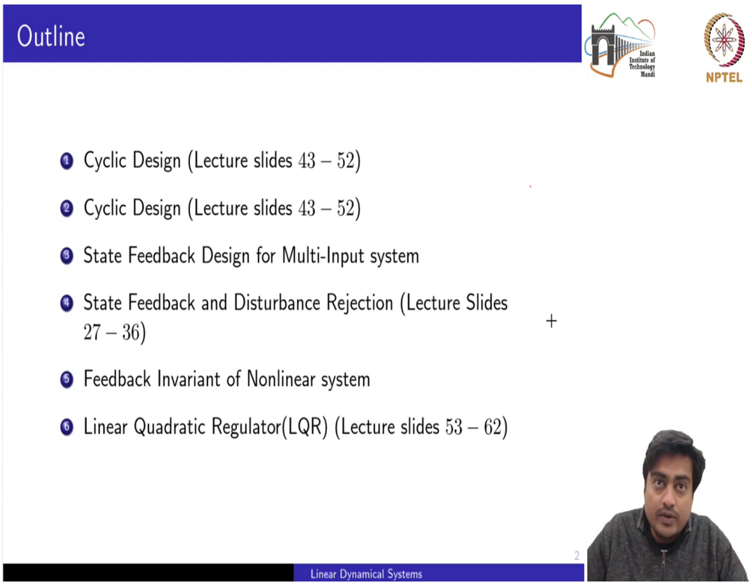
Linear Dynamical Systems
Prof. Tushar Jain
Department of Electrical Engineering
Indian Institute of Technology, Mandi

Lecture – 35
Tutorial on State Feedback, Part-II

So, hello everyone. We will be starting with the Tutorial on State Feedback, Part II of the Linear Dynamical Systems.

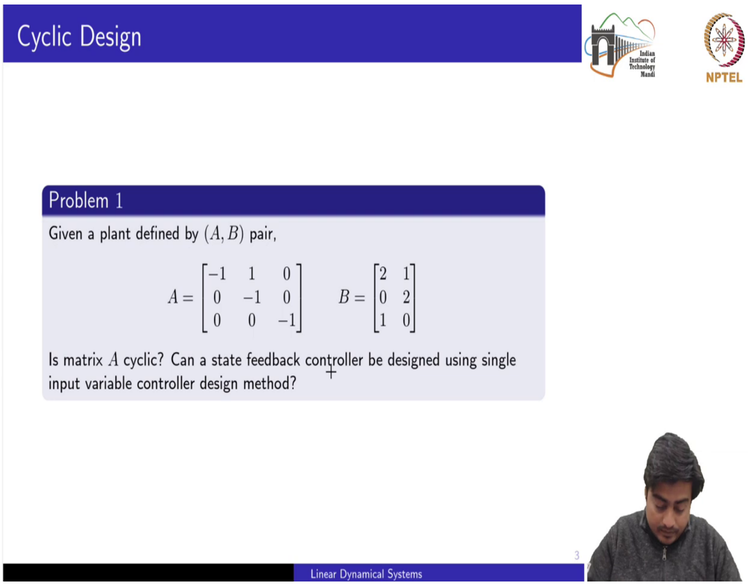
(Refer Slide Time: 00:20)



So, these are the questions which we would address in this tutorial. The first two question deals with the design of the cyclic design of the controllers, the third is the state feedback design for the multi input systems.

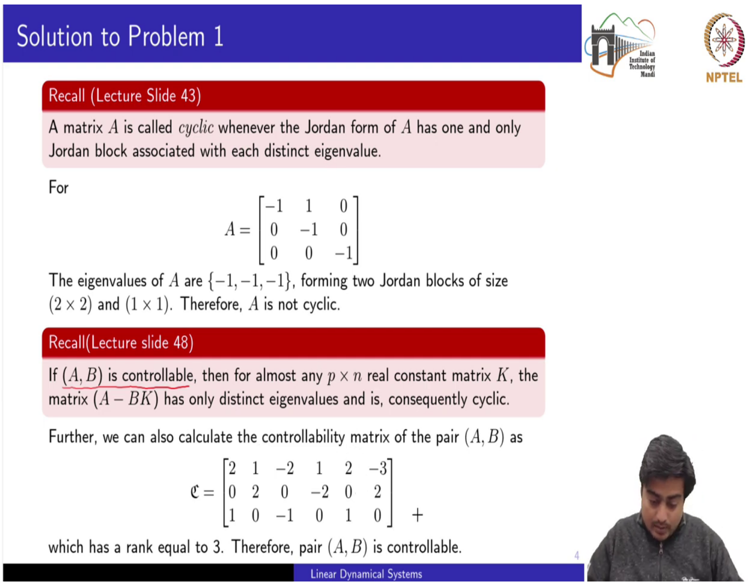
So, in that tutorial part I we had seen mainly the state feedback design for single input system. In this tutorial we will also see the problem of the state feedback and the disturbance rejection controller which we have also defined as a robust controller. And we will also discuss the linear quadratic regular problem in this tutorial.

(Refer Slide Time: 00:57)



So, in the problem 1 given a plant defined by the pair A, B, where A, B matrices are given by this. So, the first question we need to address to find out whether the matrix A is cyclic or not, and the second part of the question is can a state feedback controller be designed using single input variable control design method, ok.

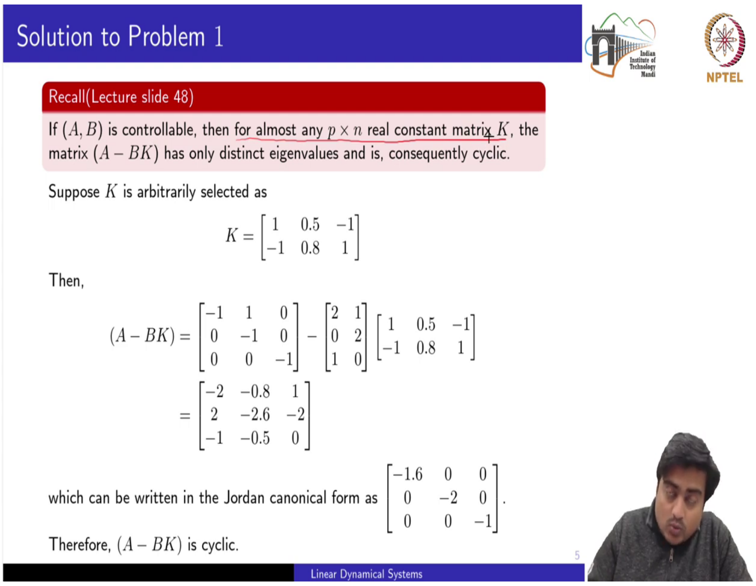
(Refer Slide Time: 01:22)



So, now recalling the lecture slide we have define a matrix A a cyclic matrix whenever the Jordan form of A has one and only Jordan block associated with each distinct eigenvalue. So, for the given matrix A, we can first compute the eigenvalues and we see that the eigenvalues are the repeated eigenvalues at minus 1 which forms two Jordan blocks of size 2 cross 2 and 1 cross 1. So, according to the definitions we had introduce in the slide the matrix A is not a cyclic matrix, ok.

Now, the second results says that if the pair A, B is controllable then for almost any p cross n real constant matrix K the matrix A minus B times K has only distinct eigenvalues and is consequently cyclic. So, first of all we need to check the controllability of the pair A comma B, and this we can do this by computing first the controllability matrix and then finding out the rank of that controllability matrix which happens to B 3 here, which is equivalent to the dimension of the A matrix. So, we ensure that the pair A comma B is controllable.

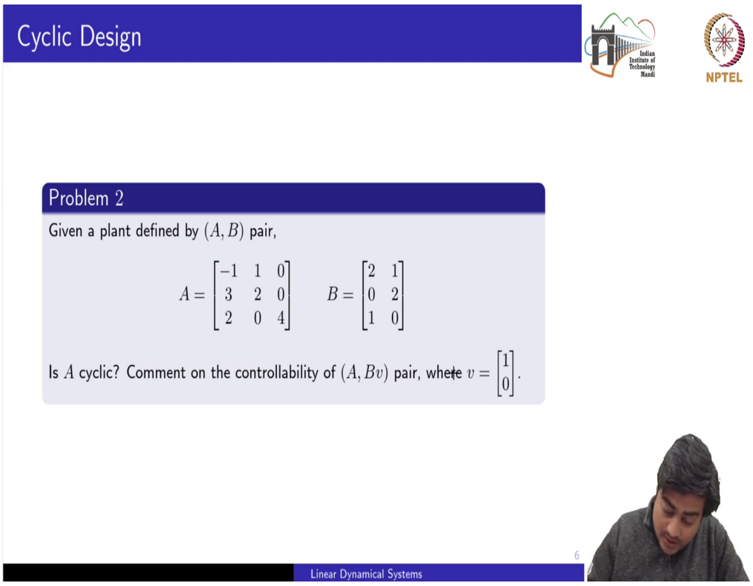
(Refer Slide Time: 02:47)



So, it calling again this results it says that for almost any real constant matrix K of appropriate dimension could ensure the of cyclicity of the matrix A minus B times K. So, here we select K is any arbitrary matrix is this one and then computing A minus B times K which happens to be this matrix and we can write this into the Jordan canonical form in this. So, we see that all the eigenvalues are place at the diagonal and therefore, this matrix is a cyclic matrix, ok.

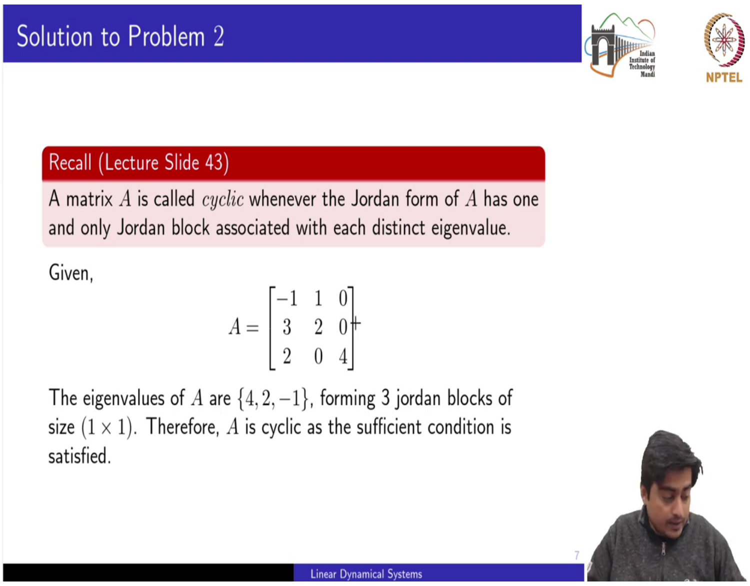
Now, just for the verification you can also select another K matrix and again verify whether for the matrix K what you have selected is this matrix A minus B times K, a cyclic or not, because we have already seen that the pair A, B is controllable. So, it means that they exist more than one K matrix.

(Refer Slide Time: 03:55)



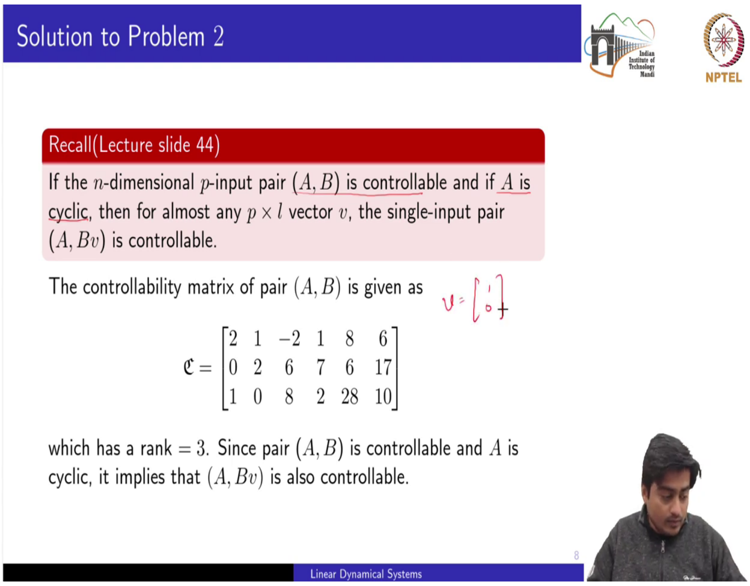
Now, the problem 2 again we are taking a pair A comma B. So, the first part remain the same where we want to check the cyclicity of the matrix A and here we are now introducing a vector v 1 comma 0, and now we want to comment on the controllability of the pair A comma B times v. So, we are discussing these two problems, so that finally we can design the multi input controller.

(Refer Slide Time: 04:25)



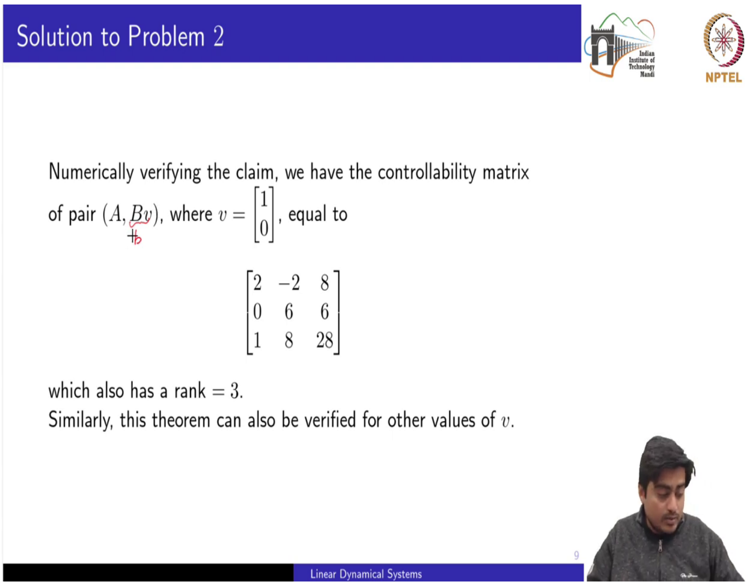
So, the first definition remain the same of the cyclic. We just compute the eigenvalues of the given matrix A which happens to the distinct eigenvalues. So, it is already ensure that the matrix A is the cyclic matrix, ok.

(Refer Slide Time: 04:40)



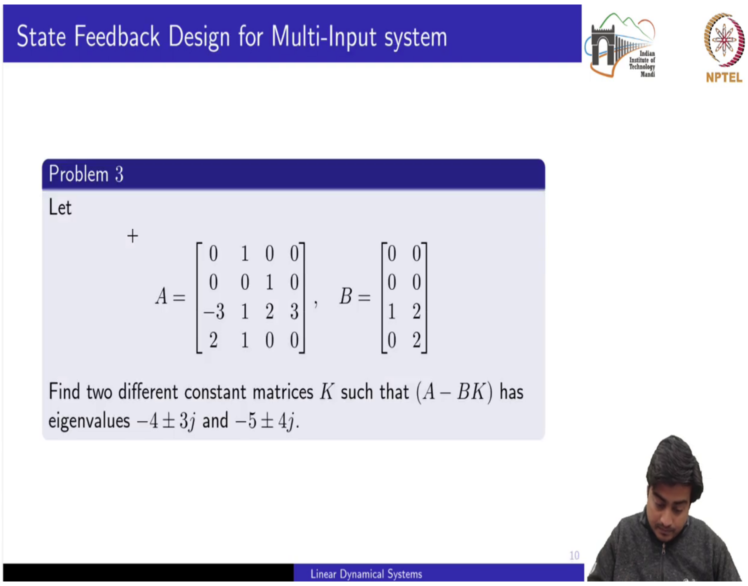
Now, there was another result which we also discussed that if the n-dimensional p-input pair A comma B is controllable and if A is cyclic, so you can also check the controllability of the matrix of the pair A, B, and by computing the controllability by computing the rank of the controllability matrix which happens to B 3. So, this pair is controllable. We have already seen that the matrix A is cyclic, so it implies that for almost any vector v, the single input pair A comma B times v is controllable. So, here in this problem the v vector is given to us is v equal 1, 0.

(Refer Slide Time: 05:29)



So, we can numerically verify this as well that if I compute another pair A comma B because now this B would be a vector and if this B is a vector we know that the controllability matrix would be a square matrix. So, which we can see here and again the rank is 3. So, for we have verified this for this v which is given into the problem now you can also take another v vector and can verify whether with you are selected v vector this A comma B pair is controllable, ok.

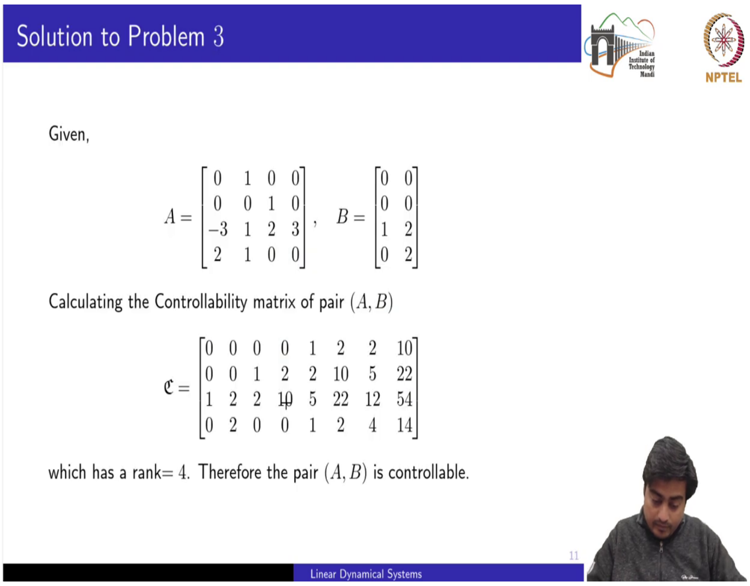
(Refer Slide Time: 06:05)



Now, this problem 3 deals with the state feedback design. So, whatever the results we had seen, the first two problems we will apply here and finally, compute the matrix K the state feedback gain.

So, here A, B matrix are given to us where A is a 4-dimensional and B is a 4 times 2, meaning to say that we have two inputs. And we want to design or we want to synthesize the constant matrix K such that this matrix A minus B K which is the state feedback matrix has eigenvalues located at these two these 4 locations.

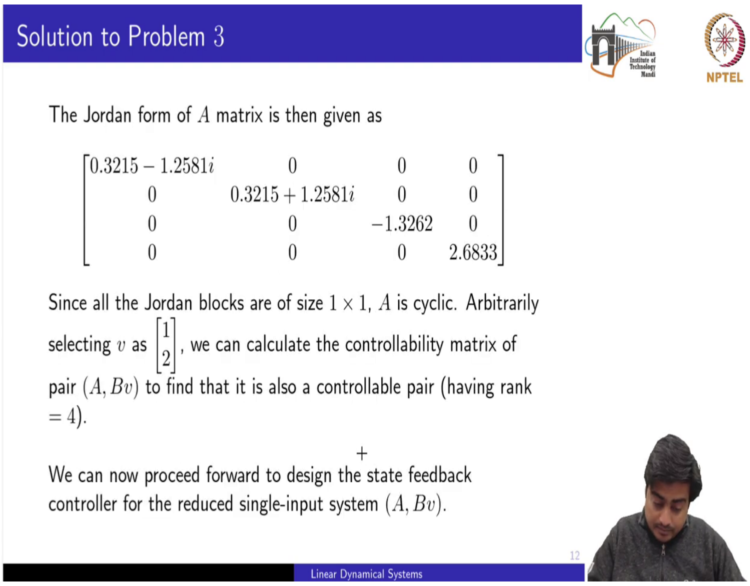
(Refer Slide Time: 06:50)



So, before coming on to the solution to this problem we would recall a couple of points. So, here we see that the system contains two inputs. Now, if the system is only single input we know that the state feedback gain matrix would be unique, but since we have two inputs we cannot ensure the uniqueness of the constant matrix K, although many of these K matrices would put the eigenvalues at these locations, ok.

So, here we would use cyclic design to synthesize this state feedback gain matrix. So, given this pair A, B first we can check the controllability matrix and since the rank of this controllability matrix is 4, the pair A, B is controllable.

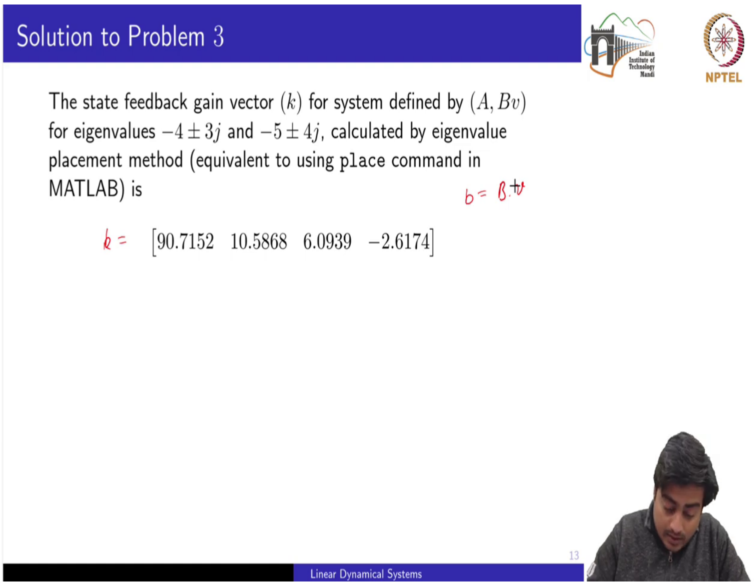
(Refer Slide Time: 07:53)



Now, writing the Jordan form of the A matrix which is given by this. So, we see that all the Jordan blocks are of size 1 cross 1. So, the matrix A is cyclic, the pair A comma B is controllable. So, we can select any v what we had done in the problem two as. So, first of all we select the v vector as 1 and 2, we can calculate the controllability matrix of the pair A comma B times v and one of the results we had seen earlier says that this pair would also be controllable.

So, this pair is the controllable pair. So, now, we can proceed forward to design the state feedback controller for the reduced single input system A comma B times v.

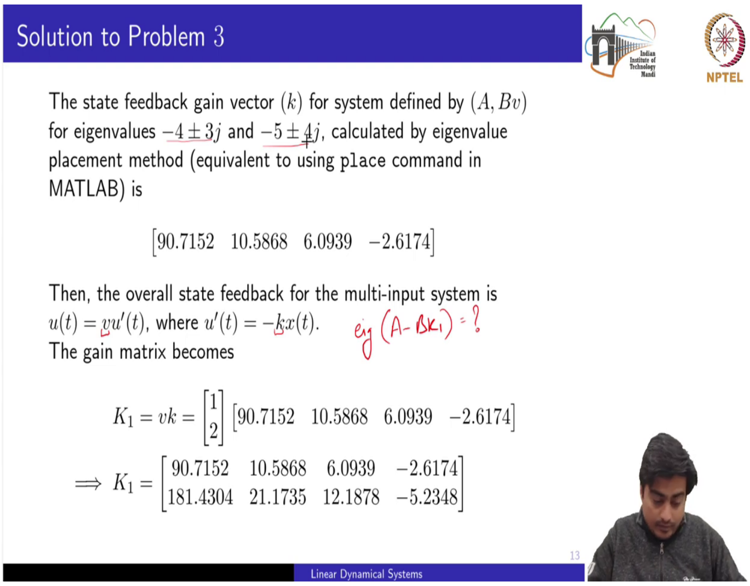
(Refer Slide Time: 08:39)



The state feedback gain vector k because we want to synthesize this is a vector k, so that the eigenvalues could be placed at these locations and we can do this by using the eigenvalue assignment method, we had discuss in the lecture or you can use the place command in MATLAB as an alternative. So, this k vector we have computed.

Now, this is for the single input system, in the single input system is where we have taken this b as capital B times v, ok.

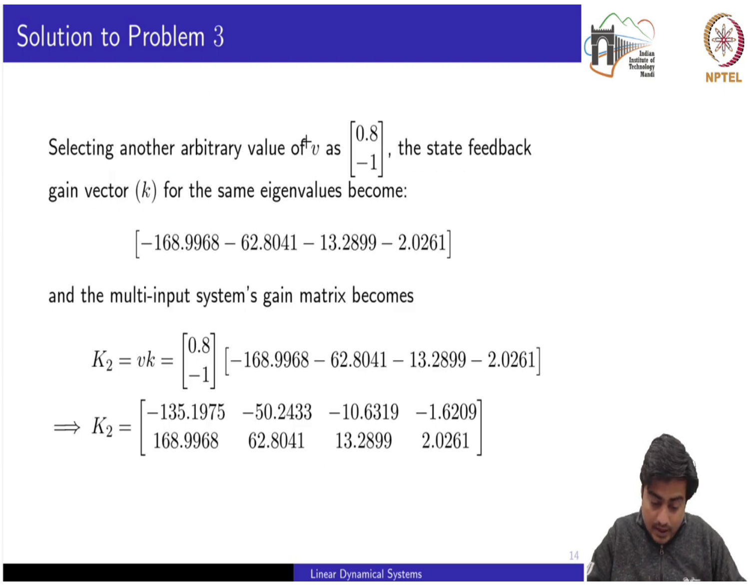
(Refer Slide Time: 09:20)



But we want to synthesize this k and we can do this by computing the original control input signal u as v times u dash where u dash is now minus k times of x, ok. So, now, if we compute this K 1 which would be the multiplication of this v vector or v column vector and this k row vector, so we can compute this K 1 matrix, ok.

Now, you can also compute the eigenvalues of this matrix, with by plugging this K 1 matrix here you would see that all the eigenvalues are located at these 4 locations.

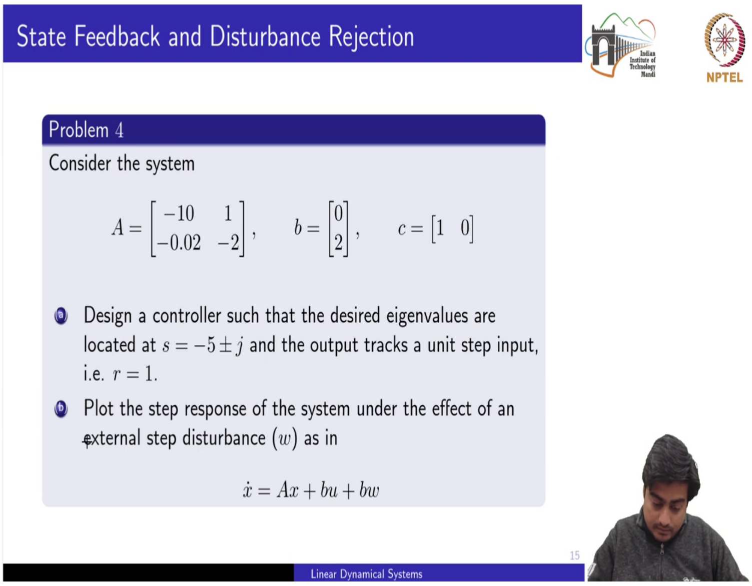
(Refer Slide Time: 10:10)



Now, the second matrix second state feedback gain matrix we can compute by selecting another arbitrary vector v, column vector v which is we have selected here as 0.8 and minus 1. Then following the same procedure first of all computing the state feedback gain vector k for us reduced single input system and then computing K 2 as the multiplication of the column vector v, what we have selected here is and the row vector K, what we have compared here.

So, this could be another state feedback gain matrix. Again you can compute the eigenvalues of the state feedback matrix by plugging this K 2 and you will see that all the eigenvalues are place at the desired locations.

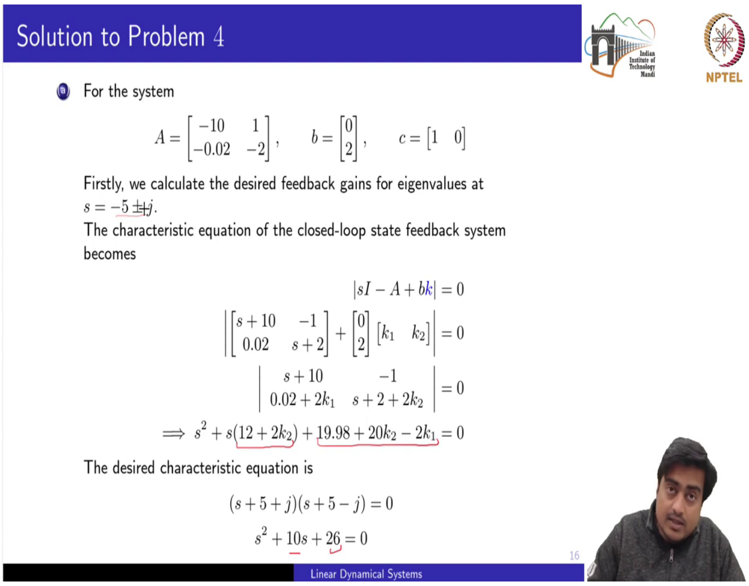
(Refer Slide Time: 11:58)



So, in this problem we would see the two points. First of all we will design the state feedback gain which ensure the stability, second we design the feed forward gain such that the another objective which is the tracking objective can also be fulfill, this is part A. And in part B we would see that if there are some external step disturbances happens to be on this system then how the response of the closed loop system would behave, ok.

So, first we will deal with this part A where we want place the desired eigenvalues on the left hand side and the output should track a unit step input which is r equal to 1.

(Refer Slide Time: 11:50)

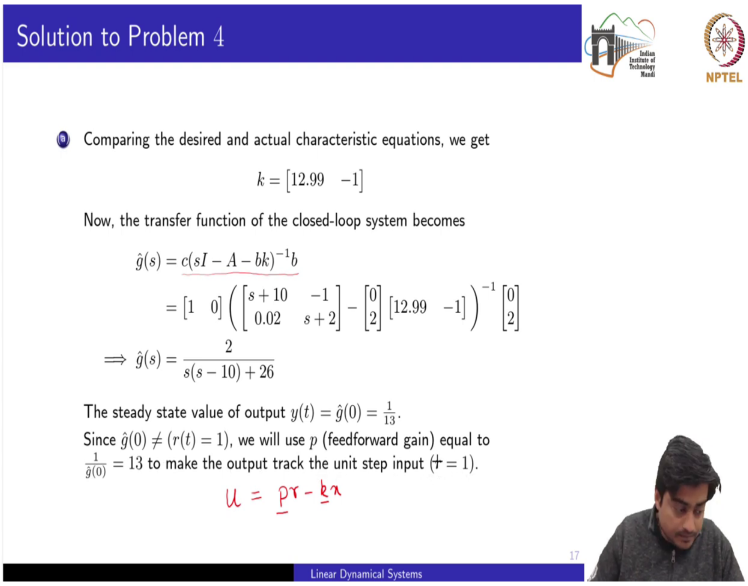


So, for the given information A, b, c we can calculate this eigenvalues, we can calculate the desired feedback gains for the eigenvalues this one. And there are number of methods we had discussed because. Again this is a input controller design. So, you can use either of the designed method, either the eigenvalues assignment approach or the Liapno design base approach, ok.

But here there is another way what we are following here which is which you can do if your system is or having a dimension 2. So, we want to design this state feedback vector k. So, suppose we have design this k and if we compute the desired eigenvalues of the of the state feedback gain it should be equal to the desired eigenvalues.

So, now, if we compare the characteristic equation, so this part should be equal to 10, in this complete part should be equal to 26. So, now, we have two equations two unknown, as k 1 and k 2. So, if you plug this k 1 and k 2 here you would get the desired eigenvalues located at these locations, ok.

(Refer Slide Time: 13:14)

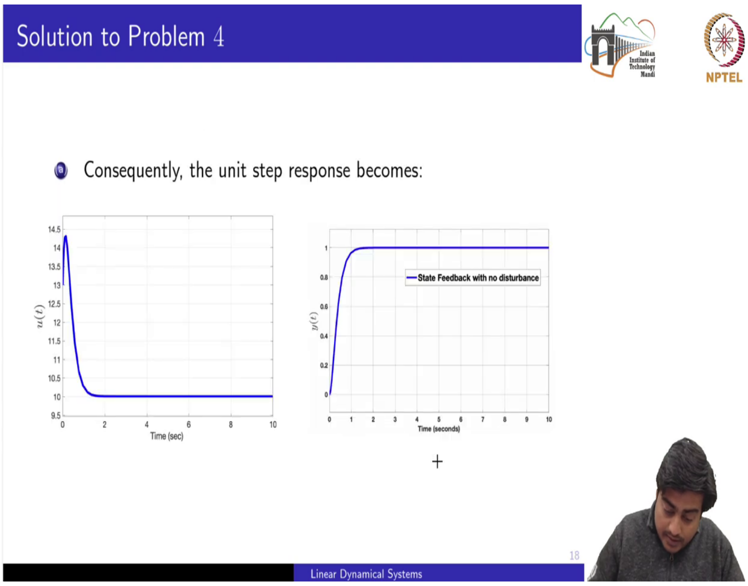


So, computing this k 1, k 2, we computed 12.99 and minus 1. So, once we put this k into the closed loop the closed loop stability has been ensured, but it is not corrected whether the output would going to track the reference single. For this if you recall the lecture slides we need to investigate about the transfer function or the DC value of the feedback system. This we can do by computing the transfer function and then computing the value of the transfer function at s is equal to 0 that is to say the DC value or the steady state value of the transfer function.

So, the transfer function can be computed by using this formula where we have we know all these vectors and matrices c, A, b, and b; k we have computed. So, we see that the closed loop system is given by this, so the steady state value of the output y you would see is given by 1 by 13. So, even if you are supplying the reference signal r is equal to 1, it will not be able to track only with this the state feedback controller.

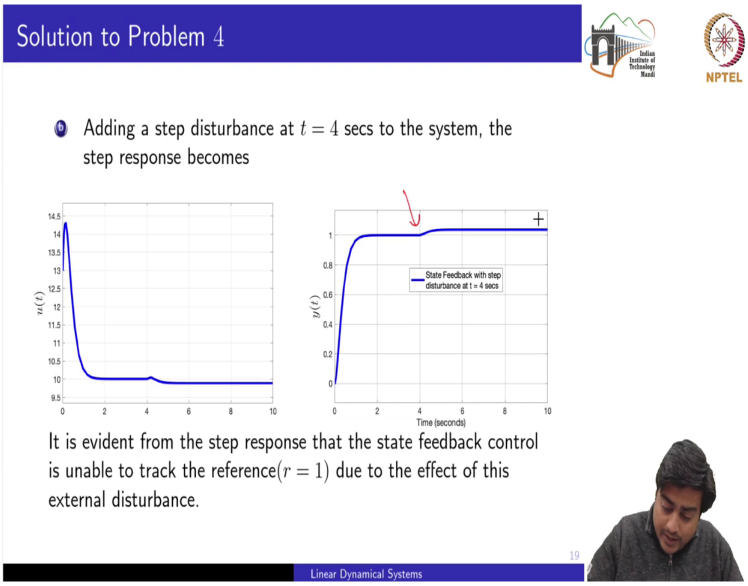
We need to provide feedfoward gain, so the control input or the control law would change to p r minus k times x, where k we have computed which ensures the stability and now we want to compute the p such that r would become equal to y, ok. So, since we know the DC value and we can put the DC value as the inverse or else the inverse of this value which is 13 to make the output track the unit step input r is equal to 1.

(Refer Slide Time: 15:07)



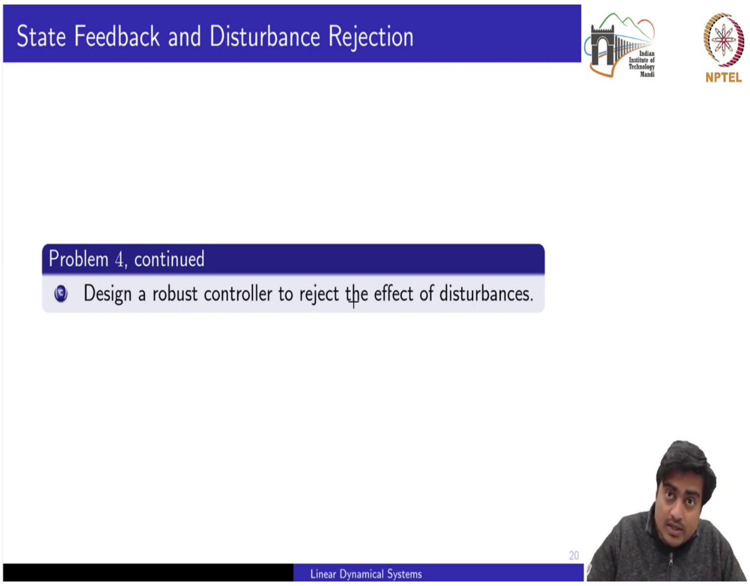
This we can see in simulations also that once I put the controller as u is equal to p r minus k x, the output of the closed loop system tracks the reference r is equal to 1 and this is the control signal, ok. Now, the second part of this problem deals that if there is some disturbances acting on to the system whether the we need to identify whether the output still able to track the reference signal.

(Refer Slide Time: 15:44)



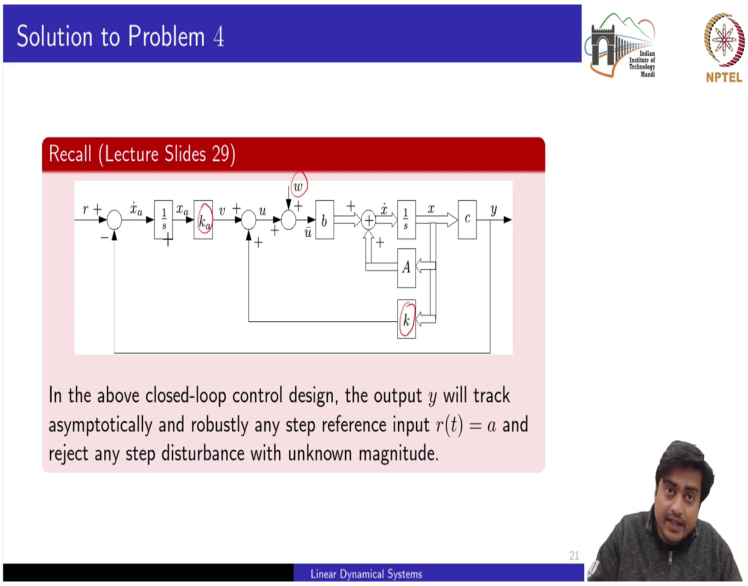
So, this we you can do in simulations also that we have added a step disturbance at t is equal to 4 seconds to the system and the step response or the output of the system is now given by this. So, at the time when the disturbance was inserted into the closed loop system the output was not able to track the reference signal, ok, due to this external disturbance. So, disturbance has some predominant effect on to the closed loop system.

(Refer Slide Time: 16:21)



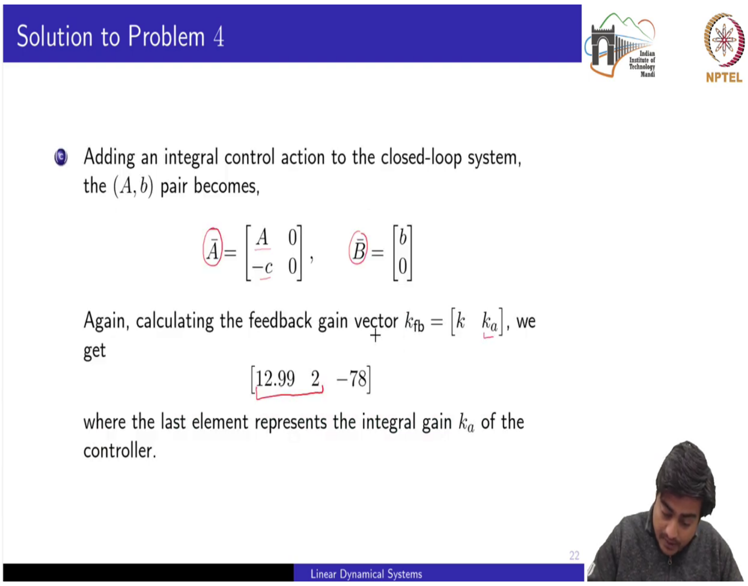
So, now we want to design a controller. You can call it a robust controller or a disturbance rejection controller, so that the effect of the disturbances on to the closed loop or the response of the system can be nullified.

(Refer Slide Time: 16:38)



So, we recall the lecture slides where that is 29 on which we have discussed another closed loop architect, another closed loop architecture which ensure that the this external disturbance or the effect of this external disturbance on to y would be nullified, so that two design parameters are k and k a.

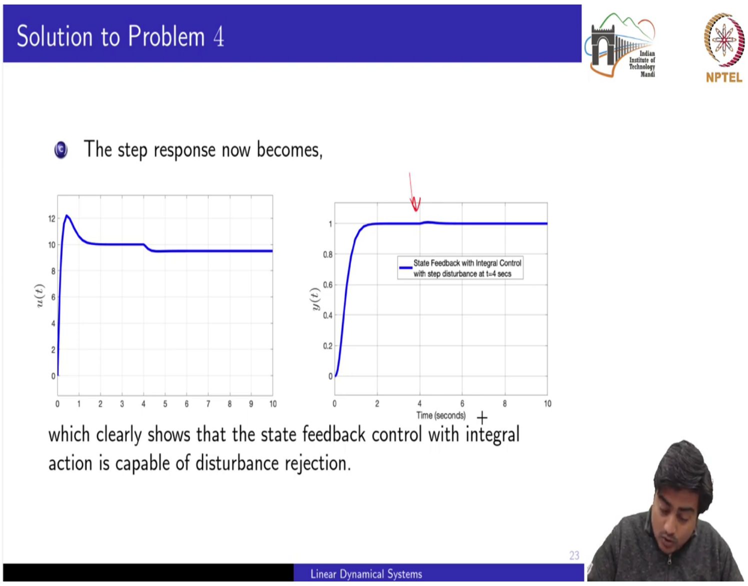
(Refer Slide Time: 17:10)



And to design this first of all we compute another pair which is nothing, but the augmented pair with the original system and with this integrator because this integrator would now become a part of the plan and this happens here. So, here we computed two different matrices A bar, B bar and now we can calculate the feedback gain vector.

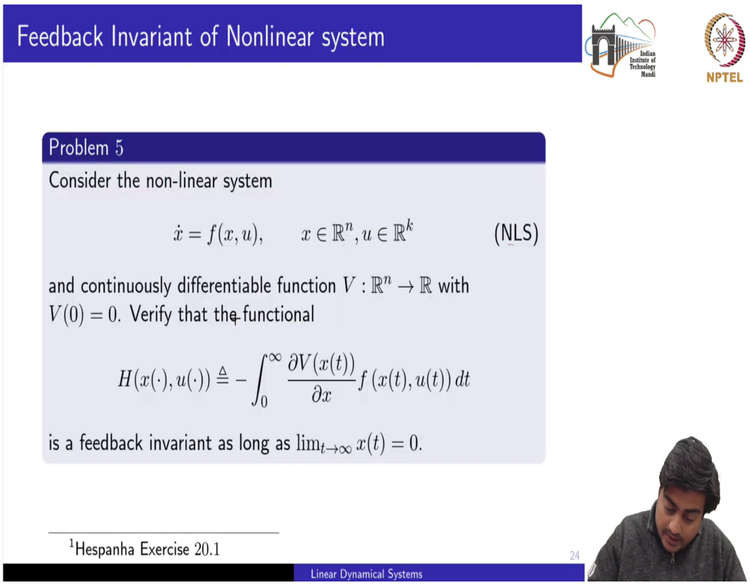
For this new pair A bar, B bar and we can partition it as k and k a, where k this part is the feedback one and k a which is being acted on to the forward path from the reference signal, ok. And you can design and you can either of the method, the eigenvalues assignment method or the Liapno base method.

(Refer Slide Time: 18:09)



So, now if we pay attention to the closed loop signals of the system on which the disturbances being acted. So, you see when the disturbance is been added at t is equal to 4 the output almost tracks the reference signal now, ok. Because of this new structure we are able to reject the effect of the disturbances on the output signal path. And this happens we know already implicitly because of this integral action which is able to diminish the error between the reference and the output, ok.

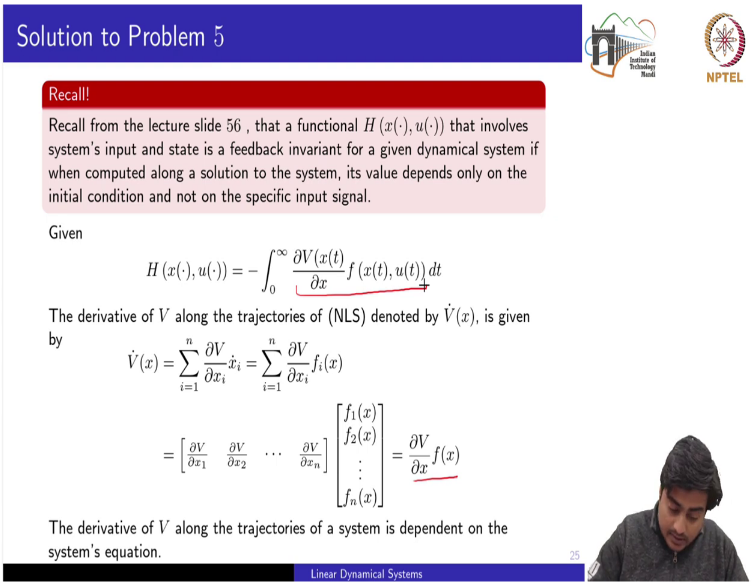
(Refer Slide Time: 18:52)



So, this problem 5, it is we want to compute the feedback invariant. So, here we are we have consider a non-linear system which is which generic representation can be given by this x dot is equal to f x comma u, which is a non-linear system. And there are some properties associated to some function V which we are introducing that a continuously differentiable function V with V of 0 is equal to 0, ok.

We need to verify or we need to validate that whether this function H defined as this one is a feedback invariant as long as the x t is equal to 0 as t tense to infinity, ok. So, this problem was taken from book by Joao Hespanha and this problem is exercise number 20.1, ok.

(Refer Slide Time: 19:49)



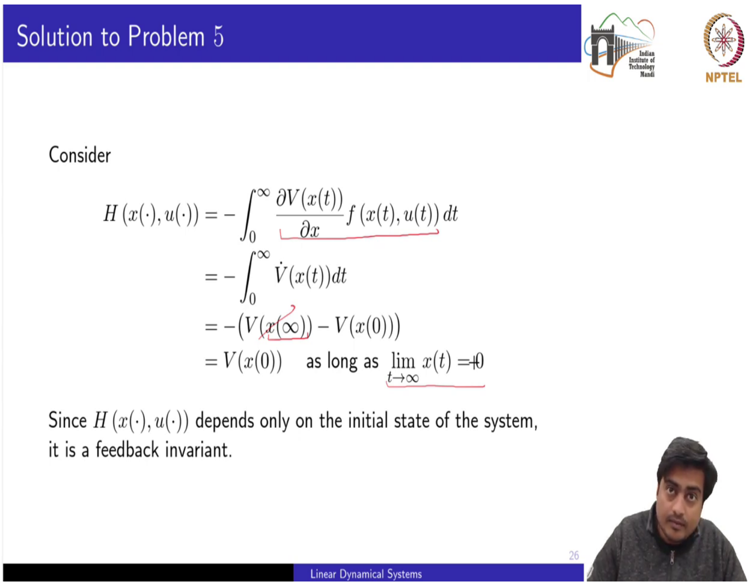
So, recall from the lecture slide number 56, we have defined a functional H is a feedback invariant whenever if when computed along a solution to a system to the system its value depends only on the initial condition and not on the specific input signal. So, this is how we defined any function which involve system input and state as a feedback invariant, ok.

So, we are provided this function H on to the left hand side, on to the right hand side as the integral from 0 to infinity the partial derivative of V with the respect to x and a function f x comma u dt, ok. So, we want to show this function is nothing, but function of only the initial condition and not on the or it does not depend on the specific input signal.

So, first of all we will treat this V or we do the some analysis on this V to reduce the integrant for better visualization. So, the derivative of if I compute the derivative of V along the trajectories of non-linear system which we denote by V dot x it is given by x because V is a functional, and which depends from x 1 to x n. So, I can I need to compute its derivative with respect to each x and this and since V is a function of x it would be multiplied by the derivative of x itself, ok, and it should be summed up because V if you see in the definition it maps from n space to R, R n to R, ok.

And we know that x dot i, would be equal to f i of x where f is a vector differential function. So, I can represent this summation as this row vector and this column vector which finally, could be written as del V by del x dot f x, ok. So, the derivative of V along the trajectories of a system is dependent on the system equation. Now, you would see some similarity. So, this part would now happens to be equal to this part, right.

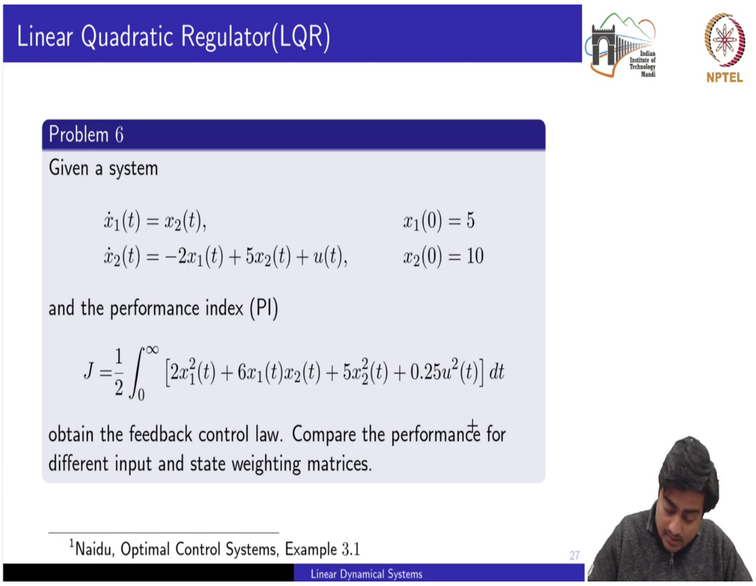
(Refer Slide Time: 22:30)



Now, let us see in more details. So, consider again this one. I can write this whole part what we had seen as equal to V dot of x and if I integrate this V dot I would get V of x at infinity minus V of x at 0, ok. So, now, if I use this condition that my x would become equal to 0 x t tends to infinity. So, this inside part would become equal to 0 and I know from the property of the function V that V of 0 is equal 0. So, this part would go away and the remaining part is V of x at 0.

So, by computation we see that this functional H depends only on the initial condition. So, this is a feedback invariant function. And this, it becomes a feedback invariant function only under this condition otherwise it will not be possible, ok. And this if you recall it is similar to the condition of the asymptotic stability, right.

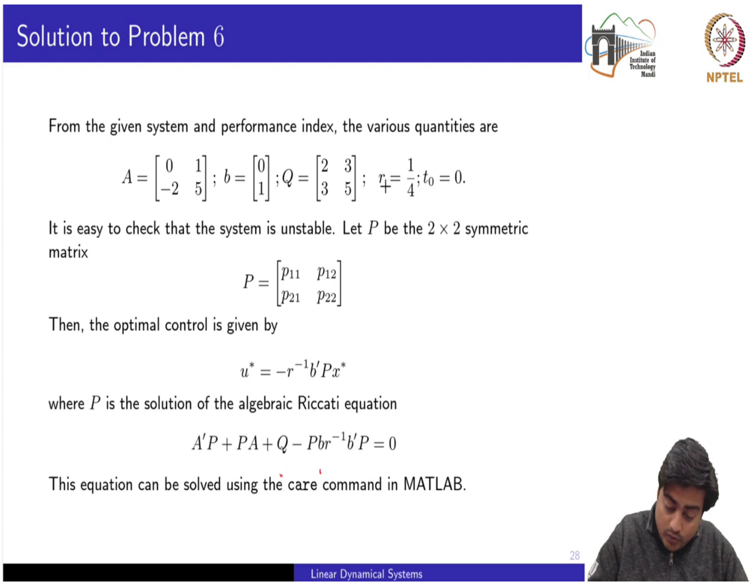
(Refer Slide Time: 23:44)



So, the last problem is about the linear quadratic regulator problem. So, we have provided this state base system which is single input and two state system and the performance index is now given as here. So, far we have considered about the performances as the stability or the tracking. Now, here in the performance index is given by, but with this particular cost function and we want to obtain a feedback control law which minimizes this performance index. This is the first part.

Now, the second part is that with respect to different input and state weighting matrices we want to compare the dynamics of the state and input signals, ok.

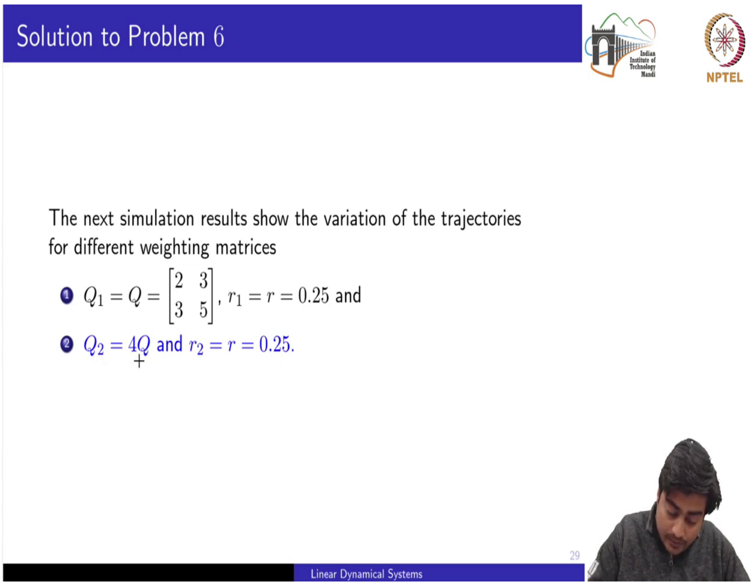
(Refer Slide Time: 24:38)



So, let us see if we write our problem into the standard form as we had seen into the lecture slides. So, the matrix A, vector b, this Q would be a state weighting matrix and r since we are using we are having a single input system, so this is a scalar and t naught is equal to 0. So, you can quickly verify that this system is at unstable system by computing the eigenvalues of the matrix A.

Now, if you recall the linear quadratic regulator control design this controller is given by this equation which is optimal minus 1 by r b transpose P times x, ok, where P is a 2 by 2 symmetric matrix which you can write it as this and you can compute by solving this algebraic Riccati equation which we have derive into the lecture slides, ok. And this you can also solved by using the care command by using this command in MATLAB, right.

(Refer Slide Time: 25:55)



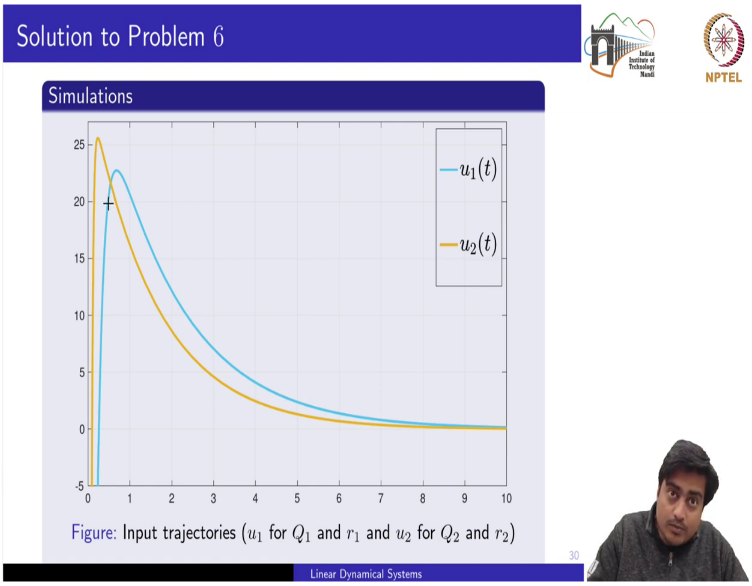
So, now we would go directly on to the simulations because this information is already given to us A, b, Q, r. Now, using this information by using the derivations we had seen into the lecture slide you can compute directly this control law, where P you can compute by this one. So, A and B are fixed because of the system dynamics. What we can see the behavior of the closed loop system with respect to Q and r, which are the state and the input weighting matrices?

So, we consider two cases in the first case we denoted by Q 1 which is the given one and this is r 1 is 0.25, which is 1 by 4, ok. In the second case we take Q 2 as 4 times of Q and we keep r 2 as equal to r. So, first of all before seeing the simulation we you should understand that what you are expected to see.

If we go to the physical significance of this Q or the significance of this Q matrix and r it defines the state the weighting matrices. So, if this if I increase this weight which I am increasing in the part 2 or in the second case meaning that I am expecting my state to go to 0 as fast as possible. I am putting more penalty to the state variable.

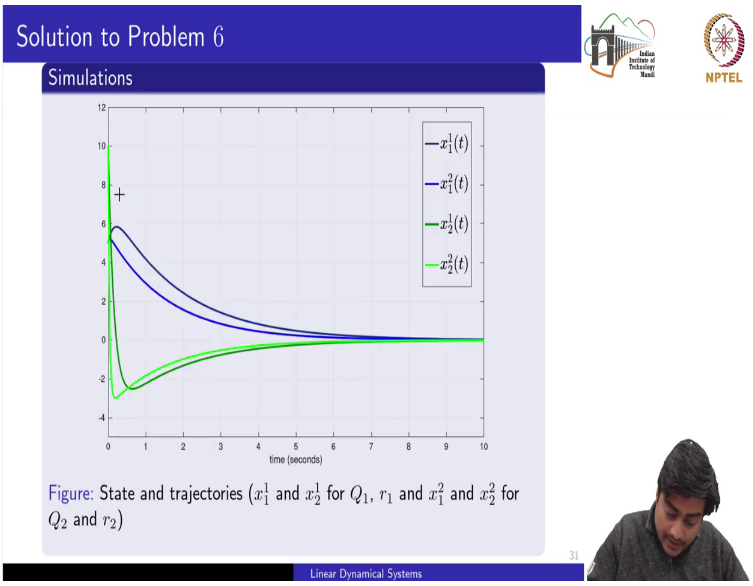
Now, if I putting more penalty to the state variable, it implicitly means that my control signal would be more aggressive. So, in scenario 2, I predict to see and aggressive control behavior as compare to the case 1, ok.

(Refer Slide Time: 27:56)



So, by putting this Q 1, Q 2, r 1, r 2 we simulated the scenario and this blue signal is the u 1 and this orange signal is u 2. So, as we have predicted that u 2 would be more regressive because the state trajectories need to reach to 0 as fast as possible, ok.

(Refer Slide Time: 28:18)



And the same behavior we would expect in the state trajectories, that this state trajectories which is the first state trajectories of the case 1 and this state trajectory the first state trajectory of the case 2. So, this reaches to the 0 as faster than the first case. Similarly, the second state trajectory, ok.

Thank you.