Linear Dynamical Systems

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Week - 05
State Feedback Controller Design
Lecture – 33
State Feedback design for Multi-input systems

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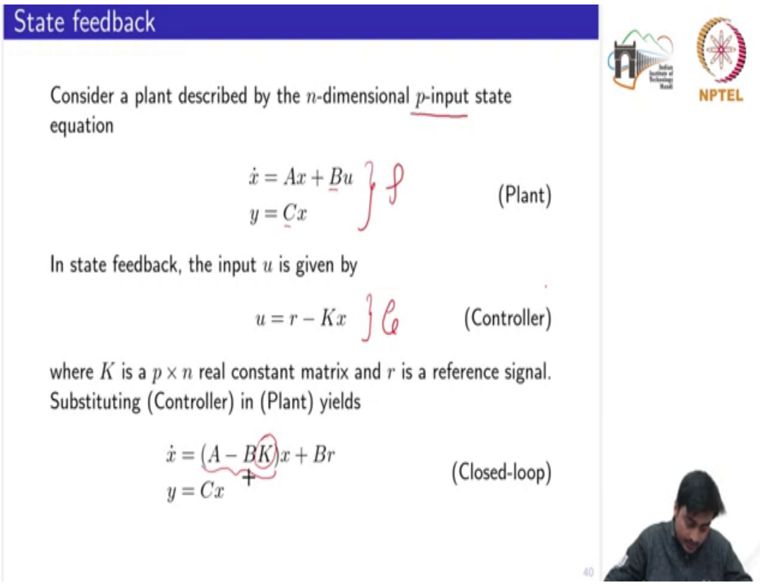

So, now we will see the case of designing the State Feedback Controller for the multi variable case. So, so far we have discussed the designing of the controller mainly for the single input variable case, where we have also discussed the robust robustness and tracking problems. So, for the multi variable case, the layout of the problem will remain the same. So, consider a plant described by the n dimensional p input state equation. So, now, here we are considering p number of inputs instead of scalar given by this state equation where ABC are the matrices.

So, you will note that when we were discussing about the single variable case; the parameters we had taken in a small letters. So, now, when we take the when we consider the case of the matrices we will consider those parameters as the capital letters as you will see B and C matrix. In the state feedback, so we denote this as the plant and the in state feedback the input view is given by this one. So, we have seen earlier for the single variable case, we have designed such kind of input, but now here the K matrix, the K would be a matrix instead of a vector.

So, K is a p cross n times a real constant matrix and r is a reference signal. So, once we put this u into this plant we obtain the state space equation for the closed loop system; that is the A matrix would now change to A minus B times K. The B matrix would remain the same, but it now becomes a distribution matrix of the reference signal and there would not be any change on the C matrix ok. So, here we will discuss cases or we will discuss some designs to synthesize this K.

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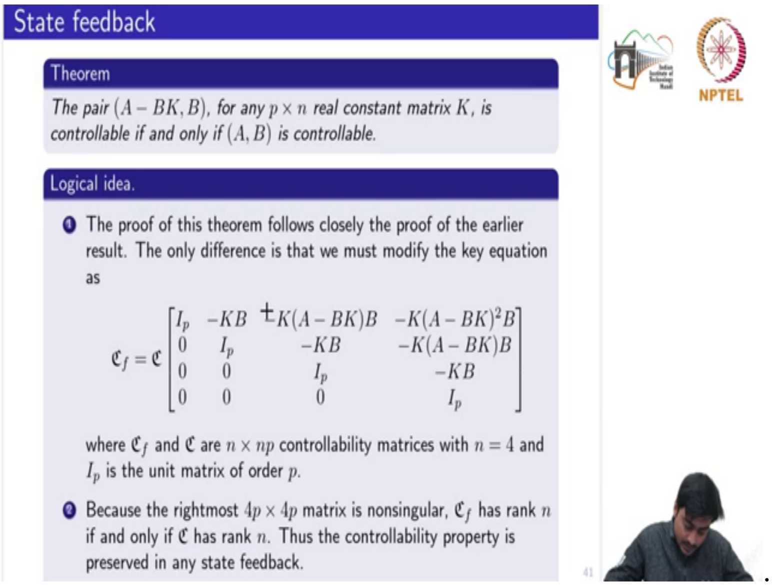

So, again recalling the important theorems or the important results we what we had discussed for the single variable case. So, for the multi variable case the pair A minus B K comma B. For any p cross n real constant matrix K is controllable if and only if the pair A, B is controllable. So, this is the key result and in fact, this is the starting point as well. So, if the pair A, B is not controllable you we cannot ensure that the closed loop pair would also be uncontrollable.

If we see a very quick snapshot of the proof of this theorem, the proof would remain almost the similar to what we had seen for the single variable case, but for the multi variable case; only this matrix would change. So, if you recall that the controllability matrix of the feedback loop, we have expressed as equal to the controllability matrix of the plant times some constant matrix.

Now, here since we have the a p number of inputs, this controllability matrix would no longer be a square matrix. But it is ensured that the rank of this matrix is full rank. So, if you see the right most part of this equation, this is an upper triangular matrix with all the diagonal elements says the identity matrix of p dimension. So, this matrix would be a square matrix and would definitely be a non singular matrix.

Now, to ensure the that the C f is also full rank we only need to have the full rank of the controllable of this controllability matrix of the plant. So, only this part would remain the would remain change, otherwise the rest of the part would be similar.

So, another conclusion we made at that time that the controllability property would be preserved in any state feedback. If the original plant is of the original pair is controllable, then under any K the pair A minus BK comma B would also be controllable. But at the same time we also noticed that the control the observability property which we would discuss in the subsequently is not necessary we preserved.

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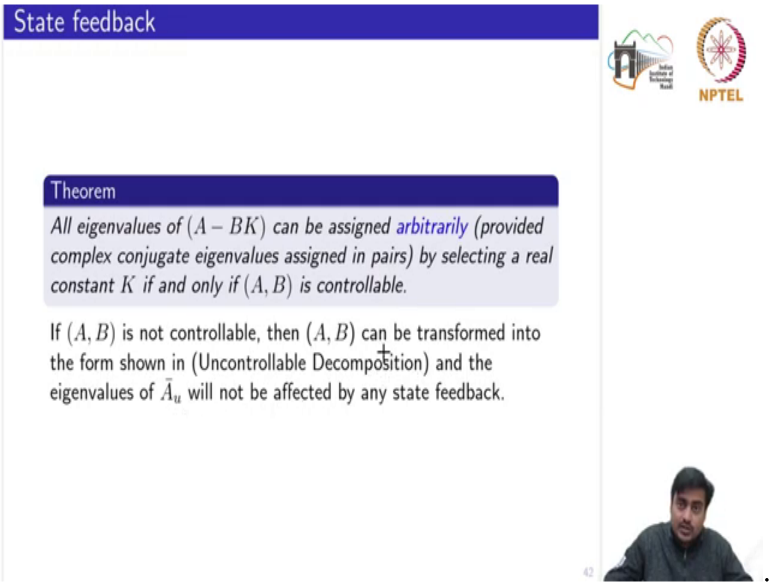

The next results says that all the eigenvalues all of the closed loop state matrix can be assigned arbitrarily provided complex conjugate eigen values assigned in pairs by selecting a real constant K, if an only if the pair A, B is controllable. So, again this has the similar result we had discussed for the single variable case, for the single input variable case. But this also holds for the multi variable case also that the if the pair A, B is controllable, then the eigenvalues can be placed anywhere.

Now we had also discussed that if the pair A, B is not controllable, then we can use the uncontrollable decomposition to apply on this pair A, B and extract only the controllable part. So, we only need to show to ensure that the uncontrollable component in that decomposition has its eigenvalues onto the left hand side right. Because, for the controllable part we can place the eigenvalues anywhere on the right hand side, but if and we cannot change the eigenvalues of the uncontrollable part.

So, if we ensure that the eigenvalues are on the left hand side then the system at least we can ensure the stability of the system.

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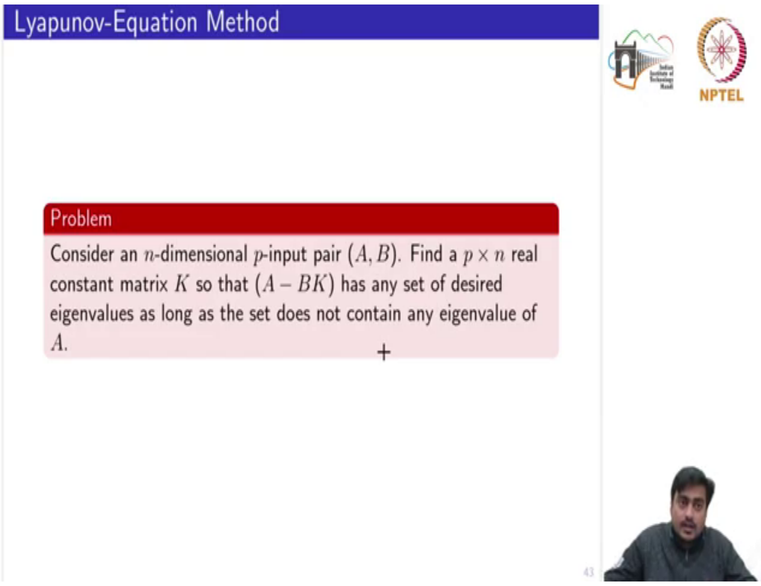

So, for the single input case we had discussed two design for the design of the for the state feedback controller. So, the first one, if you recall is the eigenvalue assignment and the second one is using the Lyapunov equation method. Now, if you recall that in the eigenvalue assignment method; we computed the feedback gain by taking the inverse of the controllability matrix. Now, since the controllability matrix here is not a square matrix we cannot take its inverse. So, the eigenvalue assignment approach cannot be applied for the multi variable case to synthesize the gain matrix A.

So, but in the Lyapunov equation method; we had seen we do not require the inverse of the controllability matrix, but it is the inverse of some another square matrix which we will see that whether that method applies directly for the multi variable case. So, if we see this problem of designing a feedback controller says that consider an n dimensional p input pair A comma B, we need to compute a p cross n real constant matrix K. So, that A minus B K has any set of desired eigenvalues.

Now, this method has some restrictions that the desired eigenvalues cannot be placed at the origin and also the set of desired eigenvalues should not contains the eigenvalues of the A matrix also yeah.

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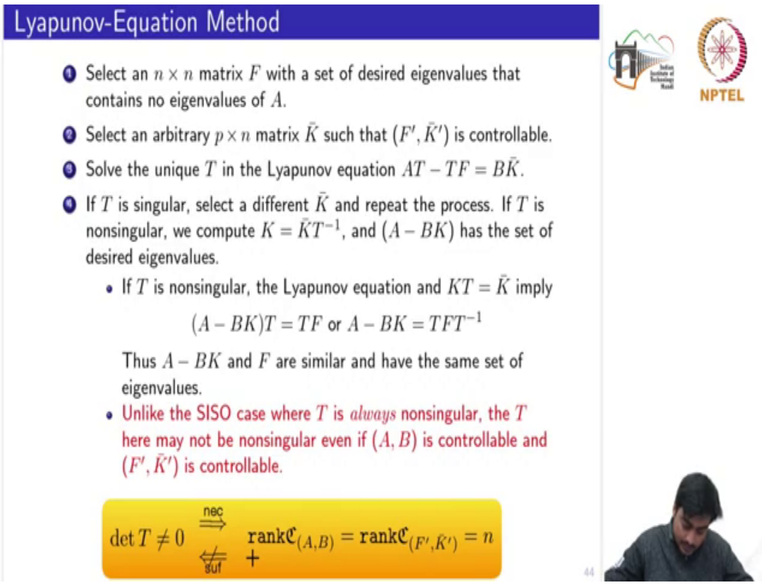

Let us see the design of the gain matrix K. So, the first 3 steps remains almost similar which says that first of all we need to form a matrix F which is a square matrix of dimension n which contains a set of desired eigenvalues. And at the same time, it should not contain the eigenvalues of the original a matrix.

So we had discussed number of forms for selecting a matrix F; one is the model form when another is the companion form you can choose the matrix F in any of the form. The second step we need to select an arbitrary p times n matrix K bar such that the pair F transpose K bar transpose is controllable. That step; once we have found such K bar then using this Lyapunov equation, we can solve over all unique T, unique matrix T. Now, from here we once we have computed this T matrix, then the design of the gain matrix was given by this one.

And most and the key result hinges upon the inevitability of the or the non singularity of the T matrix. So, we need to ensure that for the multi variable case is it always possible that the matrix T is non singular, so that we can take its inverse. So, we know that if the matrix T is singular then, we cannot carry forward this design. Because, we cannot compute this equation. So, in that case since K bar is chosen arbitrarily we would keep repeating the design until we find a T which is non singular ok.

And if the matrix T is non singular; we can compute this and at the same time we ensure that the closer loop state matrix would have the set of desired eigenvalues. And the proof of this one, we can see a straight forwardly that we just need to put K bar is equal to KT into this equation and we obtain this. And finally, we can write the closed loop state matrix as T into F into T inverse which is basically, the simulated transformation which ensures that whatever the eigenvalues this F matrix has the closed loop state matrix would also have those eigenvalues ok.

Now, the most important aspect here that, that under the SISO case, the Single Inputs Single Output case; we had ensured that the T matrix would always be a non singular matrix. But for the multi variable case it might not be possible that the matrix is non singular even if the pair A, B and F transpose, K bar transpose are controllable ok. So, we will see the proof of this one that why we cannot ensure always the non singularity of the matrix T in the multi variable case.

So, in fact, this leads to the only the necessary condition; meaning to say that, if the matrix T is non singular or the determinant of the matrix T is not equal to 0. Then it implies the both the pairs A comma B and F transpose comma K bar transpose would be controllable. But if this condition holds that both the pairs are controllable that then it might not be possible that the matrix is non singular ok.

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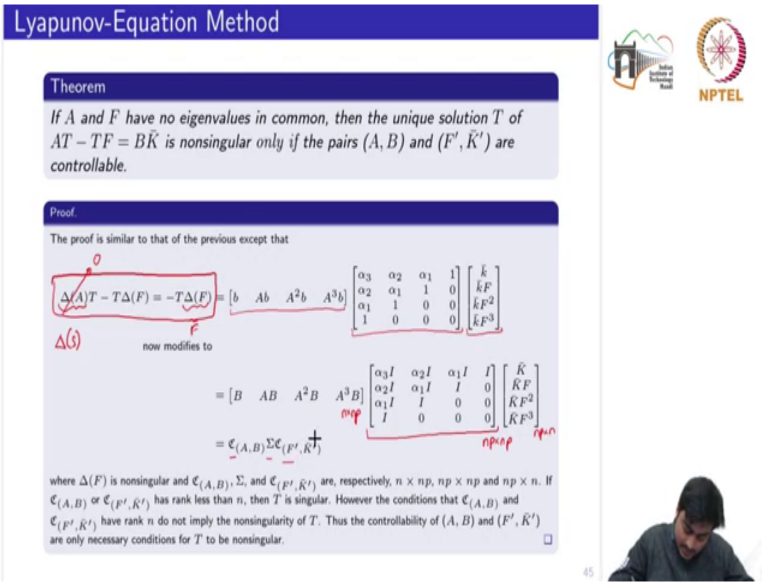

So, this is only a necessary condition, but not a sufficient condition. We can also write this statement and as one of the results of which the proof we will see. That if A and F have no eigenvalues in common, then the unique solution T of this Lyapunov equation is non singular only if the pairs A comma B and F transpose K bar transpose are controllable. So, which is only a necessary condition.

So, we can see the proof which is again the same as what we had discussed for the single input variable case. So, if you recall we had obtained this equation as the final equation to show the matrix T or the to comment on the non singularity of the matrix T.

Now, this part delta of A and delta or delta of S is basically the characteristic polynomial and by using the Cayley Hamilton theorem this part would go to 0. So, the remaining part is minus T times delta of F. So, this is another matrix which we had defined if you remember this by F tilde and on the right hand side we have the controllability matrix of the pair A comma B, the controllability matrix of the pair F transpose K bar transpose and in the middle we had a square matrix which is a non singular.

So, now if you see this for the multi variable case, all the small bs would change into the capital B's. Similarly, here all the small k bars would change into the capital K bars and all these ones would be would change to the identity matrix of appropriate direction dimension. So, this can be represented as finally, the controllability matrix of the pair A, B times some sigma matrix which is this one, times the controllability matrix of the pair F transpose K bar transpose ok.

Now, this matrix would be having a dimension n p times n p, this matrix would have the dimension n p times n and this matrix would have the dimension n times n p ok. Also note that that this is an upper triangular matrix, so the this matrix would always be a non singular matrix and the determinant would be equal to identity ok. Now, the rank of this matrix is n because it is controllable, the rank of this matrix would also be n. So, and on the left hand side; delta or F tilde we had already shown with this matrix is a non singular matrix ok. So, we can take its inverse and it would be a square matrix also.

So, now if this is control if this has full rank, this is a square matrix and this is also a full rank matrix then we cannot ensure that the matrix T would still be a non singular matrix. But if the matrix T is a non singular matrix we can ensure that all these matrices would be of full rank right we can so, we can see one example also.

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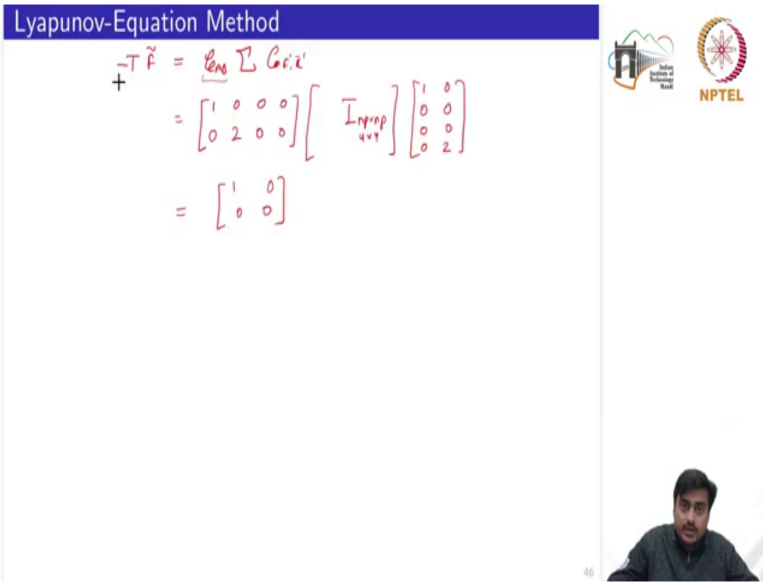

Let us see the controllability matrix or let us write minus T F tilde which is delta F as the controllability matrix of the pair A, B sigma and the controllability matrix of the pair F transpose comma K bar transpose.

So, let us write this let us take some basic example to demonstrate this effect. So, we take 2 states and 2 inputs. So, in that case this matrix would have the dimension 2 cross 4 and that 2 cross 4 matrix can be written as this one ok. Now, this matrix is a full rank matrix of dimension 2. This we can choose an identity matrix of dimension n p times n p sigma and this controllability matrix of this pair would be of dimension 4 time 4 cross 2 ok.

And n p is 4, so basically, this would be 4 times 4 and this we can write as 1 0 ok. So, this matrix has full rank, this matrix has full rank, this is a non singular matrix. So, now, if we see the product of all these matrices it would be 1 0 right.

So you see on the right hand side we do not have the full rank matrix though all the 3 individual matrices are of full rank. So, we cannot ensure that the T matrix would be a non singular matrix ok. So, that is why this is only a sufficient condition oh sorry a necessary condition, but not the sufficient condition. So, now, the problem arises that either we keep iterating to find those K bar such that 2 conditions are satisfied, that the pair F transpose K bar transpose is controllable and the matrix key is non singular ok.

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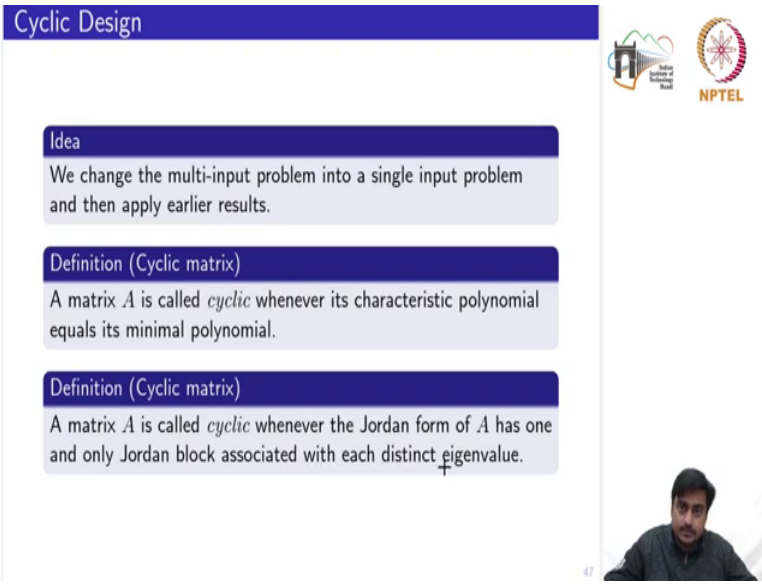

Or we should search for another design method by which we can compute in one shot the matrix K. So, for that we will carry out another design which we also call the cyclic design. So, the idea here is that we change the multi input problem into a single input problem and then we apply earlier results to finally, compute the matrix K ok. Because, for the benefit of the using the cyclic design that once we have represented a multi variable system or multi input variable system into a single input variable system. Then all the previous results the eigen values assignment and the Lyapunov base design, we could apply to finely compute state feedback for a multi input problem.

So, we define a cyclic a matrix A as a cyclic matrix whenever its characteristic polynomial equal its minimal polynomial. So, characteristic polynomial we have been discussing a number of times, but the minimal polynomial we had discussed during the stability possibly. In some of the earlier lectures we had discussed that what polynomials we can say that this polynomial is a minimal polynomial.

Now, the same definition we can also apply by transforming the matrix into its Jordan form and then applying the then seeing the conditions being satisfied by the Jordan form of the A matrix. So, matrix A is called a cyclic matrix whenever the Jordan form of A matrix has one and only Jordan block associated with each distinct eigenvalue ok.

So, this is how we define the cyclic matrix either you compute the characteristic and minimal polynomial and see their equivalence or you come or you transform by using the simulated transformation into the Jordan form and then see whether there is only one and only one Jordan block associated with each distinct eigenvalue ok.

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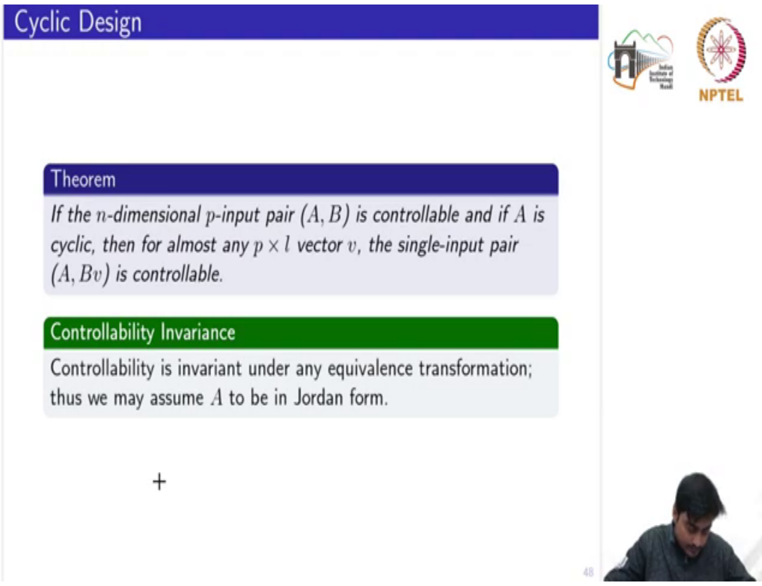

So let us see, so this is one of the important results which we will use to design the gain matrix k. So, if the n dimensional p input pair A comma B is controllable and if A is cyclic, then for almost any p cross l vector v, the single input pair A comma Bv is controllable. So, this single input pair would possible if we have this p cross 1 ok, then the single input pair A comma B times v is controllable.

So, if we recall from the controllability week, we had discussed one result that the controllability property of the system is unvariant or is invariant under any simulated transformation. So, whether if we have transformed into another equivalent transformation, the Jordan form of the matrix A would still be satisfy the controllability of the pair A comma B ok.

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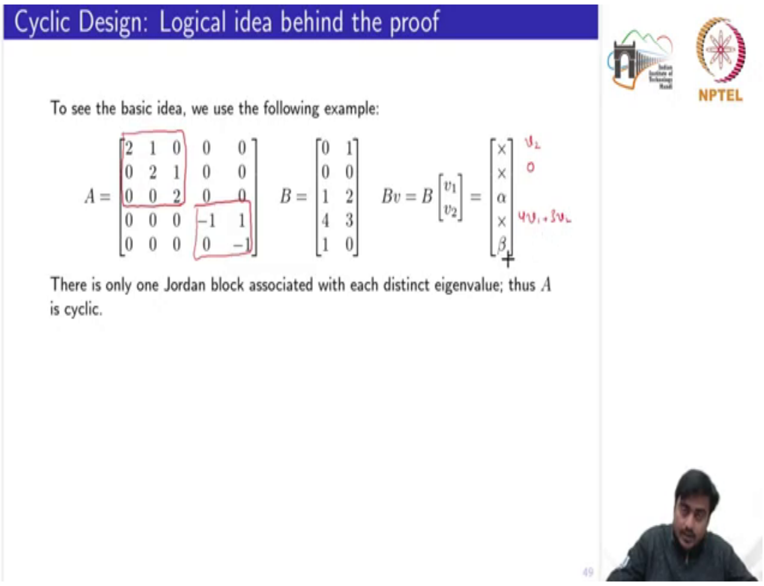

We will not go into the detail proof of this result, but we can see through some example or the logical idea behind this proof. So, for example, consider this A matrix which is a 5 cross 5 matrix and B as 5 cross 2 matrix, meaning to say that we have 5 states and 2 number of inputs ok. So, if we pay attention to this matrix A; this matrix A is already given into its Jordan form. So, the one so, and it contains 2 distinct eigenvalues in overall it should it contains 5 eigenvalues, but, but has only 2 distinct eigenvalues. One is 2 and another is minus 1. So, with respect to minus 1 we have one block which is the Jordan block and with respect to the eigenvalue 2, we have another Jordan block ok.

So, this satisfies the definition of a cyclic matrix. So, we can say that matrix A is a cyclic is a cyclic matrix. Now, let us select v which is 2 cross 1 and we can write this B times v as vector this one. So, this cross elements represent that it can represent any of the that we are not concerned with these elements. So, these elements would be pretty much straightforward if you want to compute from here. So, this 1 would be B 2, this would be 0 let us denote the third element by alpha, the fourth element would be four v 1 plus 3 v 2. And let us denote the fifth element by beta ok.

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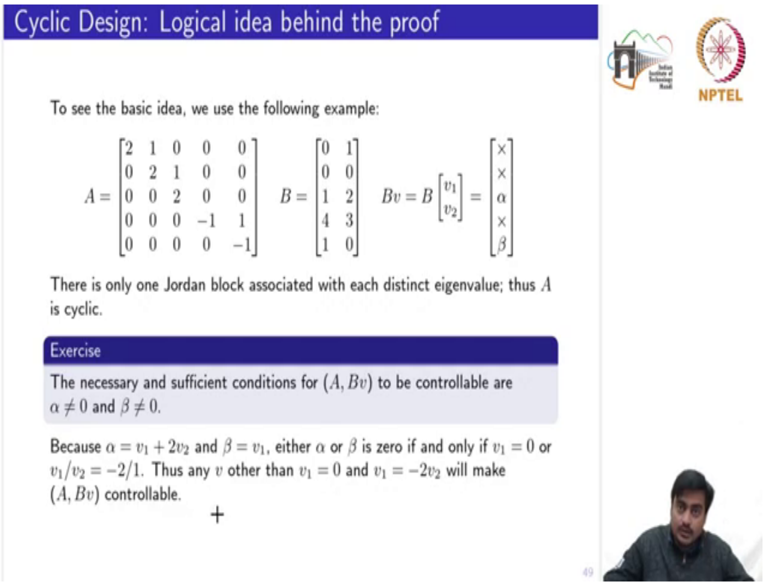

Now, one of the problems we had discussed during the tutorial of the controllability week that the necessary and sufficient condition for the pair A comma B times v to be controllable are that alpha and beta should not be equal to 0. In fact, this is also one of the results of which the proof we had seen through numerical examples during the tutorial classes or you can prove by yourself as well.

So, now see since alpha is given by v 1 plus 2 into v 2 and beta is nothing but, only v 1. Now, either alpha or beta is 0 if an only if v 1 is equal to 0 or v 1 by v 2 is equal to minus 2 by v 1.

This you can see readily that if v 1 is 0 beta would straightforwardly be 0. If beta is 0, then the a pair A comma B times v is not controllable and we cannot carry forward our design.

Now, if v 1 over v 2 satisfies or becomes equal to minus, 2 then in that case the alpha will become equal to 0. Again according to this result; the pair A comma B times v is again not controllable ok. But see thus any v other than v equal 0 and v 1 equal minus 2 times v 2 will make A comma B times v controllable right.

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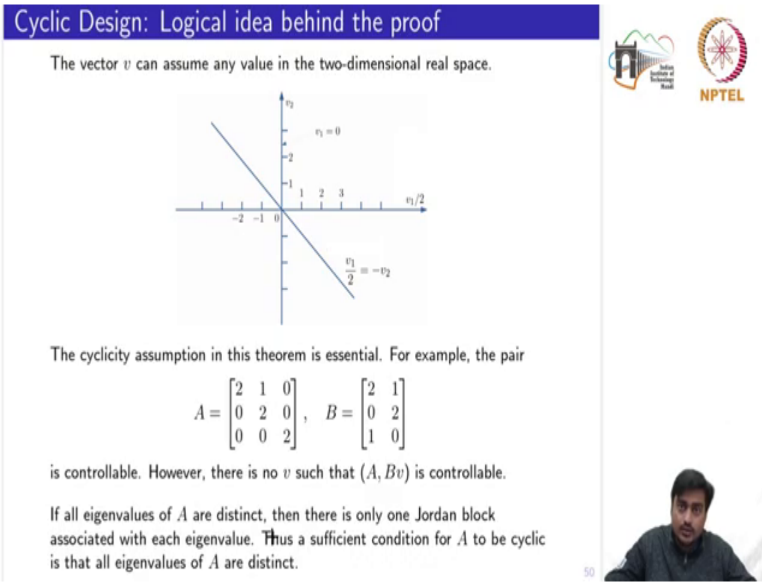

We can visualize this in a 2 dimensional space, because we have only 2 variables v 1 and v 2. So, the first condition is v 1 equals 0 which is this line ok. Along this line v 1 would always be 0 now another condition is that v 1 v 1 by v 2 should not be equal to minus v 2. So, we have normalized the x axis by v 1 by 2 and all the v s should also not lie onto this line.

Now, if this v lies any other elsewhere on this 2 dimensional space then we can then it is guaranteed that the pair a times A comma B times v is controllable ok. So, this these are the almost the rare chances. So, the cyclicity assumption is only is essential in this theorem, we can take another example also. Let us take this pair as this one which is a 3 cross 3 dimension in B again contains 2 number of inputs.

Now, this A matrix has eigenvalues has 3 eigenvalues located at 2 ok, but it has 2 eigen 2 Jordan blocks with respect to one eigen value. So, one block is this one another block is this one. So, the here although the pair A comma B is controllable, but the matrix A is not cyclic. So, for any v so there is no v such that you can make A comma B times v controllable. So, we need the controllability of this pair to carry forward our design of the gain matrix K.

If all the eigenvalues of A are distinct, then it is already ensured that there would be only one Jordan block associated to every eigenvalue. So, thus a sufficient condition for A to be cyclic is that all eigenvalues of A are distinct right. So, this is pretty much straightforward. It is only a problem if the matrix A has repeated eigenvalues. So, we need to see whether there is only one Jordan block associated to the repeated eigenvalue ok.

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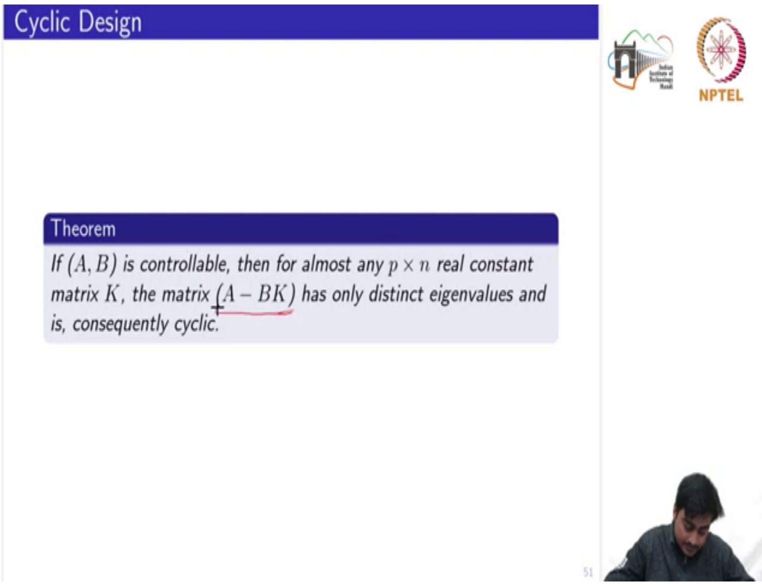

Now, the next results say that if the pair A comma B is controllable, then for almost any p cross n real constant matrix K; the closed loop state matrix has only distinct eigenvalues and is consequently cyclic. So, here we see the straightforward benefit that in the Lyapunov based design; we need to select the matrix K bar to ensure those 2 state conditions. But here we had seen that we can select almost or for almost all p v vector the pair A comma B times v is controllable.

Now, if the pair A comma B is controllable, then for almost any matrix K. We can put any matrix K such that this matrix would have the distinct eigenvalues and would definitely be a cyclic matrix. So, this is the main benefit that there we need to iterate for those K bar, but here this the become or distinct eigenvalues closed loop state matrix has only distinct eigenvalues for almost all K.

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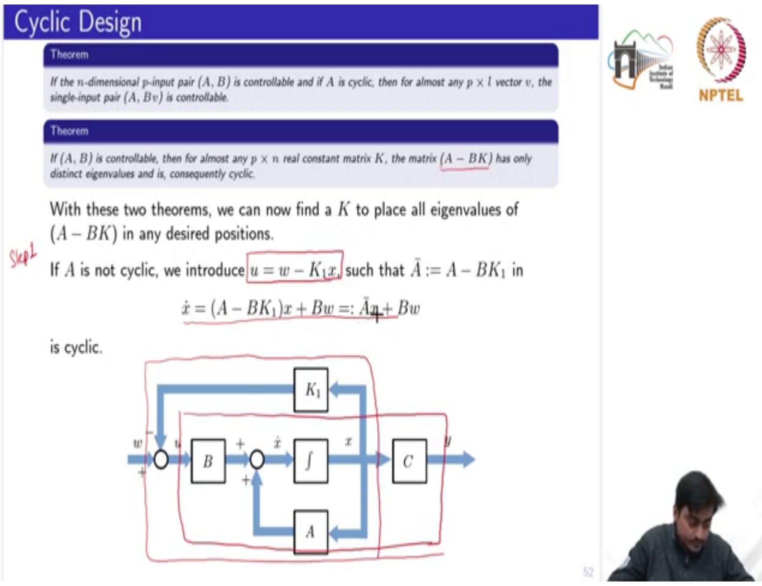

Now, with the distinct eigenvalues we also ensured with the same time that. So, this condition is pretty much straightforward that would be cyclic, because it contains all the distinct eigen values.

Now, we will use this two important results to finally, design that K the K matrix. First, but note that in both these results we need the controllability of the pair A comma B. Now, if the original pair A comma B is not controllable from the necessary and sufficient condition it is already ensured that you cannot design a K matrix who place the eigenvalues arbitrary.

So, if the pair control the pair is controllable and A is cyclic we can select any v such that this pair is controllable. Now if this pair is controllable we can select any K matrix of appropriate dimension such that this matrix has distinct eigenvalues or it becomes cyclic ok. So, we can now find the K matrix to place all eigenvalues of this closed loop state matrix in any desired position.

So let us see, so first case consider that if the matrix A is not cyclic ok. Now, if the matrix A is cyclic, we can go directly go to the step 2. This you can see as the step 1 which we need to carry out if the matrix A is not cyclic. Now, by using the second result; we can get rid of this non cyclicity of the matrix A, because we can select any K to make this closed looped state feedback matrix to make this pair as a cyclic matrix.

So, let us see how this is implemented through the block diagram, this is the original plant ok, this is the original plant. Now, this step 1 is carried out if the matrix a is not cyclic. So, first we introduce a feedback by this equation where w is some variable u is equal to w minus K 1 x, such that the matrix A bar which as defined as a minus B times K 1. In this state equation becomes cyclic.

And this is a result or this is a consequence of the second result this one second theorem ok. So, now, we had ensured two things that the that this if I see the map from w to y or w to x, from w to x. So, we I can carry out my design on this map from w to x instead of u to x. Because u to x, in this map u to x we have we do not have this matrix a as a cyclic matrix. But in the map from w to x we have the state matrix as a cyclic matrix.

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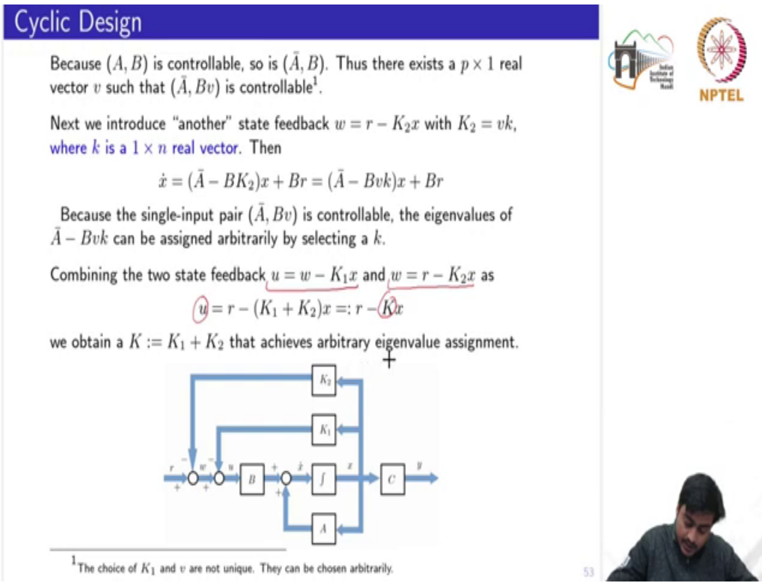

Now, because the pair A comma B is controllable, so A bar B would also be controllable, thus there exists a p times 1 real vector v such that a bar comma B times v is controllable. Now, note that here the selection of the matrix K 1 and the selection of the vector v is not unique because these this was the conclusion of the previous two results also ok. So, the choice of K 1 and v are not unique and they can be chosen arbitrarily.

Let us carry forward. So, next we introduce another state feedback where now we compute w as the reference in an r which is given minus K 2 times the state x with K 2 defined as v times k. So, you now you see that K is now a vector which we need to synthesize finally, instead of the K matrix were k is a 1 cross n real vector, then applying this controller to the previous system we obtain this state's space equation of the closed loop. So, in the block diagram you would see from the x. So, we have added this block or this path to finally, compute the step.

So, the step 1 is to carry out this part and this step 1 was carried out if the matrix K is not cyclic. Now if the matrix A is cyclic we can directly carry out the step 2 ok. So, this would become as the closed loop state matrix A bar minus B times v times k and B becomes the distribution matrix of the reference signal r.

Now, because the single input pair A bar comma B times v is controllable. The eigenvalues of this matrix of the closed loop state matrix can be assigned arbitrarily by selecting a k. So, since we only need to design this vector k which we can carry out by either using the eigenvalue assignment approach or the Lyapunov base approach where the where both the conditions would be satisfied. First the controllability matrix would be a square matrix, second the matrix T, the Lyapunov design would always be non singular.

So, now if we combine these two state feedback. So, first was the this 1 w is equal to r minus sorry, the first one was u is equal to w minus K 1 x and a second one is the w minus w is equal to r minus K 2 x. So, combining this one because we finally, need to compute the signal u is given by r minus K 1 plus K 2 times x and K 1 plus K 2 would give me the complete matrix K ok.

Now, with the help of this K matrix we can achieve arbitrary eigenvalue assignment ok. So, this is how we can carry out the design.