Linear Dynamical Systems
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Week - 07
Observability and Minimal Realization
Lecture – 36
Output feedback and observability

So, welcome to the 7th week of the Linear Dynamical Systems. So, in this week we will discuss about the Observability and Minimal Realization Problem.

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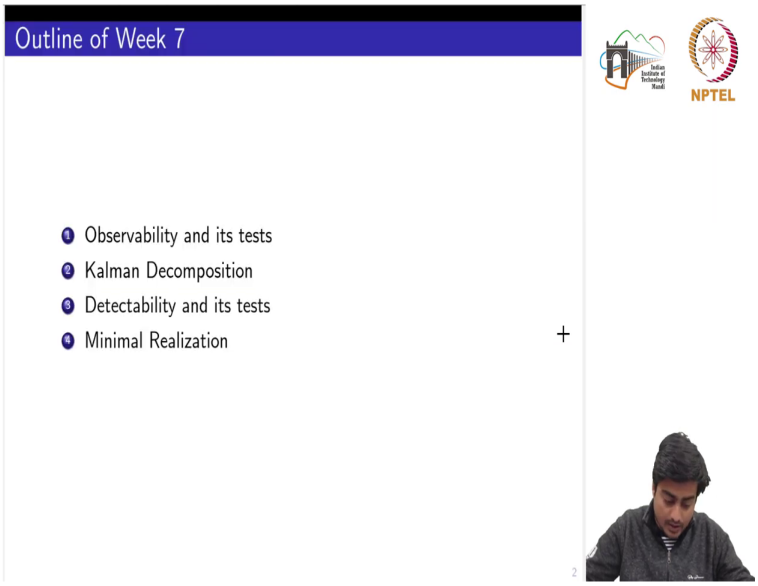

So, the outline of this week is we will start with the concepts of observability and its tests. Then we will discuss about the Kalman decomposition which will include the controllability and the observability analysis in combined. Third, we will see the weaker concept than observability that is detectability and we will also see the test. And finally, we will conclude with the minimal realization.

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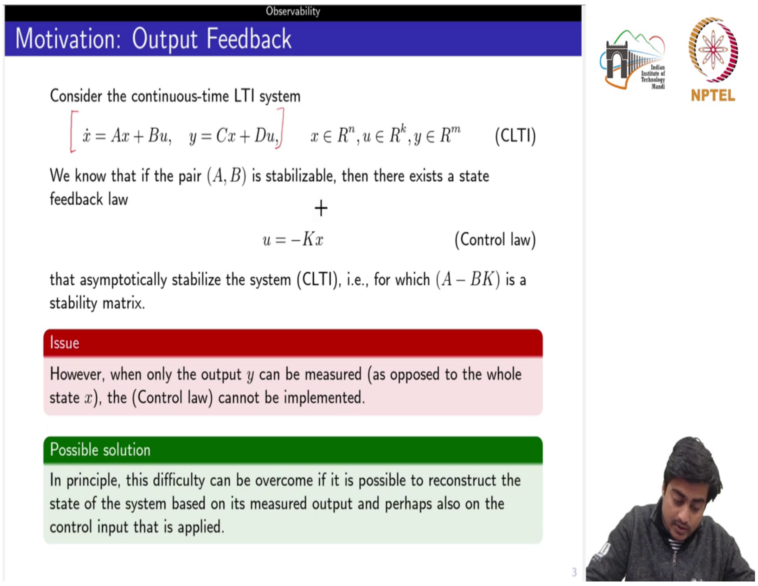

So, observability is basically related to the problem of the controllability what we had discuss in the 3rd or 4rth week. So, if you recall the problem of the controllability that we want to ensure that whether there exists a control law, such that I can steer my state trajectory from some x of t naught to some x of t 1, ok. So, here let us say if a continuous time LTI system is given by A, B, C, D matrices, where x is n-dimensional, u is k dimensional and y is m-dimensional.

So, we know that if the pair A comma B is a stabilizable then there exists a state feedback law which is given by u is equal to minus k times x, which we have defined as control law. That asymptotically stabilize the system, this one, that is for which this matrix A minus BK which is the closed loop matrix is a stability matrix, ok.

Now, if the pair A, B is controllable then we can achieve much more performance specification by designing this state feedback gain matrix K, ok. Now, here you would see that for computing the control law u you need the information of the state signal. So, however, when only the output y can be measured as opposed to the whole state x the control law cannot be implemented, right because if we do not have the direct measurement of the signal x and you only have the direct measurement of the signal y then you cannot implement this control law, you need the information about the state signal.

So, possible solution in principle this difficulty can be overcome, if it is possible to reconstruct the state of the system based on its measured output and perhaps also on the control input that is being applied, ok.

So, the idea here is now the only measured signal in this system is the external signals which we have defined earlier as the input to the plant and the output of the plant; x is an internal signal, right. So, now if it is possible to reconstruct the signal x based on the available measurements u and y, then it would be still possible to use this control law, ok. Now, there other ways of constructing this x also.

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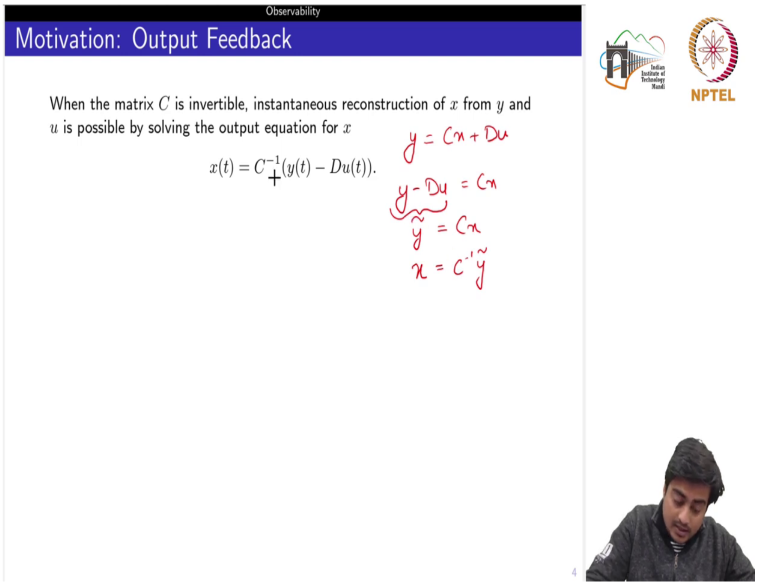

Trivially speaking, now, see the output equation; the output equation is given by y is equal to Cx plus Du, ok. The signal u is already a measured signal, so I can take this Du on to the left hand side. So, let us say y minus Du is equal to Cx and let us denote this y minus Du because D matrix is known to me u signal I am measuring and similarly y, so I can denote this as some y tilde is equal to Cx, ok.

Now, if C is invertible that is to save the inverse of the C exist then I can perfectly reconstruct the signal x as C inverse of, C inverse times y tilde. Here the information of the matrix C is also known to me y tilde I have computed by the signals y and u. So, if C is invertible that is the inverse of the C exist, then I can directly compute the signal x, ok.

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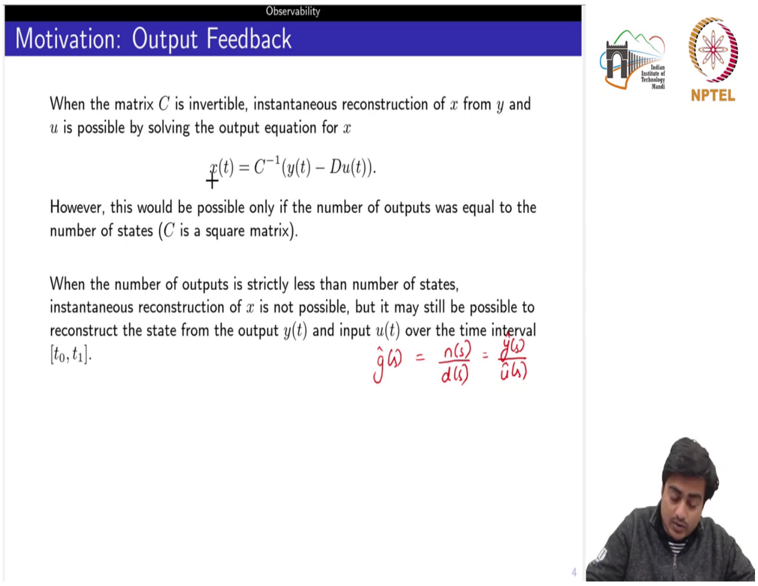

But this is only possible if the number of outputs is equal to the number of states, right thus C the first restriction is that C should be a square matrix. And this square matrix could be possible, if we have the number of output as equal to the number of states. But when the number of outputs is strictly less than number of states which is always possible or which is mostly possible, the instantaneous reconstruction of x is not possible, right. But it may still be possible to reconstruct the state again by taking the signals y and u over some time interval t naught to t 1, right.

So, this you can visualize this, say for example, if we many times we have considered a single input single output system which is of some ratio of polynomial numerator and denominator, which is nothing but y of s and u hat of s. Here we know that the output is 1 and the input is also 1. Now, depending on the degree of the polynomial of the denominator, we have so far seen the dimension of the state matrix A matrix.

So, if let us say if d s is of some the degree of the polynomial of d s is 3 then in that case we would have the number of state variable s 3, ok, but we know that the output is 1. So, in that case we cannot have the invertibility of the C matrix. But what we want to know that whether it is possible by collecting the signals y and u we can construct somehow this x or not, ok.

So, in this week we will study some conditions that under what conditions we can say that we can reconstruct this signal x, how we will going to reconstruct it we will discusses in the last week of this course.

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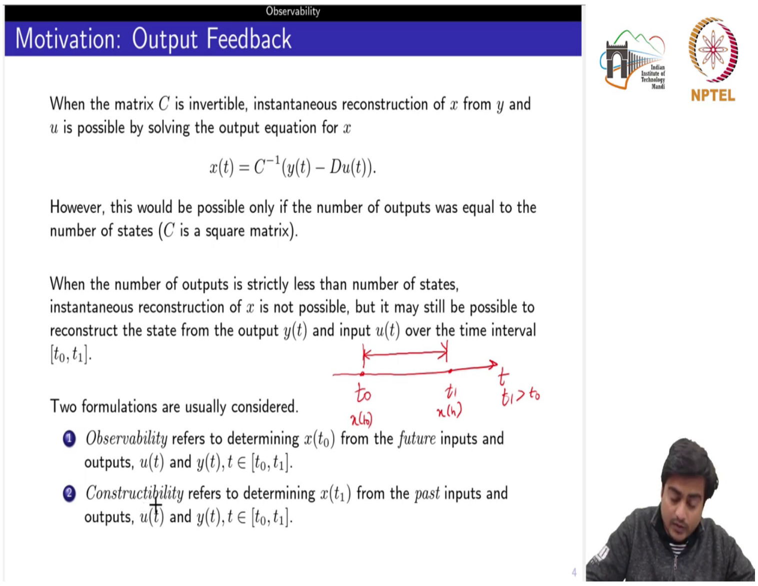

So, before proceeding forward we will we will use two formulations one is the observability and second is the constructability. The observability refers to determine the initial condition of the state at t naught from the future inputs and outputs u t and y t, observed at during the interval t naught to t 1, ok.

So, let us say we let us talk about the time axis. So, this is t, this is t naught and this is t 1 and here we have taken t 1 greater than t naught, ok. Now, start let us say, between this interval we have observed the input output signal. So, using this information of input output signal if I am computing the value of x at t naught then we call it the observability. If we are computing the value of x at t 1, we would call it the constructability, ok.

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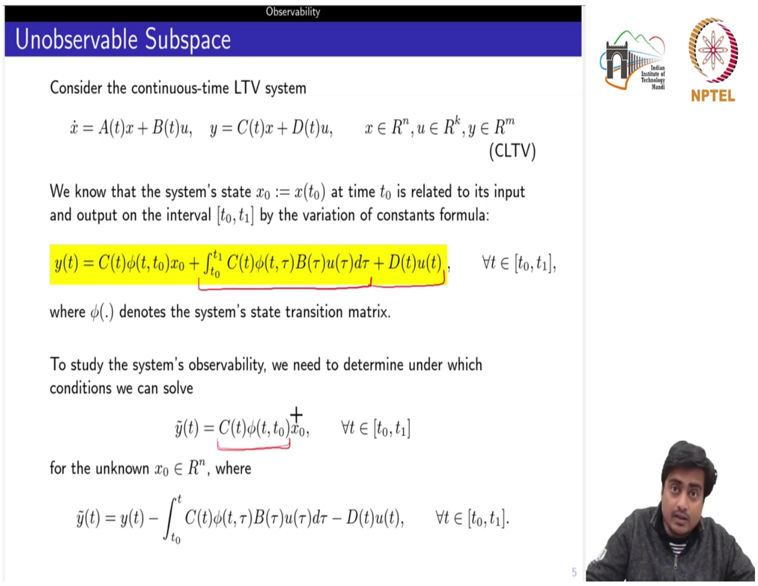

So, then we define the unobservable subspace similarly to what we have defined during the controllability week, where we have defined some subspaces and then did some characterizations. So, consider the continuous time, linear time varying system when all A, B, C, D matrices are time varying with the same dimension. So, we know that the system state x naught defined as x at t naught is related to its input and output on the interval t naught comma t 1 by the variation of constants formula what we had seen in the first week. This is the solution of the LTV system, y t is equal to C of t times state transition matrix phi, x naught and this integrant plus D t u t, ok.

So, to study the systems observability, we need to determine under which conditions we can solve this equation. So, this equation is similarly to what we had seen earlier. So, this is the solution, right. Now, if you pay attention to this term in this term you have information of all the signals and the functions, u we are measuring the signal; B, phi and C you can compute depending on the system matrices, ok.

Similarly, here we know about u and D is already known to us. So, if I take this complete term again onto the left hand side and defined as y tilde which is y t minus this whole term for all t during this interval t naught to t 1, I can specify it as y tilde is equal to C into phi into x naught, where C is my output matrix and phi is the state transition matrix, right. And I want to find out x naught say for example, ok.

Again, I cannot take the inverse of this part because I know phi is non-singular in the square matrix, but since it has been pre multiplied by the output matrix which is still possible that it could be a non-square matrix. So, the overall matrix here would still be a non-square and I cannot take the inverse of this one to directly compute this x naught, ok. But we will see some spaces some subspaces and for which we can characterize that for what or for what elements x naught the we can construct this signal, ok.

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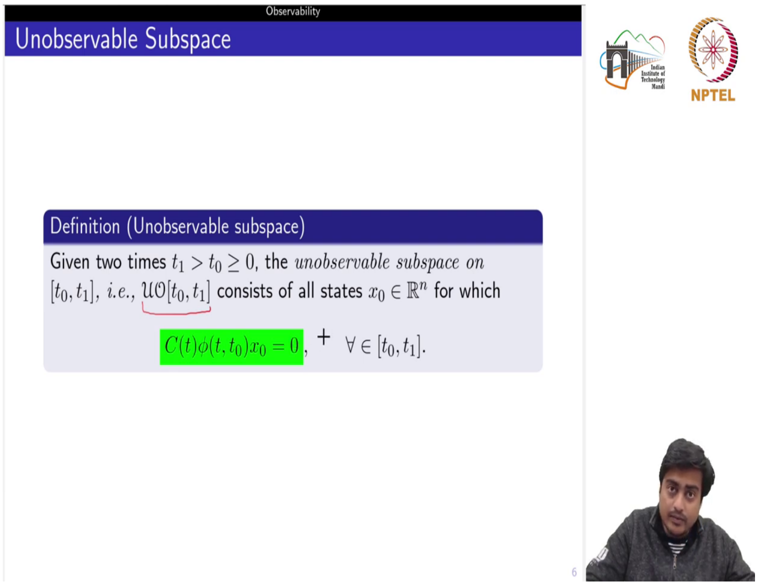

So, the first of all we will define unobservable subspace that given two times t 1 greater than t naught which is greater than or equal to 0, the unobservable subspace on t naught to t 1 be denoted by this by the scalar graphic U and O, unobservable t naught comma t 1 consists of all states x naught for which this condition is satisfied, right. So, there is a direct implication from the previous equation by which we can define the unobservable subspace. So, let us see.

So, this is the equation we had seen earlier y tilde is equal to C phi into x naught, ok. Now, if my y tilde which is basically the subtraction of the output and a function containing input if this subtraction becomes equal to 0, then all those x naught would becomes, would belong to my unobservable subspace because I cannot observe them at, all right. So, this is what this definition says that if the C phi x naught becomes equal to 0 then all those x naught for which this equation is satisfied would belong or would make me or would give me the unobservable subspace, ok.

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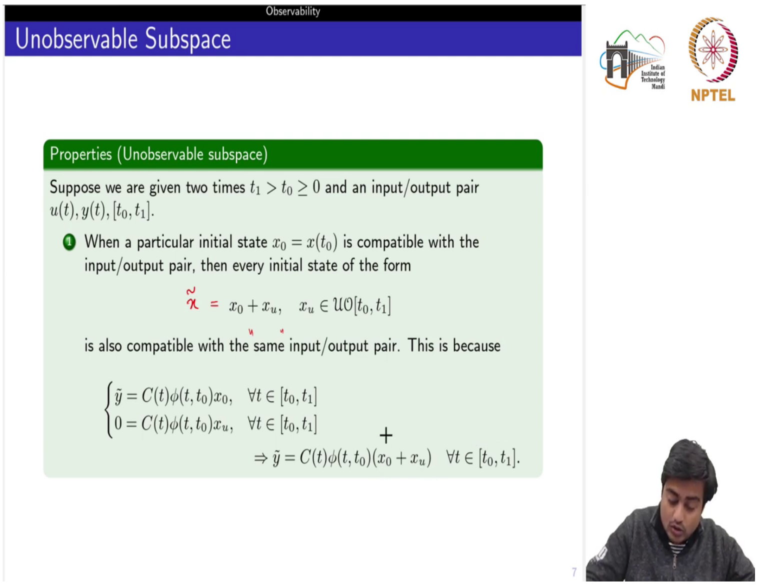

Now, let us see some properties of this unobservable subspace. So, suppose we are given two times again t 1 greater than t naught and an input output pair u y over this time interval t naught to t 1. Now, when a particular initial state x naught that is x defined at t naught is compatible with the input output pair then every initial state of the form, so you can define this as x tilde, that x tilde is defined as x naught plus x u, where x u belongs to that unobservable subspace, right.

So, would also be compatible with the same input output pair. Why? Because say for example, x naught belongs to that input output pair then it means that y tilde will not be 0 for x naught. So, I can write y tilde is equal to C into phi into x naught and since x u belongs to the unobservable subspace it would have, we I would have 0 is equal to C phi times x u. So, if I sum these two equations, I would get y tilde is equal to C phi x naught plus x u. So, I cannot differentiate or I mean to say this one would also belong to the same input output pair.

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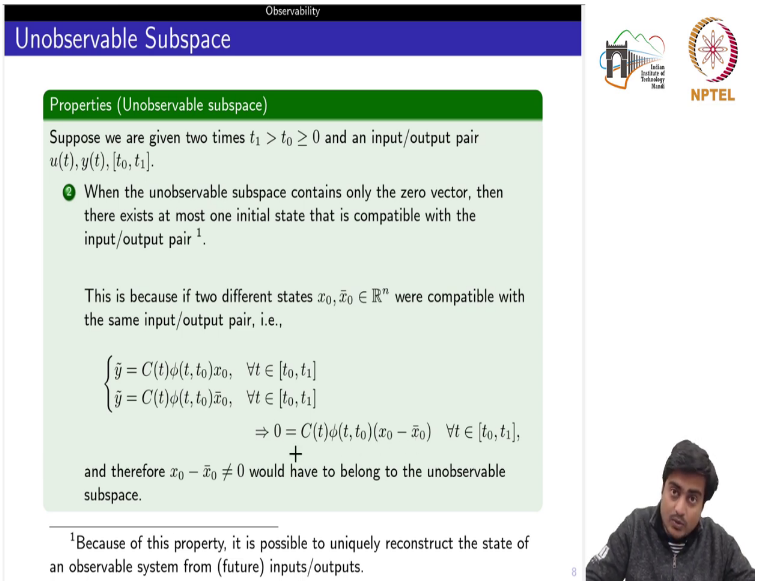

The second property which is the most important one, again considered two times t 1 greater than t naught and the input output pair u y over this interval. So, when the unobservable subspace contains only the 0 vector, then there exists at most one initial state that is compatible with the input output pair, right. So, because of this property only it is possible to uniquely reconstruct the state of an observable system from future input and output. Let us see.

So, consider two different states x naught and x naught bar, which are compatible with the same input output pair. So, I can write the same equation for x naught and x naught bar, right.

Subtracting these two equations I would get 0 onto the left hand side and C phi times x naught minus x naught bar.

Now, this equation means that this x naught minus x naught bar which is not equal to 0 belongs to the unobservable space subspace, because this equation has been satisfied as we have seen the definition. So, there exists at most one initial state, they cannot exist two initial states, right. So, that is why it helps us to uniquely reconstruct the state of the system.

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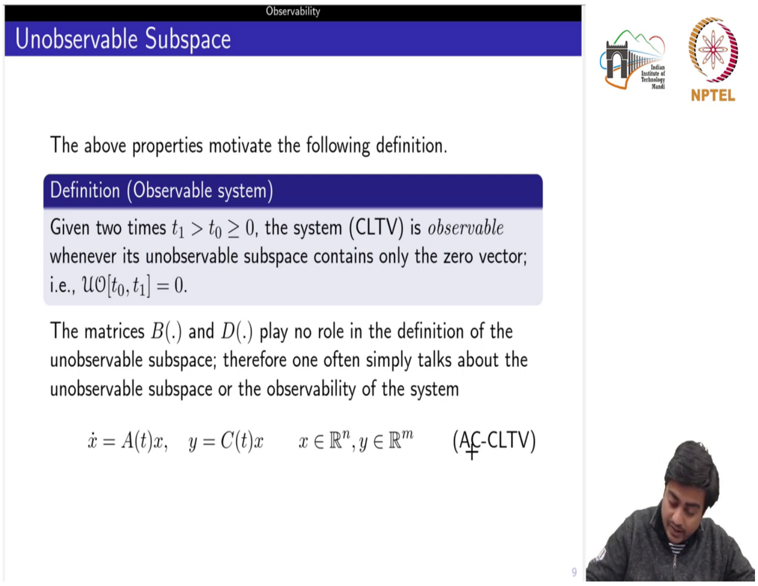

So, the above properties motivate the following definition. So, given two times t 1 greater than t naught the system continuous time linear time varying system is observable whenever its unobservable subspace contains only the 0 vector, right. Meaning to say that there does not exist x naught for which that previous equation is satisfied, right. So, the matrices B and D, we had seen that they do not play any role in the definition of the unobservable subspace. So, we will generally talk about the unobservability or the unobservable subspace or observability of the system or let us say the pair A comma C, which we define pair AC of CLTV.

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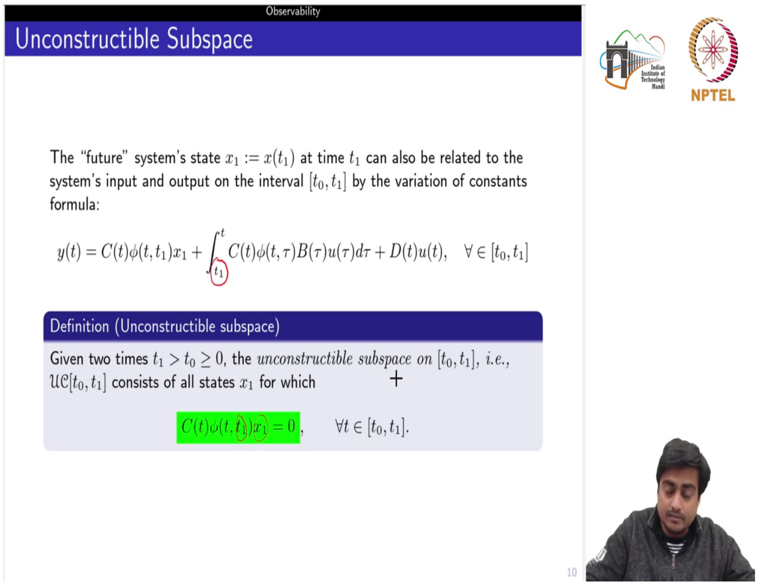

Now, similarly we can define the unconstructible subspace. Now, here we had seen only the value of the state at some particular time based on that we have define the unobservable and unconstructible subspaces. So, here the future system states x 1 defined as x of t 1 at time t 1 can also be related to the systems input and output on the interval by the variation of constants formula by using this one, the only difference is that instead of t naught now we are having t 1 at every places, ok.

And by keeping the same definition that the unconstructible subspace on the time interval t naught to t 1 consists of all states x 1 for which this equation is satisfied. Again, the differences of the t 1 and the value of x at t 1 which we have defined as x 1, ok. So, all those x naught would belongs to the unobservable subspace, if this equation is satisfied for some x naught with phi t comma t naught and now for all x 1 and phi t comma t 1 if this equation is satisfied, then we call it the unconstructible subspace, ok.

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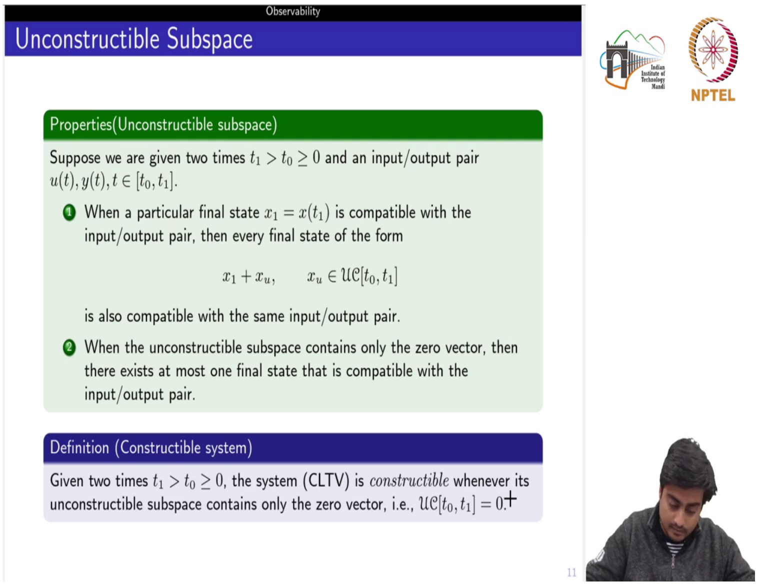

The property would remain the same, that when a particular final state x 1 is compatible with the input output pair then every final state of the form x 1 plus x u because x 1 is compatible and x u belongs to the unconstructible. So, the summation of that would also belong to or would be compatible with the same input output pair, right. Second when the unconstructible subspace contains only the 0 vector, then there exists at most one final state that is compatible with the input output pair and this ensures the uniqueness also.

So, if you talk about now the system the constructible system that the system is constructible whenever its unconstructible subspace contains only the zero vector that is UC is equal to 0, right, that there is no element in that subspace.

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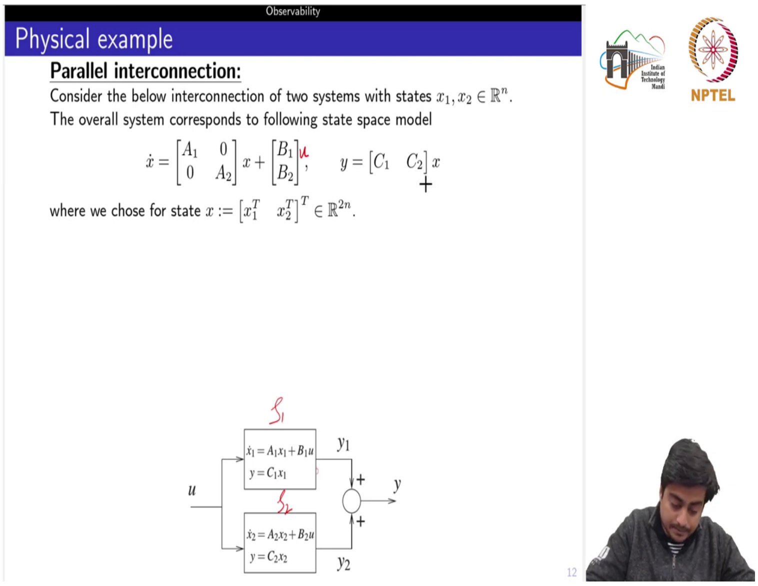

We can see one example let us talk about the parallel interconnection. So, this is one system, let us call its system 1 and this is system 2, and we are supplying the common input to both the system and taking output as the summation of y 1 plus y 2. So, you can represent this parallel interconnection by using this combined state space equation where x is x 1 and x 2 and A1, A 2 would be in the diagonal elements and B1, B 2 and this would be u which is missing, ok. And y is equal to C 1, C 2 x, right.

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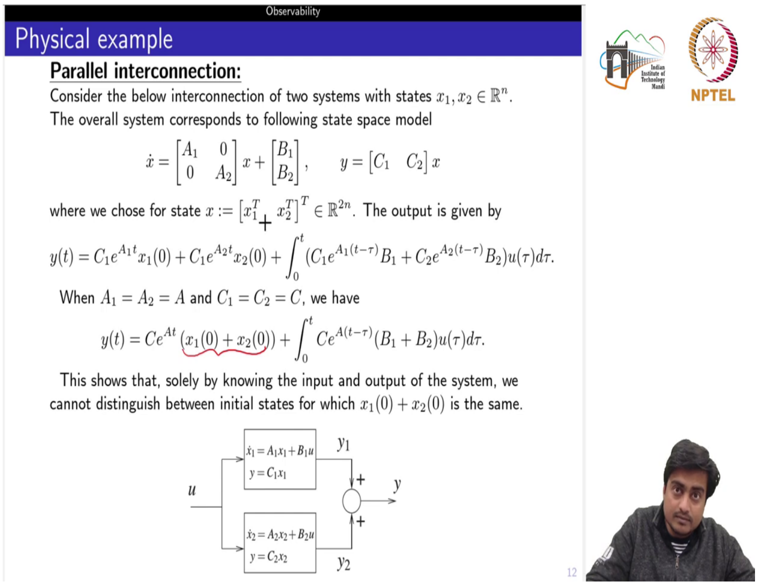

So, the output you can also write by computing y 1 which would be the y 1 of this system and y 2 of this system and I can sum them up to compute y, and you can write it as C 1. It should be this C 1, and this integral is y 1. And this should C 2, so this and this would be your y 2, right.

Now, suppose if your C 1 and C 2 become equal to C and A 1 and A 2 become equal to A, then this complete equation I can write this as this one, ok. And you would notice that that by knowing only the input and output we cannot distinguish between the initial stage for which x 1 of 0 plus x 2 of 0 is the same, right. Whatever I would compute let us say all the conditions of the observability has been satisfied and I have been able to compute this x, but that x would be the summation of both the states. And the concept of observability says that I need to know x 1 and x 2 individually and here I am getting the summation. So, this whenever you are having this parallel interconnection the system might not be observable always, ok.

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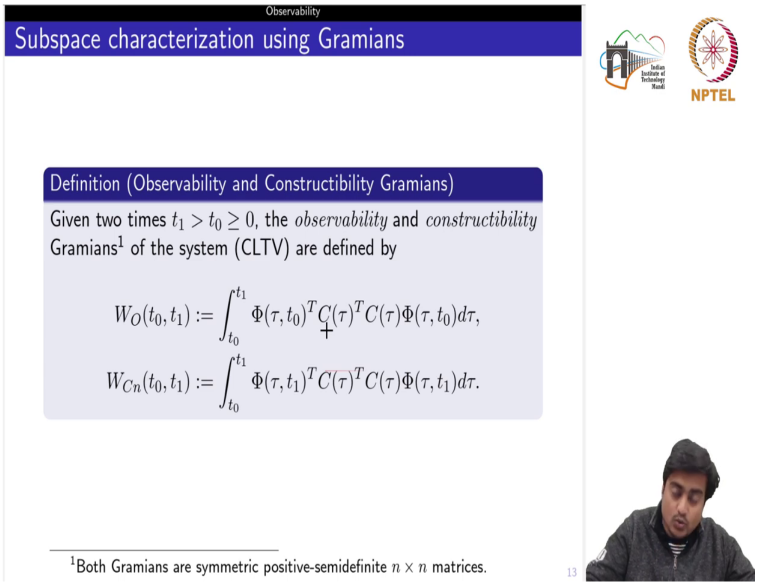

So, similarly to what we had seen during the controllability week that we had introduce the Gramians and we did the subs the characterization of the discussed subspaces using those Gramians.

So, first of all we will define the observability and the constructability Gramians says, that given two times t 1 greater than t naught the observability and constructability Gramians of the system are defined by this W o and W Cn over t naught to t 1, ok. So, this here phi is the state transition matrix, C is the output matrix and yes. So, both Gramians here are symmetric, positive, semi-definite, square matrix of dimension n, ok.

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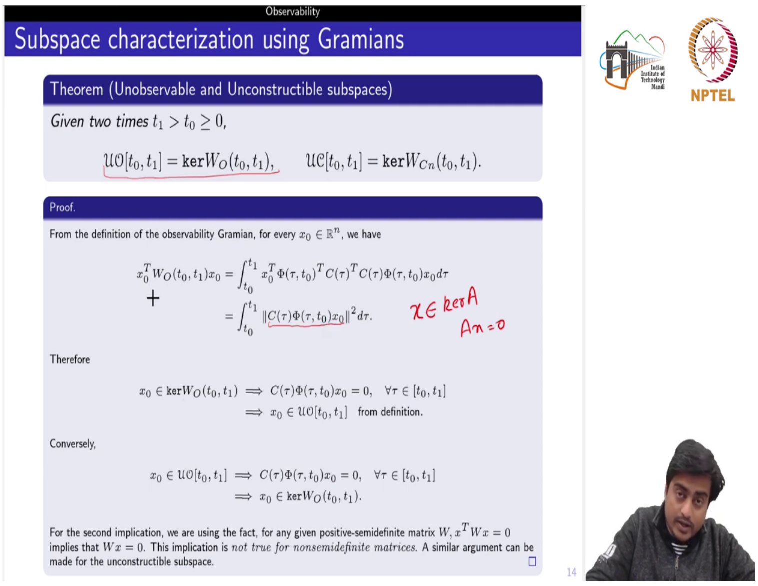

So, based on this Gramians we can give our first results which basically characterizes these subspaces. So, the first trivial method is to find out all those x naught for which the previous equation is satisfied that is C phi into x naught is equal to 0.

Now, if you want to characterize the entire subspace that subspace is basically given by the Kernel of the control the observability Gramian matrix W o, ok. Similarly, the unconstructible subspace is equivalent to the Kernel of the constructability Gramian W Cn, ok. We can see a quick proof.

So, from the definition of the observability Gramian we can write it in the first of all we can write it in the quadratic form as x naught transpose W o into x naught and then substituting the value the matrix W o here, I would get this one. And then using the property of the norms and the quadratic forms I can compress this long expression into thus squared norm of this term, ok. So, this is basically C phi x naught and this is the transpose. So, I can write it the squared norm of this term, right.

Now, let us see because now we want to show that if an element belongs to this unobservable subspace then that element would also belong to the Kernel of the observability Grami, ok. So, let us see. So, therefore, x naught belonging to the Kernel of this W o implies, so if you remember that the if an element belongs to this Kernel of any matrix A, then it means that A of x is equal to 0, ok.

So, if x naught belongs to this Kernel of W o, then what would happen? W o x naught would be 0. Now, W o x naught, if I put W o x naught 0 then this left hand side term would become equal to 0, right. Now, if this term becomes equal to 0 and this is the integral of the squared norm and this would be only 0 if this inside the norm is itself equal to 0.

So, this implies that x naught belonging to the Kernel of this observability Gramian implies that this term would be equal to 0 and by taking the definition of the unobservable subspace that all those x naught would definitely belongs to that unobservable subspace. So, all those elements which belong to this subspace would definitely belong to this unobservable subspace meaning to say that this equivalents would be is satisfied, ok.

Now, see the reverse one. Now, if we have x naught belonging to the unobservable subspace meaning to say this equation would satisfy, right. Now, if this equation is satisfied and I put it here it means the quadratic form itself is 0. Now, if the quadratic form itself is 0 it implies that this is equal to 0. Now, there is and some under some condition this this implication would hold, that is to say that satisfying this one implies x naught would belongs to the Kernel of this observability matrix and this is only possible because W is a positive semi definite matrix. Now, if W is some non-semi definite matrix then this implication will not hold, ok. So, a similar argument can be made for the unconstructible subspace.