Linear Dynamical Systems
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Week - 05
State Feedback Controller Design
Lecture - 34
Linear Quadratic Regulator

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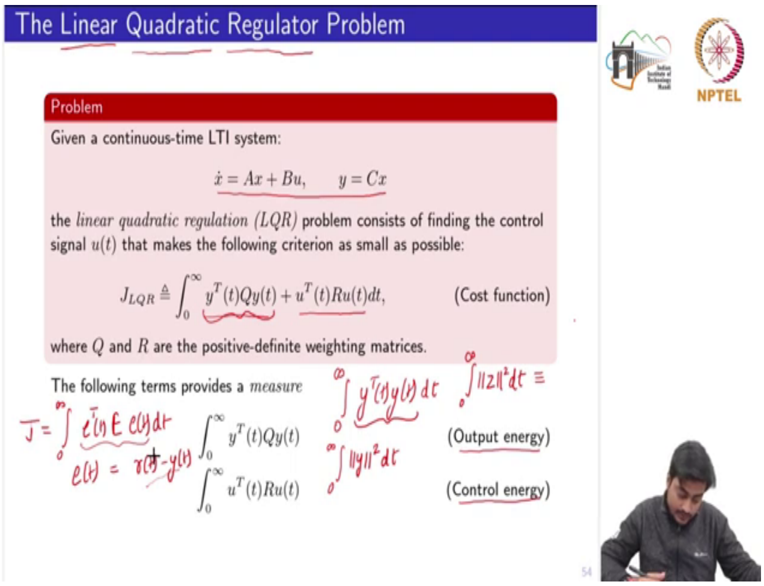

So, today we will start with the problem of Linear Quadratic Regulator. So, far we have discussed the problem of state feedback design by for the single variable case and also for the multi variable case. We have also seen the robust controller design problem also. So, the main idea in the problems whatever we have discussed lies in this having the set of desired eigen values.

So, if you recall with whether we have taken the eigen value assignment problem or the Lyapunov design problem for the controller. We have the set of desired eigen values and we also had a brief discussion on that based on the transient and steady state characteristics. You can have those information about the desired set of eigen values and whenever we specify those transients or the steady state performance criteria, they form the control objective basically. So, now, here we will see another viewpoint of designing a state feedback controller, where we now define the performance index explicitly in terms of the input output variable.

So or basically this is also a part of the optimal control design problem and today, we will see a brief overview that how the controller can be designed using that viewpoint. So, the first a standard problem is called the linear quadratic regular problem, the linear term, this linear term basically comes from the fact that we are designing a system or we are designing a controller for a linear system which is we have been discussing from the beginning that is our continuous time linear time invariant system.

A quadratic word comes basically from the fact that whatever the performance index, we will formulate or we want to optimize. It is quadratic in nature we will pay more focus to this to this aspect as we go along the lecture. The regulator we have seen one regulation problem. So, the idea remain the same that the state or the output should reach to 0 or the state impact should reach to 0 which is a regulation problem.

So, the problem in this framework can be defined as that given a continuous time LTI system which is given by this, the linear quadratic regulation problem or the regulator problem consists of finding the control signal u of t that makes the following criterion as small as possible.

So, we define the cost function which is the integral from 0 to infinity and inside the integrand, we have the quadratic functions of the output signal y and the input signal u with some weightage matrices Q and R. Now, these Q and R matrices are the positive definite matrices. In the previous lectures, we have discussed about the quadratic forms and also about the positive definite matrix. So, now, if I write only this integrand, the first integrand.

So, this integral is basically this y transpose of t and y of t say for example, forget about the Q or take Q as an identity matrix dt. So, if you recall that this part by using the property of the norm, I can define this as 0 to infinity and a norm of y square dt ok.

Now, we let us write this integral in some another variable z is going; where, z could be either state or the output variable y or the input variable u. Now, if it is an input variable u in some of the tutorial problems we had seen that this define basically the energy; the energy in the control signal or the control energy. Now, if it is only the output signal y, then we call it the output energy. So, the basic idea here is we want to minimize the energy in the input output signal and at the same time, we want to design a controller or we want to design a state feedback controller such that this, these energies are minimized ok.

So, the following terms provides a measure first is the output energy; another is the control energy. Now, this performance index it is because of its nature that is quadratic in nature, that is why it is called LQR problem which we have define in terms of the output variable and the input variable. Now, if I represent let us say this, I choose another function J in define 0 to infinity and I use e of T which is a signal and some let us say some rate matrix e dt and if I define e as R of t minus y of t; where, r is my reference signal and y is my output signal. Now, if I design if I want to synthesize again matrix K such that this performance index is minimized.

So, I am actually aiming to minimize this energy. So, that my error becomes equal to 0. Now instead, now when we see or when we incorporate the property of the norm, we know all these quadratic forms can be represented by these norms and from the property of the norm, we know that this norm is positive or it would be 0 if and only if this z is equal to 0 ok.

So, it cannot have the negative values. So, the minimum value the this can go is only 0 and when it become 0, e become 0 that associate my reference my output signal would track the reference signal ok. So, most of the control problems we had discussed so far can also be formulated in this framework. So, we will see now the solution of this problem.

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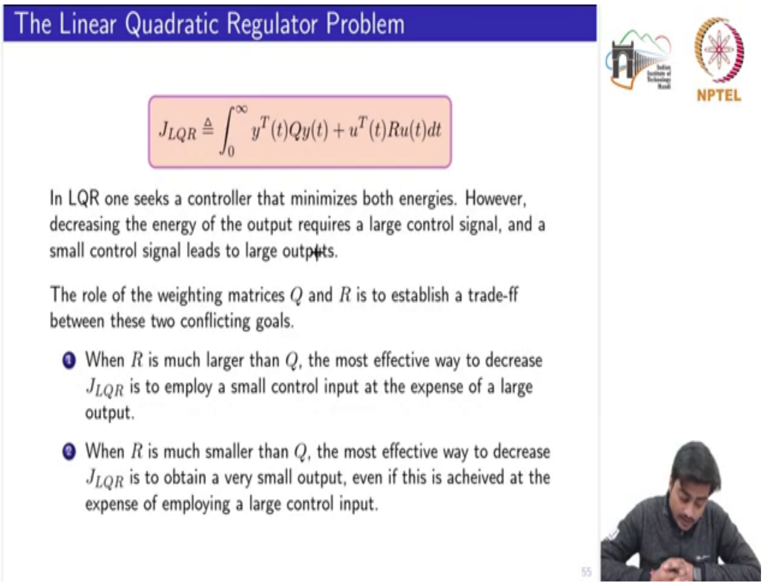

So, in LQR taking this quadratic cost function. We seek a controller that minimizes both energies. However, decreasing the energy of the output requires a larger control signal and a smaller control signal leads to larger outputs. So, we will pay more attention to it and in fact, we had seen in some of the tutorial problem also in the past couple of weeks that if we want to achieve faster tracking, then in that case the control energy is much higher.

If we want to track the reference signal with the slow dynamics, where in that case the control energy require in the or the control energy is a small. So, we can also play with those aspect by changing the these the value of these weight matrices which are the positive definite matrices. So then, R is chosen much larger than the Q matrix, what does it mean that we want to penalize more control energy.

We want to penalize more energy and the most effective way to decrease J LQR, this cos function is to employ a small control input at the expense of the large output meaning to say the difference between the reference in the where we have the origin, the difference between the origin and the output y would be having would be larger in that case ok. So, it is so there is some trade off in selecting the these matrices Q and R. If we choose R higher, meaning to say that we want to penalize the control energy more than the output energy.

If we choose Q higher, then we want to penalize the output energy, meaning to say in that case our control energy would be higher also because we need faster control action to reach towards to the reference trajectory.

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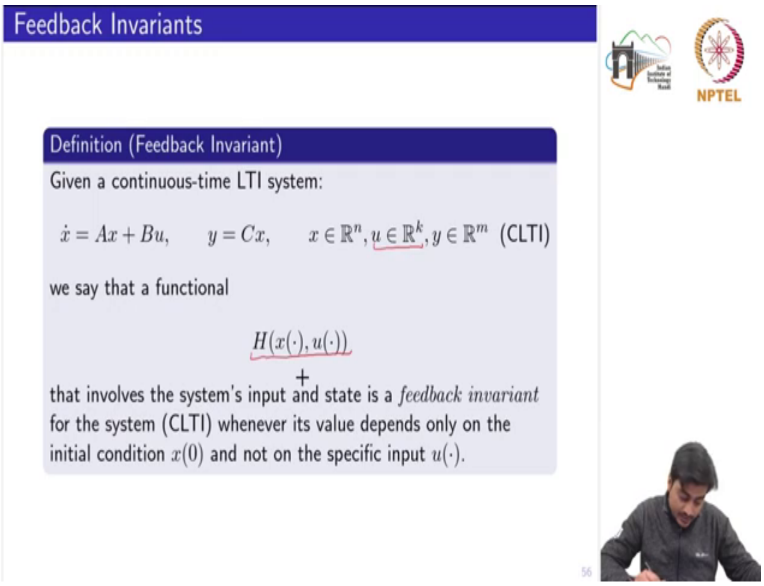

So, before solving this problem, we will introduce a couple of definitions which would help us to synthesize the feedback in matrix K. So, the first definition is about the feedback invariant, that given a continuous time LTI system this one of where the state r is having dimension n, here u is having a dimension k and y as dimension m.

We say that a functional H which is only a function of the signal x and the signal u that involves the or we call this functional a feedback invariant for the system C LTI whenever its value depends only on the initial condition x of 0 and not on the specific input ok. So, this definition has some in fact, this is the that is why the name originates of this function as feedback invariant because it is not affected by the control trajectory, but its value depends entirely on the initial condition ok.

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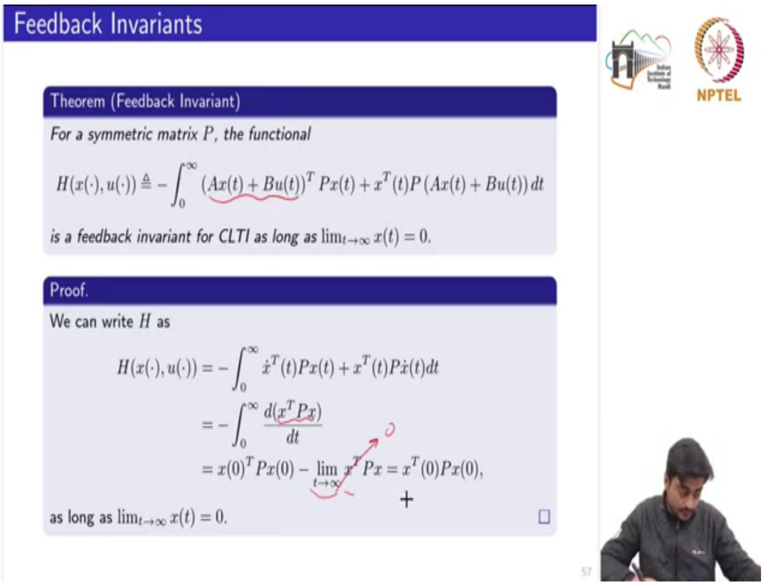

So, by introducing this functionl H, we have another result that for a symmetric matrix P, the functional H defined as the negative of the integral from 0 to infinity of this signal transpose P into x plus x transpose times P times this signal. We call this functional which is defined as this is a feedback invariant for the continuous time LTI system as long as the state trajectory x t satisfies that the value of the signal becomes equal to 0, as t tends to infinity ok.

So, this functional which we have characterized in terms of the state and the input trajectory is called a feedback invariant under this condition. So, we can see a quick proof of this one. So, if you pay attention to this part, this is nothing but the x dot ok. So, we have replaced this part by x dot, similarly here we have replaced by x dot. Now, if you recall in one of the lectures, we have discussed that I can express this overall part as the derivative of this quadratic form x transpose P x.

Now, since we have the negative sign. So, I need to compute its value of this quadratic form at t is equal to 0 minus the value of this as t tends to infinity ok. Now, if you see here, we have x of 0 transpose P times x of 0 minus limit t tends to infinity x transpose P x. We have this conditions given to us that the signal x t tend becomes 0 x t tends to infinity. So, this part would go 0 and the remaining part is this one. So, we see according to the definition that this functional becomes a feedback invariant because its value depends only on the initial condition of the state trajectory.

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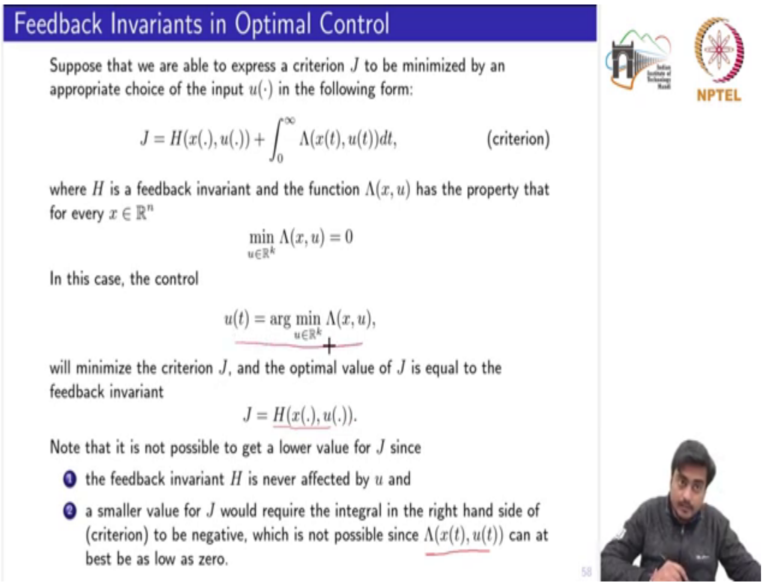

Now, suppose that we are able to express a criterion J to be minimized by an appropriate choice of the input signal u in the following form. So, this criterion is important to pay attention. So, we have expressed this J as equal to the feedback invariant function H plus integral of some function gamma ok, which is a function of x and u; where, H is a feedback invariant and this function has the property that for every x or for every n dimensional state vector, this is satisfies that the minimum value of this function over u should be equal to 0 ok.

Now, it is pretty much straightforward that under this scenario, we can write the control signal as the argument of this thing because we are basically computing those u for which this function becomes equal to 0. So, the argument of this function would become or give me the control signal u of t ok. So, we know that this u will minimize the criteria J and the optimal value of J is equal to the feedback invariant now ok.

Because the I if I want to minimize this function, minimizing this part would go only 0. So, from here I compute u. Now, once this part becomes equal to 0 and this part is independent of the time because its value depends only on the initial value of the state trajectory.

So, I would get this fixed value whenever this function is minimized or whenever this control signal is applied in the feedback loop ok. So, note that it is not possible to get a lower value for J. Since, the feedback invariant H is never affected by u. So, once we have computed or once we know the initial condition or the initial value of the state, its value is fixed ok.

And as we have seen in the previous result or generally what we have seen that this function could only get the positive value. A smaller value for J would require the integral on the right hand side of the criterion to be negative, which is not possible since this function can be can at best be as low as 0. So, this would be the minimum value of this criterion whenever we apply this control signal.

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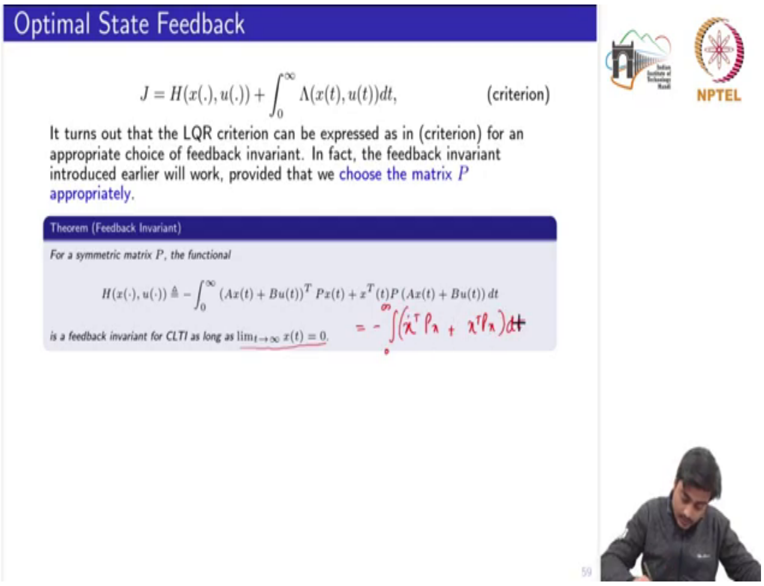

So, again proceeding with this criterion, it turns out that the LQR criterion can be expressed as this function for an appropriate choice of feedback invariant. In fact, the feedback invariant which we have introduced earlier can also work provided that we choose the P matrix appropriately. So, this function we have introduced in one of the results that this functional is a feedback invariant under this condition. This condition is important for us; we will see why. So, now, we will play with this function H to see the similarity between this criterion and the LQR criterion we started with.

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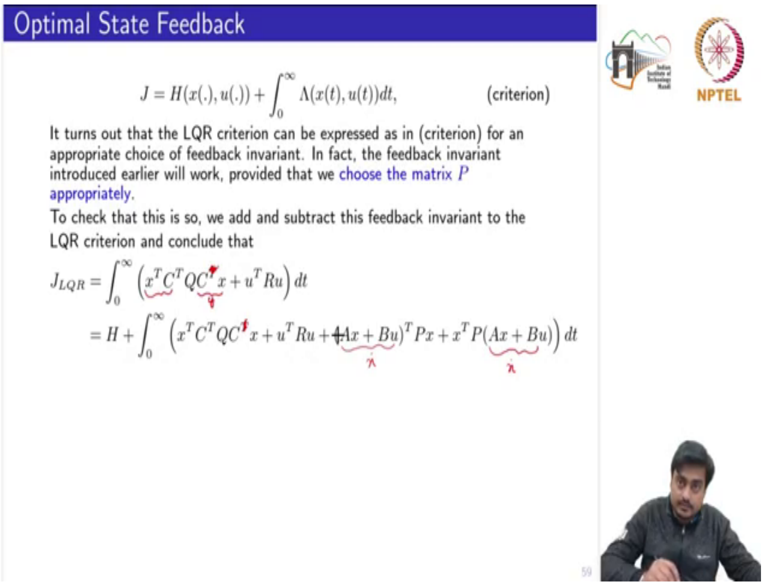

So, to check that this is so, we add and subtract this feedback invariant to the LQR criterion and C. So, J LQR, I can write or what we had since 0 to infinity. So, this part is y transpose and this is y ok. So, there should not be transpose. So, this transpose won't be there because this is y. Similarly, this transpose would not be there plus u transpose R u. So, we add this functional H and subtract this functional H.

Now, this H, we already know is expressed by this one or in the compact form, I can write as minus 0 to infinity x dot transpose P x plus x transpose P x dt. So, when I substitute with this one, I this is the same what we have written. This is x dot and again, this is x dot ok. And since, it was having a negative sign, so subtracting it would add it into the overall integral ok.

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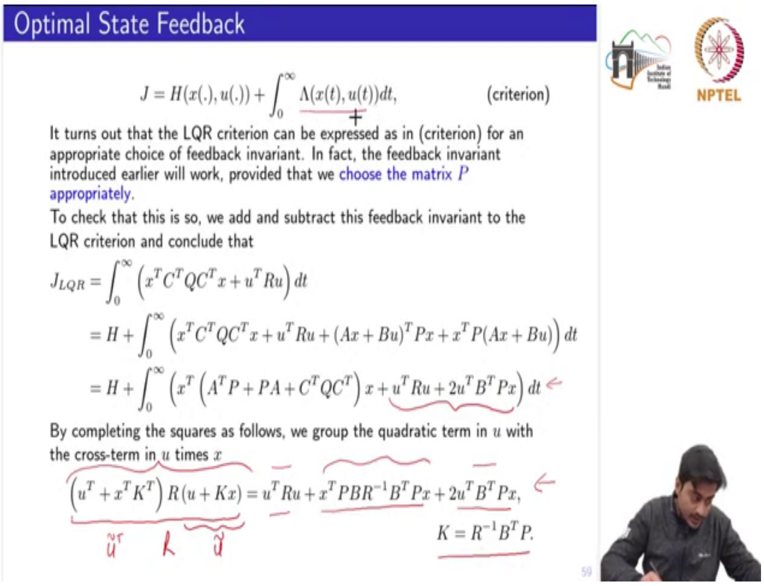

So, now we will arrange a couple of terms to visualize that what is how to simplify the things for computing the feedback gain k. So, we have coupled all the or we have combined all the terms for which the x transpose is common in the pre multiplication factor and x is common in the post multiplication factor. So, here we would have x transpose A transpose P times x.

So, if you see that onto the left hand side and the right hand side, we should have this one x transpose x. So, this part would go here. Similarly, by taking the transpose of it, we would have x transpose A transpose and P times x. So, this is the first term. So, this first term will become this one and the next term is this one which is x transpose P Ax ok. This is the third term.

Now, the rest of the term we have written here. So, this one become u transpose R u and the two terms could add up because from here, we would have u transpose B transpose P times x and x transpose P times B into u. Now, these two terms because they are scalars, they would add up. So, we have written that twice of u transpose B transpose P times x ok. Now, by completing the squares as follow, we group the quadratic term in u with the cross term in u times x.

So, now, we introduce this quadratic form where you can express this is u tilde; where, u tilde u plus K times x and this becomes u tilde transpose ok. And here, K we have replaced by R inverse B transpose into P. So, if I open this quadratic form, we would have u transpose R u plus x transpose and by putting K into this, we get this long expression plus twice into this. Now, if we see the similarity between this expression and this expression, the rest of the part this part I can express as which is also available here. If we see these two parts, I can express as this whole part minus of this full part.

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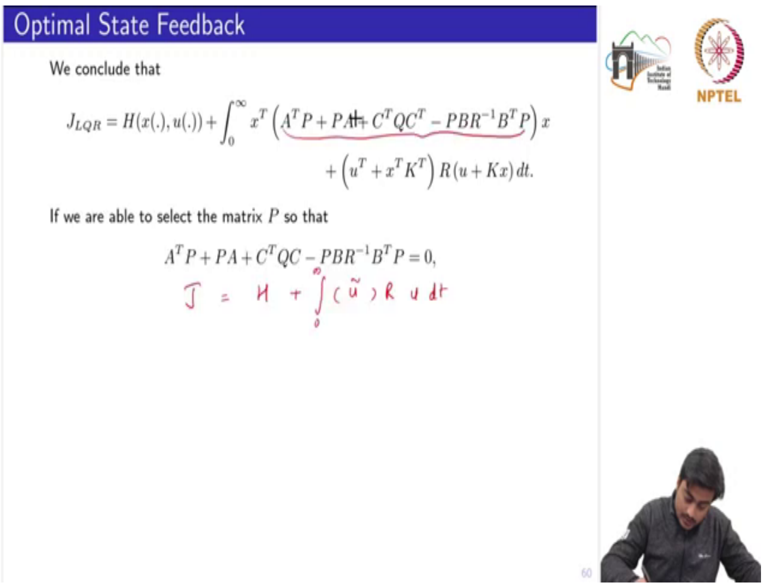

So, this is what we have written here, the J LQR is equal to the feedback invariant function H and inside the integral, this is the term which we already had previously. Now, this term we get if I replace this and this by this it minus this and since it also contains x transpose and x as pre multiplication and post multiplication vectors. So, we have combined that term inside this term minus of this one and the rest of the part as this one right.

So, now, if we pay attention to this function that if we choose this part is equal to 0 plus if we choose this part equal to 0, the rest of the part remains J is equal to H plus integral of 0 to infinity this or let us say u tilde R u dt ok. Now, if you pay attention to this form and to this form or the criterion, we had introduced inside this integral. So, this is nothing but equal to your gamma function ok.

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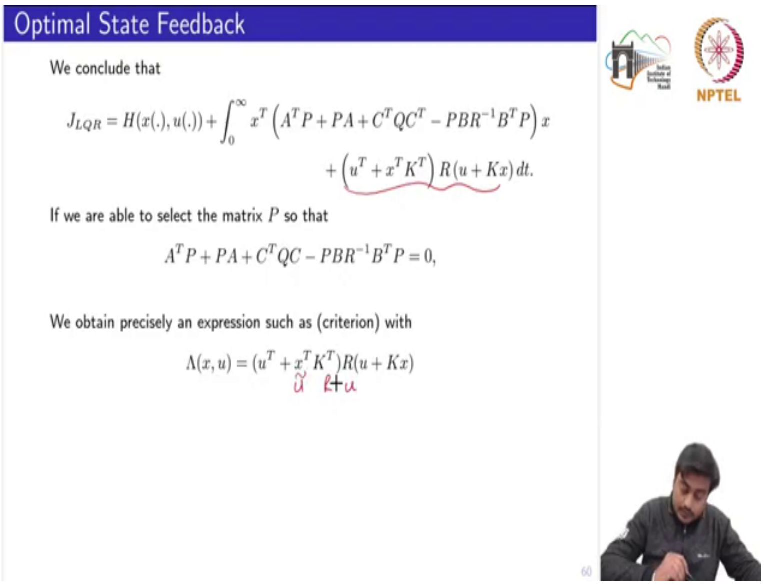

So, this is nothing but your gamma function as u tilde R u ok.

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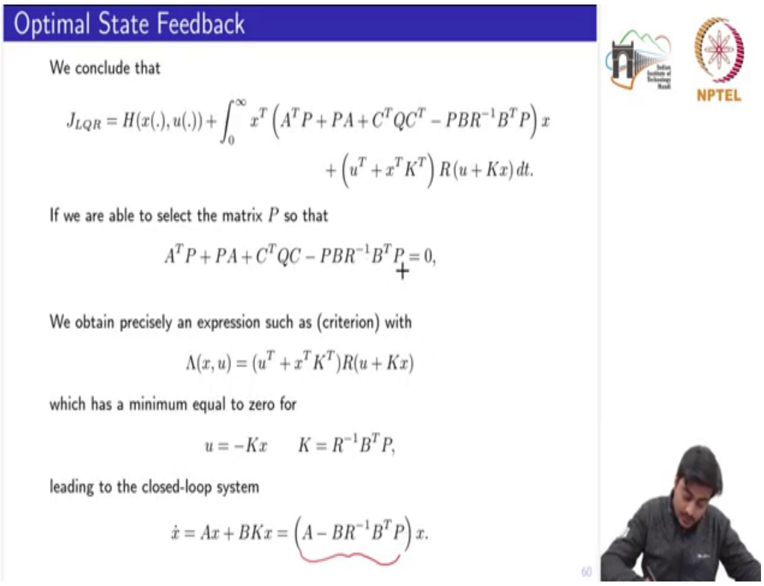

Now, which has a minimum equal to 0. So, this would be equal to 0, if we have u is equal to minus K times x and this is nothing but our state feedback, where K is expressed as R inverse B transpose P. Here, the inverse of R is possible because R is positive definite already in the definition of the cost function and P, we need to find out ok.

So, now the problem of designing the feedback gain K, the problem now boils down to compute the matrix P and the matrix P would be computed from this equation. So, when I apply this u is equal to minus K times x to the LTI system, we know already that this would become our closer loop system ok, with the closer loop state matrix A minus BR inverse B transpose P; where, instead of finding the matrix K, k we need to find the matrix P and here you would realize that this equation is a non-linear equation.

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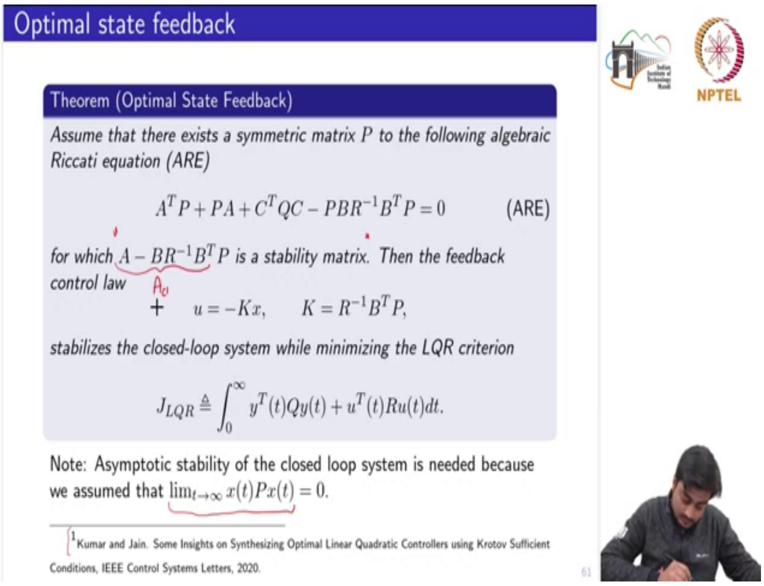

So, this we have one of the result, the optimal state feedback. They assume that there exists a symmetric matrix P to the following Algebraic Riccati Equation. So, this equation what we have find out through this whole procedure. So, this equation is basically defined as the Algebraic Riccati Equation for which the closer loop state matrix is a stability matrix. So, this point is a very very important point, we will come onto this point a bit later. Then, the feedback control law u is equal to minus K times x; where, K is defined by this, stabilizes the closed loop system while minimizing the LQR criteria ok.

So, why I say that we need to find a symmetric matrix P for which this closed loop state matrix is a stability matrix because there are many matrices P which satisfies this equation, but for some P this closed loop state matrix might not be a stable matrix or its orbits. In since, we want to minimize the cost function over 0 to infinity, we definitely want that the state trajectory should reach to 0 as t tends infinity. So, there might be some P; there might be some P for which this closed loop state matrix is not stable matrix, but still it satisfy this matrix this equation ok.

So, generally the problem of the optimal control design and the stability problem are two different problems. For optimality, we need to find one P which satisfies this equation; but for some P, if I compute this controller u and apply to the plan, the closed loop might not be stable ok. But for some P, I could still have the this matrix has a stable matrix which ensures that my closed loop would be a stable ok. So, we can also say that the symmetric P which satisfy this one for which this is a stability matrix, we can also called those P as a stabilizing solution. So, basically, we want to find all the stabilizing solutions of this equation.

So, that this closed loop state matrix becomes a stability matrix. Note at asymptotic stability of the closed loop system is needed because we assumed that this part should become equal to 0 which is equivalent to sign that the state trajectory should reach to 0 as t tends to infinity. Now, here we have I have put a one paper in the footnote, where we have discussed another viewpoint of discussing the linear quadratic problem using a Krotov sufficient conditions. So, you may look at this paper to get to dwell into more into the subject.

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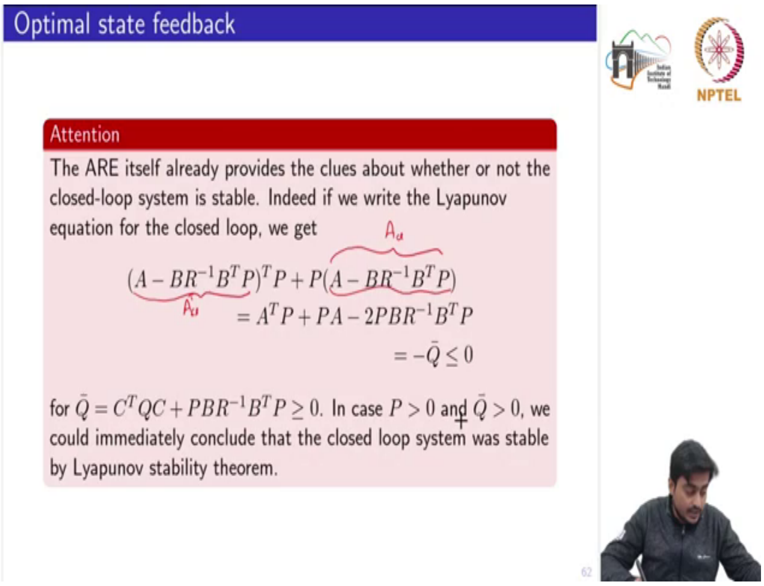

So, the ARE itself already provides the clues about whether or not the closed loop system is stable indeed if we write the Lyapunov equation for the closed loop. So, this is our let us call it A bar or AC, a matrix of the closed loop. So, this matrix becomes AC l transpose AC l transpose P plus P times or AC l ok. Now, if I simplify this equation, I would get A transpose P plus P times A minus twice of this whole part ok, which would be expresses minus Q bar which would always be a negative definite this one; for Q, positive definite ok.

In case P is a positive definite matrix and Q bar is a positive definite, we could immediately conclude that the closed loop system was stable by Lyapunov stability theorem. So, here we have recalled one of the result, we had discussed during the stability week, where we established the Lyapunov stability theorem for continuous time LTI systems.

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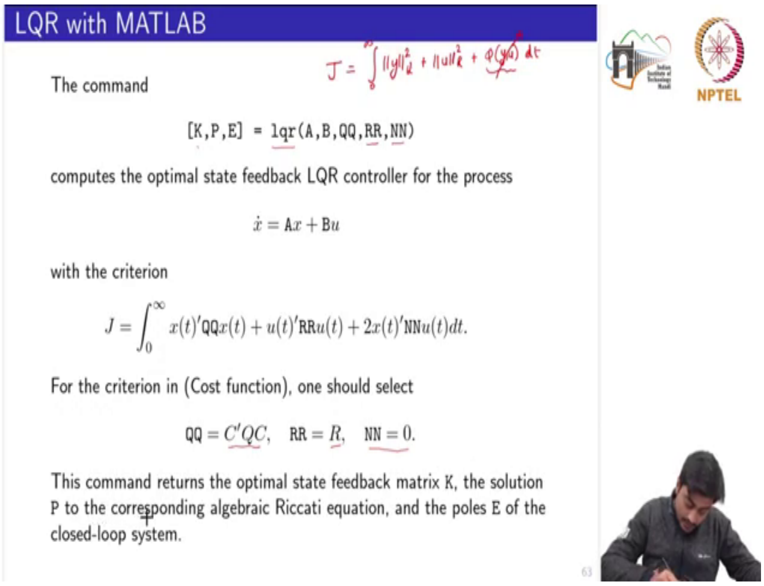

In MATLAB, u there is a direct function available which we is defined by its own name LQR. So, here on the left hand side, we give three outputs of this function whose arguments are the A, B matrices; this QQ is nothing but C transpose QC which is the weight matrix in the cost function; This RR is this the weight matrix of the of the control energy and NN, we have not discussed about this NN.

So, for the moment you can put NN is equal to 0. So, here NN. So, let us say so we have discussed this J 0 to infinity as the norm of y weighted by Q plus norm of u weighted by R. So, there are some couple of terms, let us call it phi which are the multiplication of y and u or x and u.

So, the weight matrix of those cross terms is basically NN. Now, here since we have not discussed this cross terms this part, we have you can put NN is equal to 0 ok. So, once you input all these matrices, you will get three outputs k. So, K would be the optimal state feedback matrix K. The solution P is the solution of the Algebraic Riccati Equation and E gives you the poles of the closed loop system. So, by using this method, you can also design the multivariable the state feedback in matrix for the multi variable system as well.