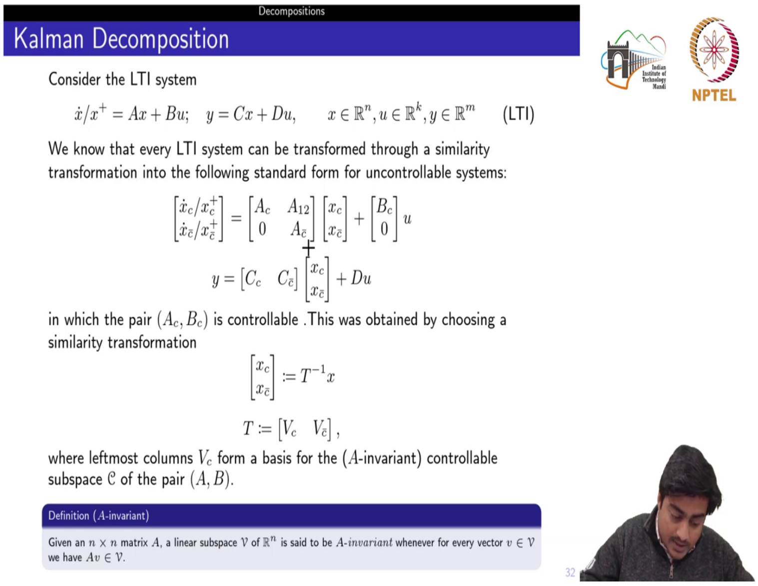

So, this decomposition is specifically called the Kalman decomposition for one specific reason, that so far we had seen the decomposition of the controllability and the decomposition for the observability, ok. Now, once you combine these two concepts into one stage space which clearly gives you the decomposition of the controllability and the observability in one representation that representation we call it the Kalman decomposition, ok.

So, starting with the transformation for the controllability, so if you recall that we have extracted out the controllable matrices, the controllable pair and the uncontrollable. So, here we are presenting the uncontrollable by C bar, ok. So, this AC bar because you would have x dot C bar is equal to AC bar x C bar. So, we cannot control this. So, this is uncontrollable matrix and A c, B c forms a controllable pair, ok. So, this was obtained by choosing a some simulated transformation matrix where x c and x c bar are the transformed vector and T we have chosen as V c and V c bar, ok.

Now, here the left most columns of this V c form a basis for the A invariant controller controllable subspace of the pair A comma B, ok. So, try to visualize this. So, here I have also recall the definition of the A invariant that given square matrix A of dimension n, a linear subspace v is said to be A invariant whenever for every vector v belonging to that subspace we have Av also belonging to that subspace, ok. So, this is a invariant control subspace. Now, the basis it is the minimum number of vectors which requires to spend the subspace, right. So, these vectors are basically the left most column of this matrix V c, ok.

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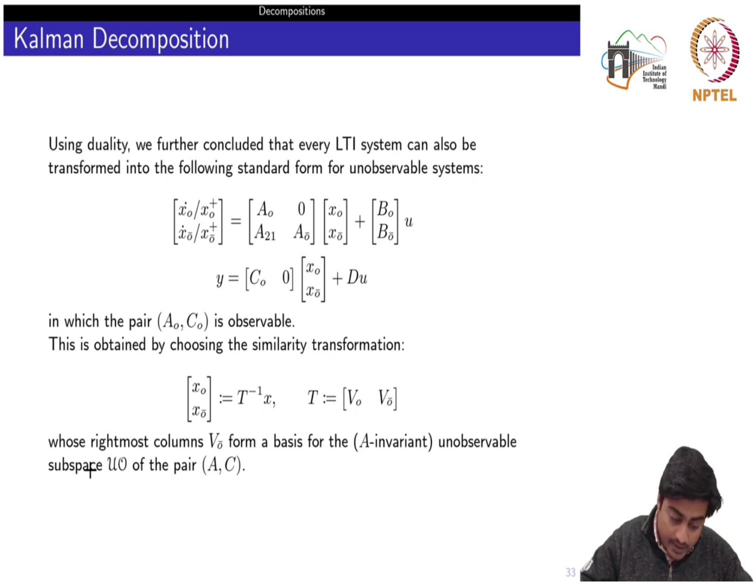

Now, coming onto the simulated transformation for the observability. Again we had seen earlier. So, instead of representing it by u, I am using O bar, where A o and C o is an observable pair and A o bar is an unobservable matrix, ok, A o bar and 0, right. And this is obtained by choosing the simulated transformation at this one where T I specifies V o and V o bar, where the right most columns of this V o bar form a basis for the a invariant observe unobservable subspace of the pair A c, ok.

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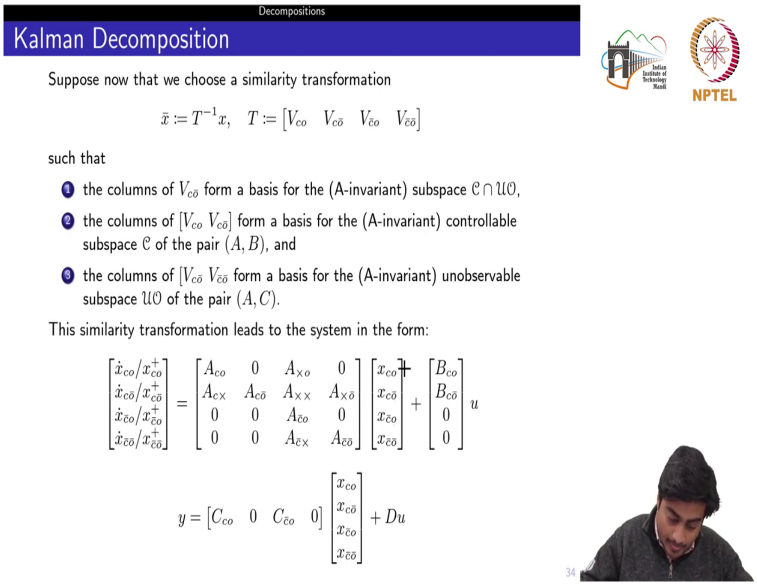

Now, if I concatenate these two vectors the controllable and uncontrollable observable and unobservable. So, I can also write this T matrix at V co, V co bar, V c bar o and V c bar o bar, ok. We will see quickly the significance of denoting this by c o and the bars because it is pretty much obvious that if c is without bar then it will also the controllability and if it belongs to some bar then the uncontrollability and the unobservability, ok.

So, the columns of this V co bar from a basis for the A invariant subspace which is an intersection of the controllable subspace and the uncontrollable subspace, right. This is what it denotes V co bar. c is the controllability controllable and o bar is unobservable, and I want the intersection of these two subspaces, ok. The columns of this V co and V co bar from a basis for the A invariant controllable subspace of the pair A B and this V co, V co bar and V c bar O bar form a basis for the unobservable subspace u bar of the pair A C.

Now, writing it into the combined form I would get this complete stage space representation where we have this vector x co. So, x co denotes all the states which are controllable and observable. x co bar all the states which are controllable, but not observable. Similarly, here the states which are not controllable, but observable and these are all the states which are neither controllable nor observable, and the matrices corresponding to those vectors, ok.

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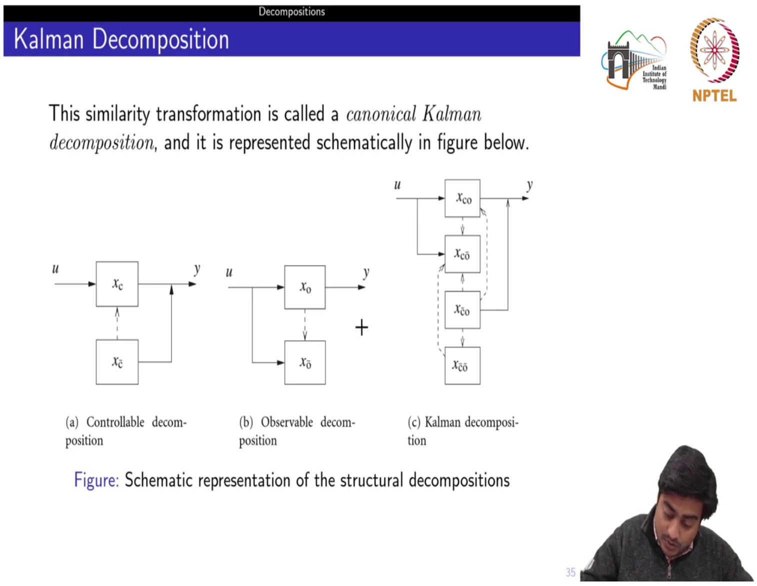

So, if you want to visualize in the using the figures. So, this controllable decomposition we had seen that only x c is controllable, but x c bar is not, because x c bar is not in being influenced anyhow by the input. For observability we had seen that both xo and xo bar are being influenced by u, but they are not, but the unobservable state is not receiving any information from y, ok.

Now, if I combine these two I could represent the states which are controllable observable the and vice versa. In the sense, that whatever the states are controllable and observable they would have the x as to u and y completely. The controllable and unobservable states u, but not y. Similarly, uncontrollable and observable not to u, but to y and x c bar o bar is completely independent because they would receive n information from u and y. So, they are purely autonomous systems, ok.

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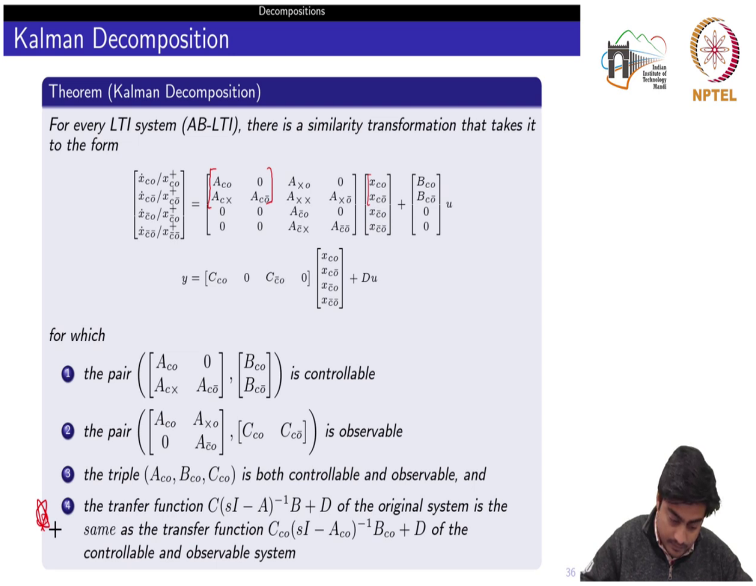

So, we have the result on this Kalman decomposition that for every LTI system there is a simulated transformation that takes it to the form this one of which we had seen in the detailed analysis in terms of the block structures and with their significance for which the pair A co 0. So, this pair this pair because both these states are controllable, we have x co and x co bar. So, this pair is controllable. The pair this bar, so belong with respect to the x co and x c bar o which correspond to the matrices A co and 0 and A cross o and A c bar o.

Similarly, this C matrix is observable that triplet A co, B co, C co is both controllable and observable and the transfer function which we obtain using this Laplace transform of the original system is the same as the transformation of the system which is being formed by using this controllable and observable pair, right. This is the most important result because we will use this result in the last topic of this week which is the minimal realization to set up the relationship between the controllability, observability and at the same time the minimal realization, ok.

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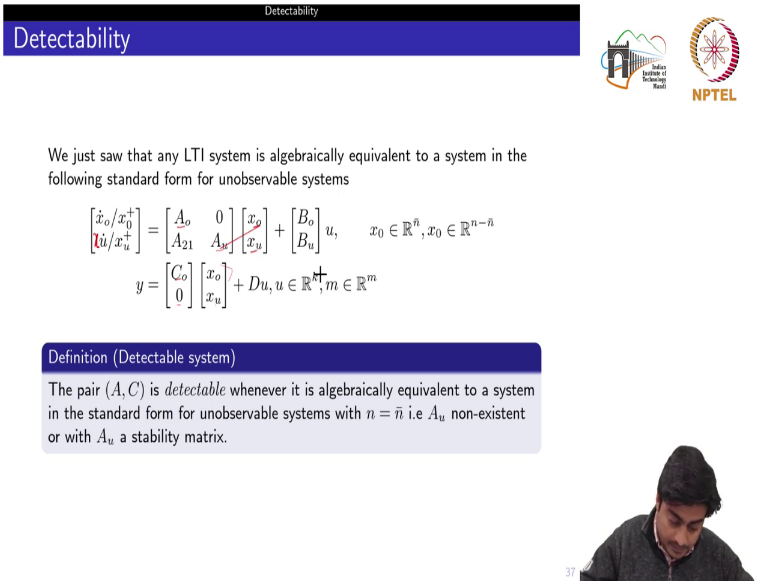

Now, after this decomposition we will go towards obtaining the weaker results than the observability. So, we had seen that the LTI system is algebraically equivalent to a system into the following standard form for the unobservable systems, right where we have decompose the states. So, this is x u dot, right. So, do not confuse, this is x u dot as the observable and unobservable decomposition. So, the pair A comma C is detectable whenever it is algebraically equivalent to a system in the standard form for unobservable system with n is equal n bar that is A u is nonexistent, or with A u a stability matrix, right. What does it mean?

See here, so we have A o and C o. So, these states x o is completely observable, right we just need to see that what happens with the matrix associated with the unobservable state, right because the C matrix associated with the unobservable state is 0, right. Either this matrix A u should not exist or if it exists then it should be a stable matrix, so that whatever be the state trajectories are with respect to this one goes to 0 as T tends to infinity, right.

So, this detectability has a very practical significance in the sense that if you want to observe some of the variables or let us say some of the state trajectories, then whatever you are observing depending on the observability of the system you might not getting the complete information of the state variable, right. So, there might be some trajectories which are reaching to infinity. So, from there you cannot absorb all the state variables, right.

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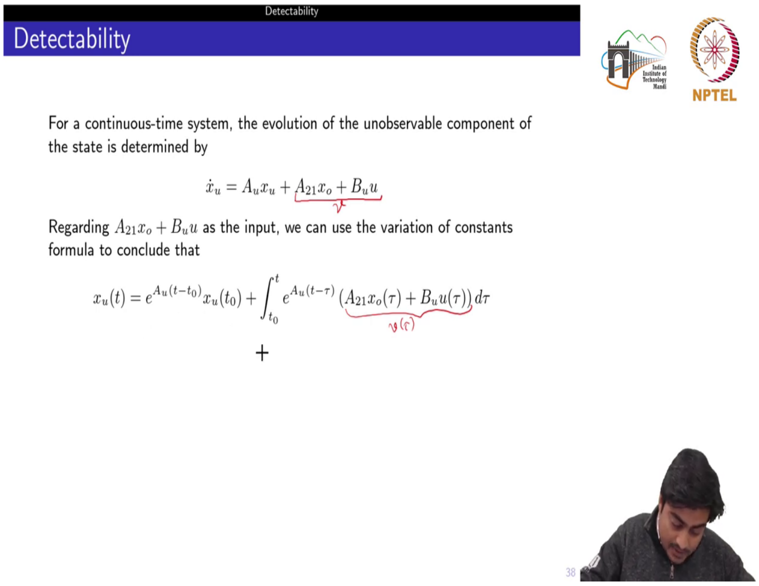

So, for a continuous time system the evaluation of the observable component of the system is determined by this one. So, I have just write this equation by opening it. So, I would have x o dot as A o x o plus B o u, and sorry the unobservable; A 21 x o plus A u x u plus B o u, ok, only the unobservable component. So, I can treat this component as another input, let us say v. This is what we had also seen earlier. We can use the variation of constants formula to compute this x u, right. So, this one is basically B tau, ok. So, x u is given by this one.

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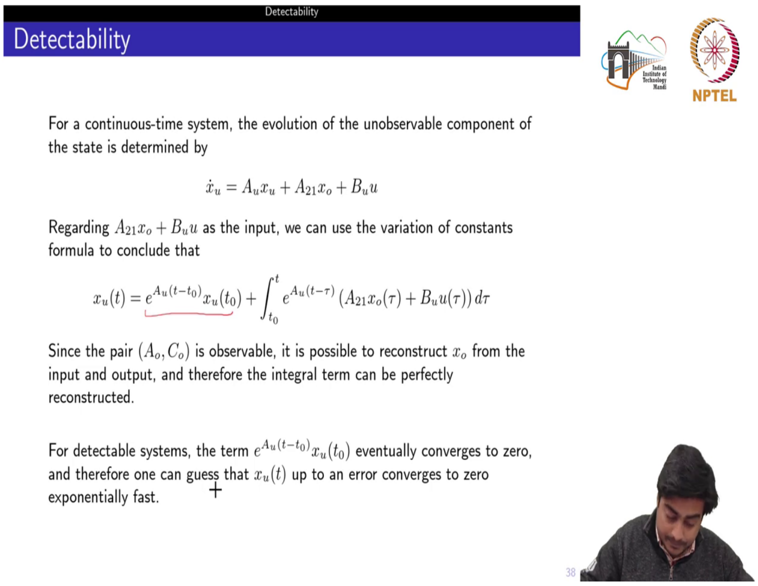

Now, since the pair A o C o is observable, it is possible to reconstruct x naught from the input output pair and therefore, the integral term can be perfectly reconstructed, because it contains two important variables; x o which is observable and u of which you have the information which is directly measurable. So, actually I can compute this integral once I have the knowledge of this A u matrix.

For detectable systems, the term this one, eventually converges to zero because we have ensured that A u is a stability matrix this is in the definition of the detectability and therefore, one can guess that x u goes up to an error converges to zero exponentially fast.

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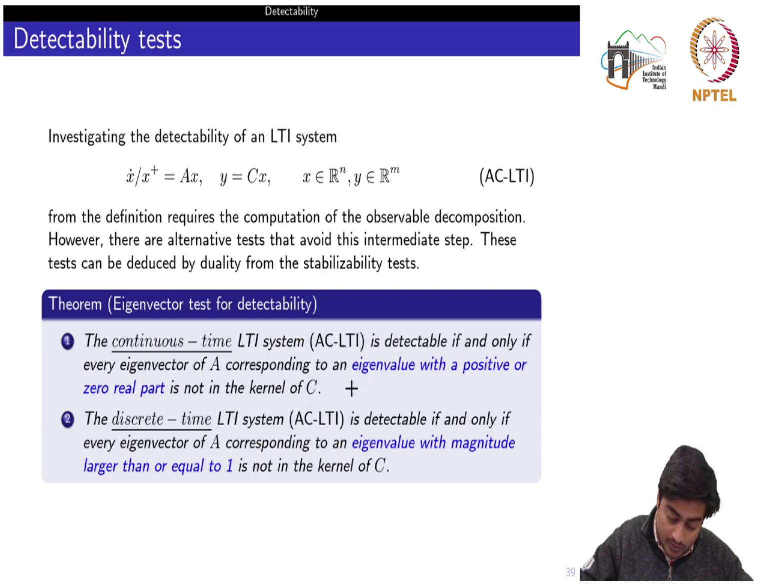

Coming on to the test for detectability. Again, we would consider only the pair AC because the B and D matrices plays no role in determining the either the observability or detectability. So, one way of determine the detectability is to do the observable decomposition and then check the stability of the unobservable matrix, ok.

But again similarly to the stabilizability, we have the alternative results where we can skip this decomposition part, right. So, the continuous time LTI system is detectable if and only if every eigenvector of A corresponding to an eigenvalue with the positive or 0 real part is not in the Kernel of C. So, if you recall the observability test or the eigenvector test for observability, it says the eigenvector of A is not in the Kernel of C, without any association with the eigenvalues.

Now, this results is only with the eigenvalues which are either on the right hand side or having the real or on the imaginary axis, right. For the discrete time it would be the eigenvalues with magnitude larger than or equal to 1 is not in the Kernel of C, ok. So, this would be change only.

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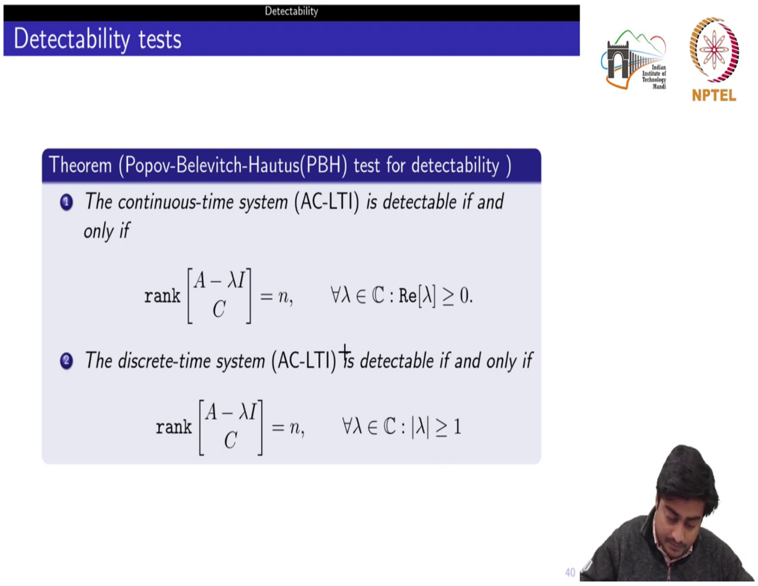

So, the PBH test. The rank conditions would remain the same the only difference is that where we are taking the real part of the eigenvalues. So, there we had the for all eigen, for all lambdas belonging to the complex set of numbers, but here we would consider only those lambdas which are having the real part greater than or equal to 0, ok.

Similarly, the magnitude of this lambda which is an eigenvalue greater than or equal to 1 this is just a simplification of the eigenvector test, which is labeled as PBH test.

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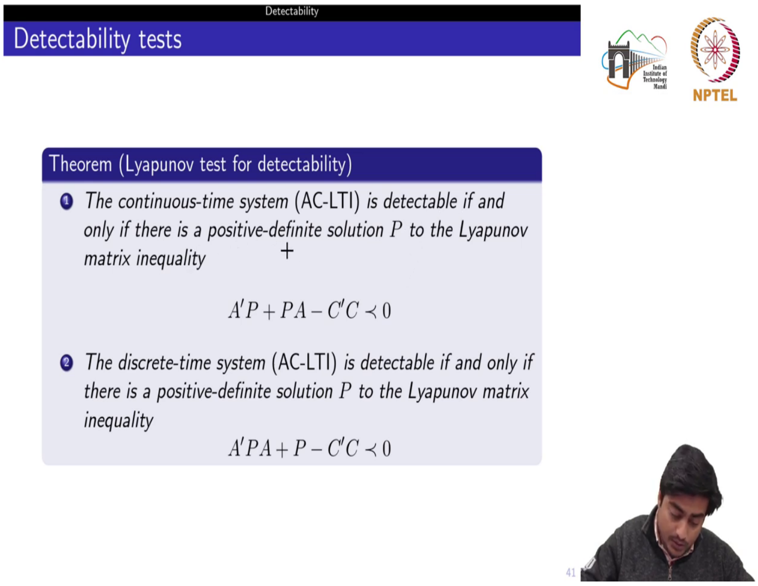

The in the Lyapunov test. The Lyapunov matrices would change is detectable if and only if there is a positive definite solution P to the Lyapunov matrix inequality this one in the continuous time and this in the discrete time, ok.