Linear Dynamical Systems
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Week – 07
Observability and Minimal Realization
Lecture – 37
Duality and Observability tests

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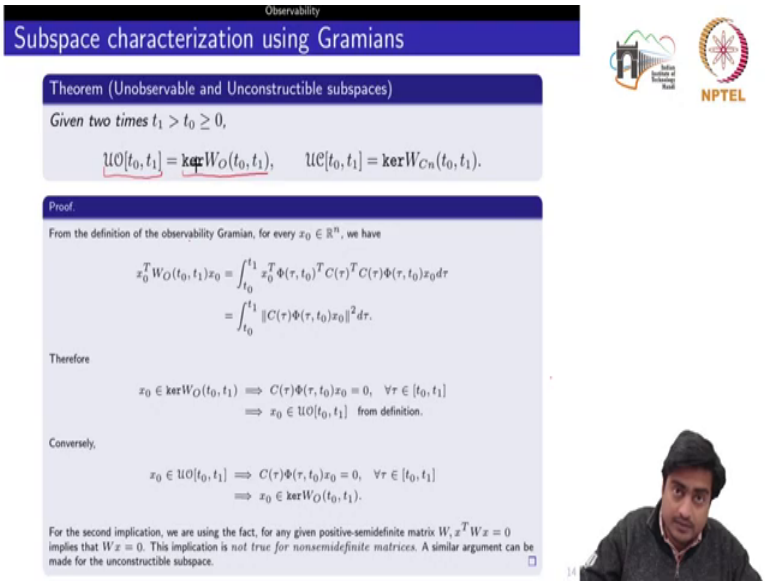

So, far we had seen about the subspace characterization using Gramians. So, whatever the result we have obtained in this theorem, we can directly apply to determine whether the system is observable or not.

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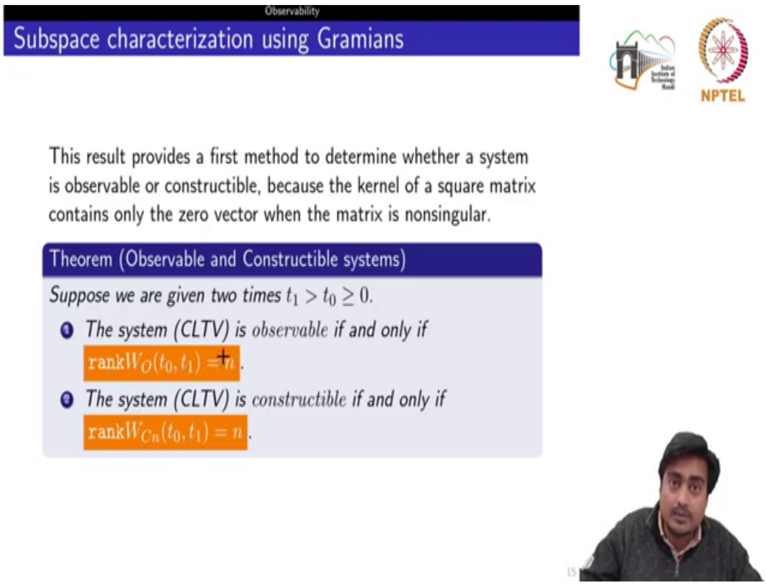

So, this results not only provides a first method to provide to determine whether a system is observable or constructible, but it also provides the characterization of the sub spaces which we would use in the later concepts. So, this Gramian has a direct implication. So, let us see. So, it says that the unobservable sub space from t naught to t 1 is equivalent to the kernel of this a square matrix W o dimension n.

Now, if we go back to the dimension the definition of the observable observability, it says the unobservable subspace should contain zero vector right then the system is observable. Now if it contains the zero vector then the kernel of that matrix should be 0 it should not contain any vector and this will only possible if the matrix itself is non singular. Because in the previous theorem we had seen that the Gramian matrix is a positive semi definite ok.

So, based on this we have the next result that suppose we are given 2 times t 1 greater than t naught greater than equal to 0 the system continuous time LTV is observable if and only if the rank of the observability Gramian is equal to n similarly its for the constructible system. In satisfying this rank condition means that the square matrix are nonsingular ok.

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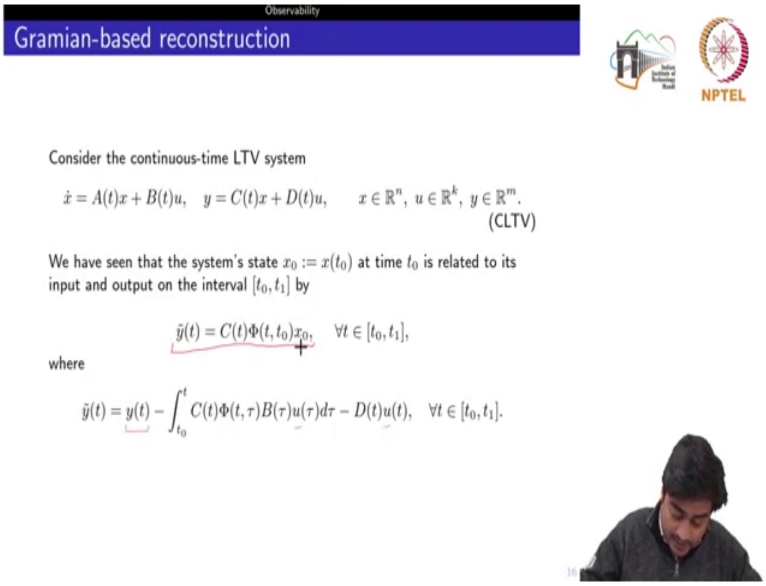

Now we can again use this Gramian to actually reconstruct the state. So, consider again this LTV system given by this pair ABCD, we have seen that the system state x naught at t naught is related to its input and output on the interval this one and this we have obtained from the variation of constants formula we had seen earlier. So, y tilde is nothing, but a subtraction of the terms containing the output signal and the function containing the input signal ok.

So, basically we want to construct this x naught and we had seen earlier difficulties that if my C is a non singular matrix or its in is not invertible, then I cannot compute this x naught, but using the Gramians we would say we would see that how we could do the reconstruction of the x naught ok.

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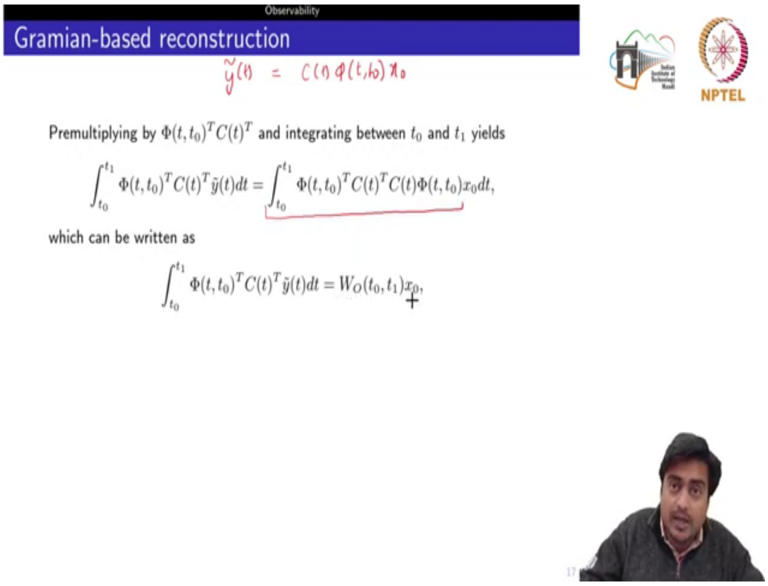

So, we will premultiply by this vector where phi is a straight translation matrix and C is the output matrix and so, let us write it here once again. So, that there is no confusion we have y tilde this is the question we had obtained. Now if I pre multiply this above equation by phi transpose and C transpose and then integrated I would obtain on the left hand side phi transpose C transpose y tilde dt and on the right hand side I would obtain phi transpose C transpose multiplied by C phi and x naught and here x naught is a constant. So, I can take it inside the integral or outside the integral right.

Now, keeping the left hand side as it is and notice this whole term this is nothing, but your observability Gramian. So, I can replace this on the right hand side as observability Gramian multiplied by x naught ok.

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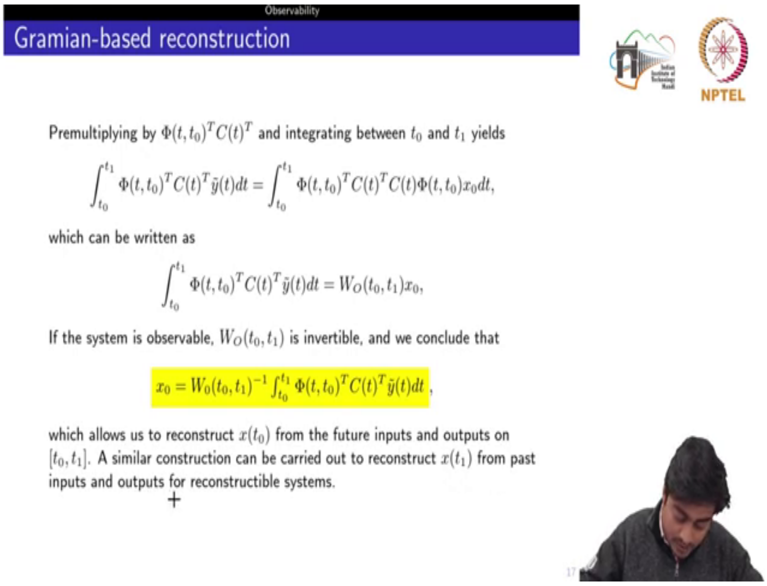

Now, taking the property of the observability which says that the observability Gramian is invertible or it is full (Refer Time: 04:36) then it is possible to take the its inverse which is a square matrix to perfectly reconstruct this x naught using this formula ok. Similar construction can be carried out to reconstruct x of t 1 from past inputs and outputs for reconstructible systems right.

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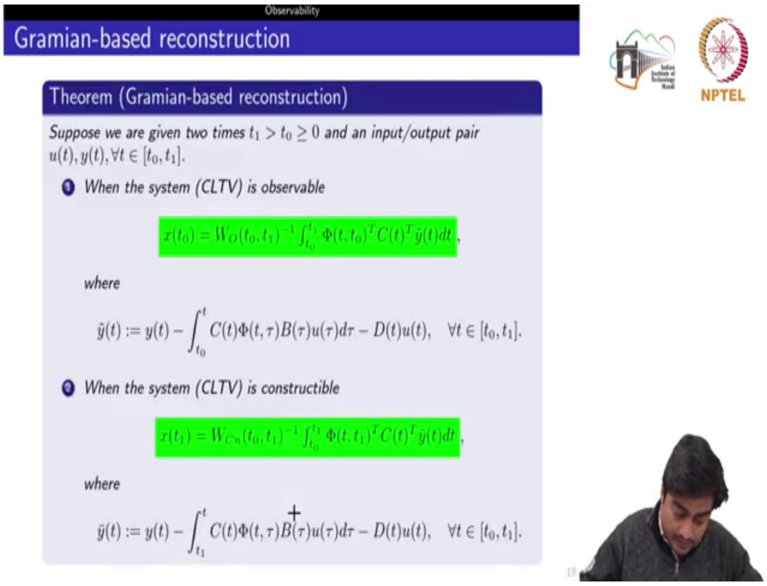

So, we have the immediate result that suppose we are again given 2 times t 1 t naught and an input output pair u y for all time t belonging to a t naught and t 1. So, when the system CLTV is observable we can compute x of t naught using this formula where y tilde is given by this one and when the system CLTV is constructible again we can compute this x of t 1 using this constructible Gramian. .

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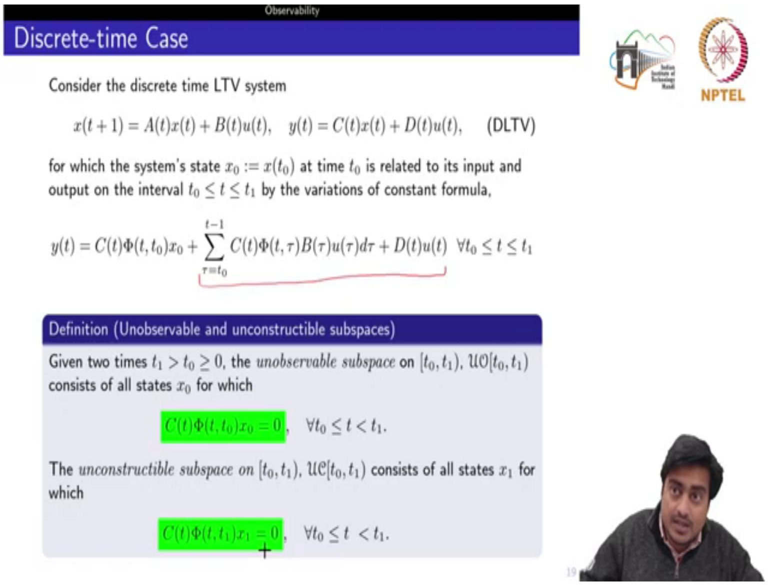

Now generalize whatever the results we had seen we its in the continuous time domain, we can generalize all those results and the discrete time domain also. So, consider this ABCD again the time varying matrices where t is now an integer. So, by using the variation of constants formula we had seen in the first week the solution of the discrete time LTV system; where again phi is a state transition matrix and ABCD or CBD are the parameter matrices of this LTV system ok.

So, along the same line of the continuous time domain we can define these two sub spaces unobservable and unconstructible on t naught to t 1 consisting of all states x naught for which C phi x naught is equal to 0 right because again I can take this entire part onto the left hand side and then denoted its y tilde so I would have y tilde is equal to C phi x naught and if my C phi x naught becomes equal to 0.

So, for all those x naught for which we have C phi x naught is equal to 0 would below comprise the unobservable subspace. Similarly the unconstructible subspace on t naught to t 1 consists of all states x 1 for which this equation is satisfied.

Now, in the continuous time domain and the discrete time domain we had seen a key difference in computing the state transition matrix that in the continuous time domain we can compute the state transition matrix in both the directions ok, but in the discrete time domain we had seen some limitations, when this limitation arises because if you pay attention to this state transition matrix which is phi t comma t 1, where t is from t naught to t 1 which means that I need to compute the state transition matrix backward in time.

Say for this one phi t comma t naught where t is t naught to t 1 again. So, the initial point is t naught and the final point is t which is greater than t naught. So, I am computing here state transition matrix forward in time, but here my initial point is t 1 and t is from t naught to t 1.

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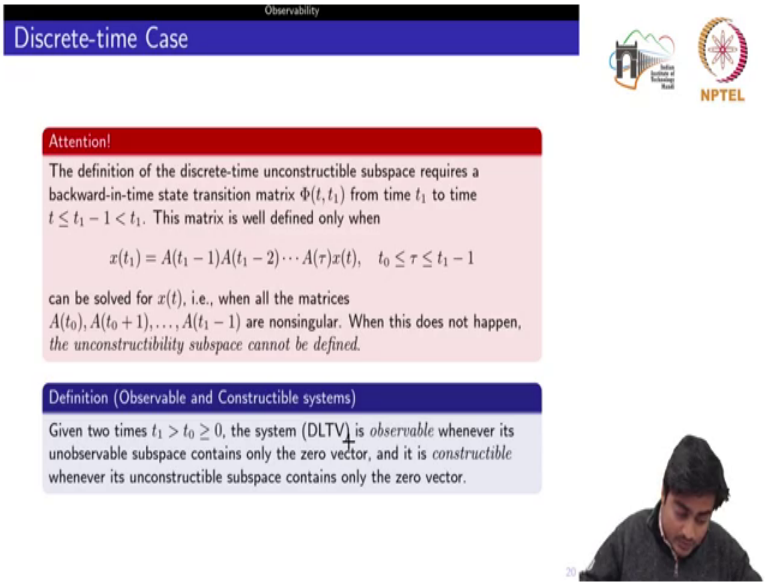

So, here I need to compute the state transition matrix backward in time and this is only possible that this matrix is well defined only when I compute x of t 1 which is represented by this one for all x ts then and this is only possible when all the matrices at different time from t naught to t 1 minus 1 are nonsingular. This is the same restriction what we had seen in the first week while computing the solution of the discrete time scenario. So, when this does not happen the unconstructability subspace cannot be defined ok, but the there is no limitations in the unobservable subspace right.

So talking about the systems, now given two times t 1 t naught the system DLTV is observable whenever its unobservable subspace contains only the zero vector and its constructible whenever its unconstructible subspace contains only the zero vector and the same definitions, but we had used while understanding the concepts of the continuous time case.

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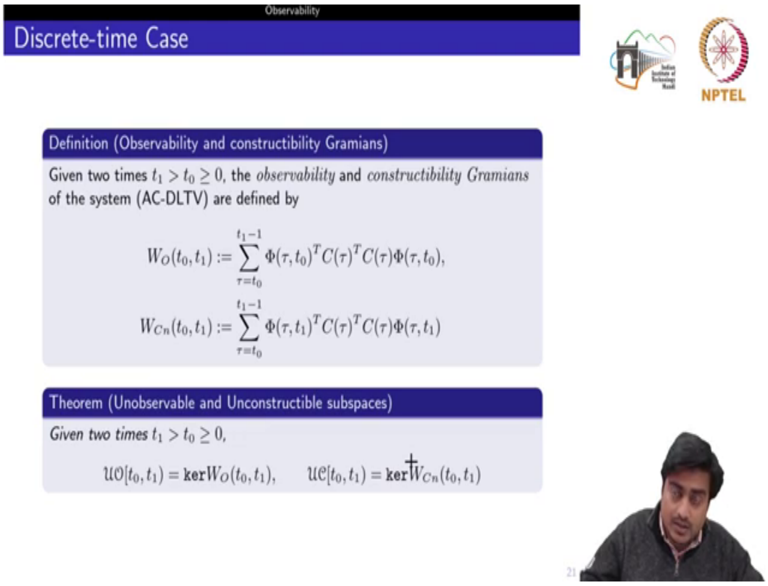

The Gramians the Gramians would remain the same with just the summation would appear instead of the integration and also the time is in integer instead of the real number. So, this is the Gramians the observability Gramian and this is the controllability Gramian and the key results which characterizes these sub spaces in terms of the Gramian is given by it remains the same in also the derivation remains the same.

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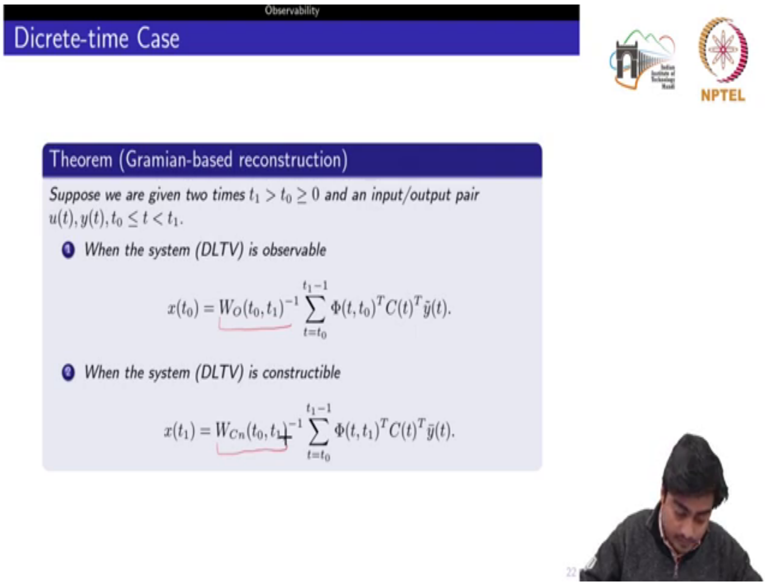

So, we will not go into the derivation of these results ok. Now in the discrete time case if I want to reconstruct this is the formula, you can use it if your system is observable to reconstruct x t naught if your system is constructible, you can compute x of t 1 using this formula where we had seen key use of the Gramians the respective Gramians ok.

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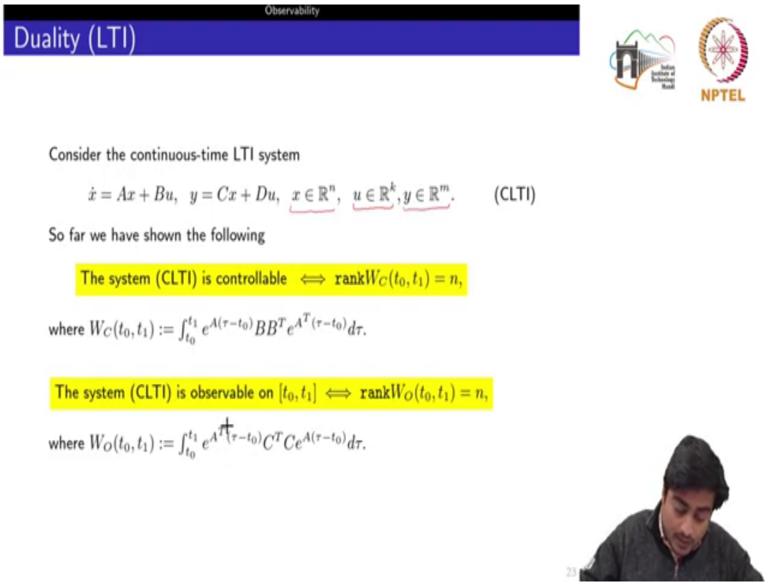

Now, the next concept in this observability is duality. So, duality is a very important concept by which we will see our relationship between the results what we have derived during the controllability and also the test for the observability for the LTI system. So, let us recall first of all the controllable systems for the LTI specifically. So, given these ABCD matrices note that here that x is n dimensional u is k and y is m ok.

We will give some importance to that in the next slide that is so, far we have seen that the CLTI is controllable if and only if the rank of the controllability Gramian is equal to n where controllability Gramian is defined by this ok. Now, the CLTI is observable if the rank of the observability Gramian is this. So, here we made use of the AB matrices and here we made use of the AC matrices right.

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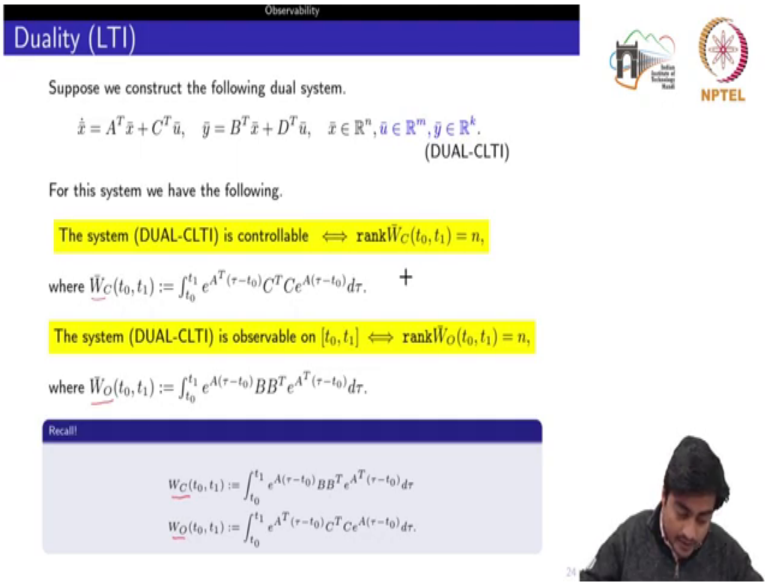

Now, I will define a dual system in the sense that the x bar would be of n dimensional, but u bar would be the dimension of the output. So, if you see previously we have y n number of outputs but in my dual system I am representing it by u bar m number of u bar and k number of y bar and I represent my dual system is x bar dot or by A transport C transpose B transpose and D transpose where, C transpose in this dual systems becomes my input matrix and B transpose here becomes my output matrix and the state matrix and the input matrix and the output equation are taken as that transposes ok. Now let us see the controllability and observability for this dual system.

So, for this representation again I need the conditions would remain the same it just I need to compute the controllability Gramian and the observability Gramian ok. So, by taking A transpose and C transpose to compute the controllability Gramian I would obtain this one ok.

Now for observability I need to see A transpose and B transpose and using that pair I can write the observability Gramian in terms of that AB matrices ok.

Now try to see the relationship between this dual CLTI and the actual LTI system. So, recall these controllability Gramian and the observability Gramian which we had seen in the last slide. So, I put this thing on the same slide so, that we can easily visualize the relationship between the controllability of the original system and the observability of the dual system. So, this WC and WO are the gramians of the original system and this WC bar and WO bar are the gramians of the dual system ok.

So, here WC you see the a matrix the pair AB matrix. So, this WC and this WO bar basically is the same right. Similarly WO that is the controllability Gramian of the original system and the controllability Gramian of the dual system also remains the same.

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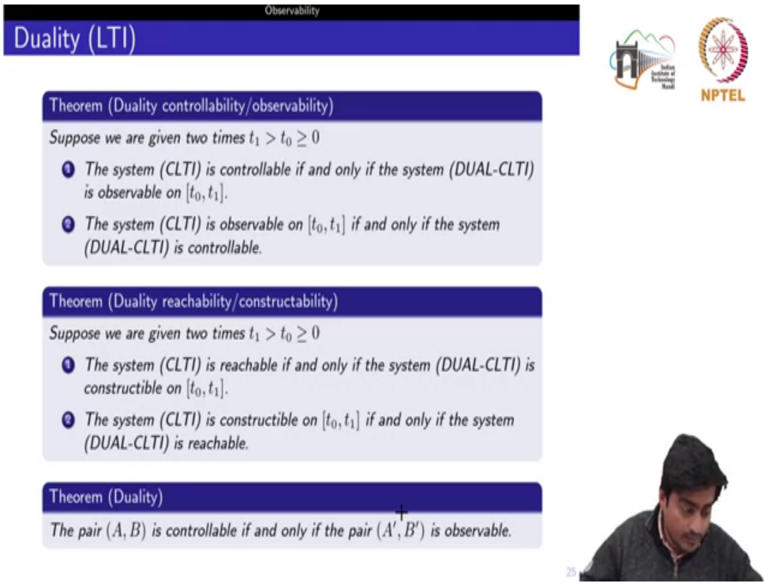

So, based on this we have the immediate result that the system CLTI is controllable if and only if the dual system is observable right. Similarly the CLTI is observable if and only if that dual system is controllable. So, they both are equivalent the result would remain the same for the reachability or the duality between the reach ability and the constructability.

So, we can also summarize these two results in the form that the pair if we only talk about the pair. That pair AB is controllable if and only if the pair A transpose B transposes observable and this is only valid for the LTI system we will see that how these conditions varies for the LTI system.

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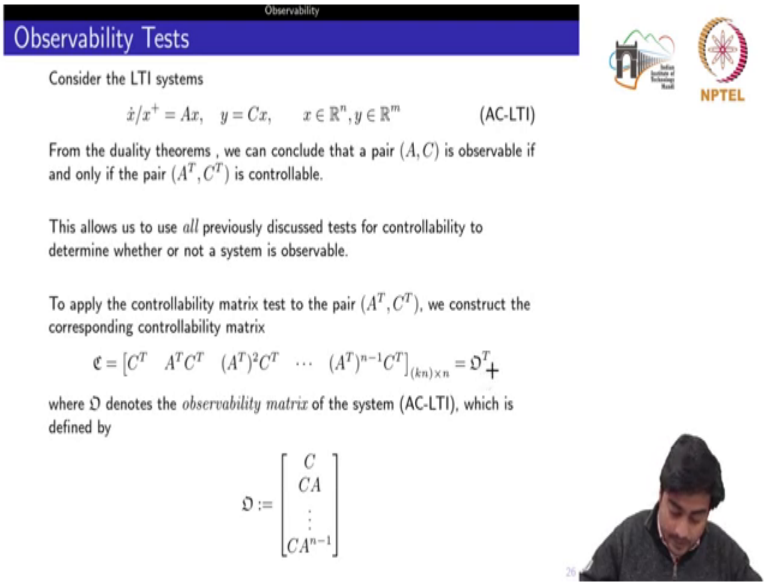

So now moving forward to the observability test. So, we had seen that there exists there is a duality between the controllability and observability. Meaning to say that whatever the results we had obtained for the controllability they would be directly applicable to analyze the observability. So, we will list out all the results of the observability without going into the proofs of those results.

So, consider the LTI system. So, here we use the notations so, that we can represent the discrete time and the continuous time with the same state equations. So, from the duality theorems we can conclude that a pair AC is observable if and only if that pair A transpose C transpose is controllable or because in the previous we had seen AB is controllable if and only A transpose B transposes observable it just the pair.

So, here we have replaced B by C and the results would remain the same. So, the this allows us to use all the previously discussed tests for controllability to determine whether or not the system is observable particularly for the LTI system. So, this we can visualize also.

So on this pair A transport C transpose if I compute the controllability matrix this would be given by this one of dimension kn cross n. Now I can define this matrix as the transpose of the observability matrix where my observability matrix is given by this one ok. So, the controllability matrix of the pair A transpose C transpose given by this and whatever the matrix I have obtained after doing some simplification, I defined the observability matrix as this one. That is to say the controllability matrix is actually equivalent to the transpose of the observability matrix ok.

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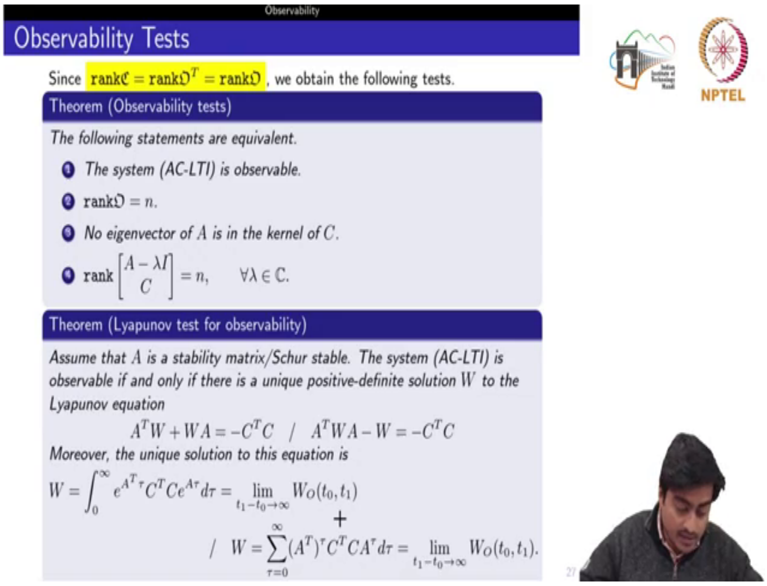

So, based on this equivalence we know that if I compute the rank of the controllability matrix and it is of full rank, then the rank of the transpose of that observability matrix would also remain the same and the rank does not change under any transpose ok. So, we obtain the following results that the following statements are equaling the system AC-LTI is observable, rank of the observability matrix is n which is the dimension of the a matrix.

So during the controllability we have defined this as the matrix test, the third is that no eigenvector of A is in the kernel of C this is the eigen vector test and the simplification of the eigenvector test which we have defined as the PBH test that the rank of this matrix is equal to n for all lambda belonging to the set of complex numbers ok. Now there was one additional results that is the Lyapunov test for the observability which a assumes that the matrix A is at stability matrix.

So, we can say that the system is observable if and only if there is a unique positive definite solution W to this Lyapunov equation in the continuous time and this Lyapunov equation in the discrete time and moreover the unique solution to this equation is given by this one.

So, we had seen that detailed results or detailed proof of these results during the controllability week. So, the equivalence or the duality between the controllability and observability particularly for the LTI systems remains the same right. So, the proof remains the same as what we had seen earlier.

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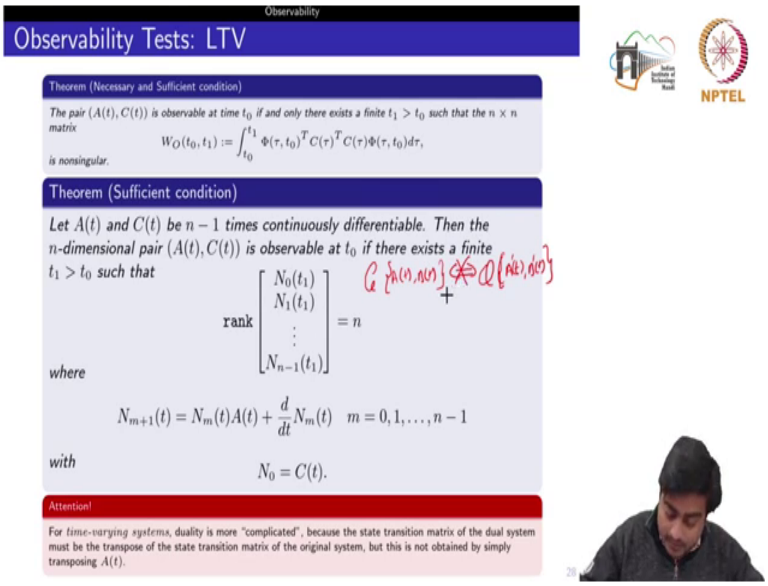

Now talking about the LTV systems. So, the result we had seen earlier. So, this is the same result what we had seen earlier that the this WO matrix should be nonsingular either for LTI system or for the LTV system. But in the controllability week we had seen that for the time varying system it is a bit difficult to compute this controlling the observability coming why? Because it requires the computation of the state transition matrix.

For the LTI system the state transition matrix is basically exponential, but for time varying system it is not necessary that it would be or it requires detailed computation. So, but so, this is if an only if this is if and only if because this is the necessary in sufficient condition for the observability. So, if you recall that for the for analyzing the controllability of the LTV system we take a different approach and we had defined only the sufficient condition and similarly we can define the in a similar way for the observability that like A and C be n minus 1 times continuously differentiable.

Then the n dimensional pair AC is observable at t naught if there exists a finite t 1 such that the rank of this matrix is equal to n. So, now, instead of ensuring the nonsingularity of this observability Gramian we compute the matrix n and then using those matrix n for 0 to n minus 1 be computed strength and this is how you can compute the n the n matrices ok. So, this is only a sufficient condition, but not the necessary condition.

So, for time varying system duality is more complicated because the state transition matrix of the dual system must be that transpose of the state transition matrix of the original system right, but this is not obtained by simply transposing the A matrix right. So, because of this reason we cannot say that the controllability of the pair A t B t or let us say the controllability of the pair A t B t is not equivalent to the observability or A transpose B transpose right.