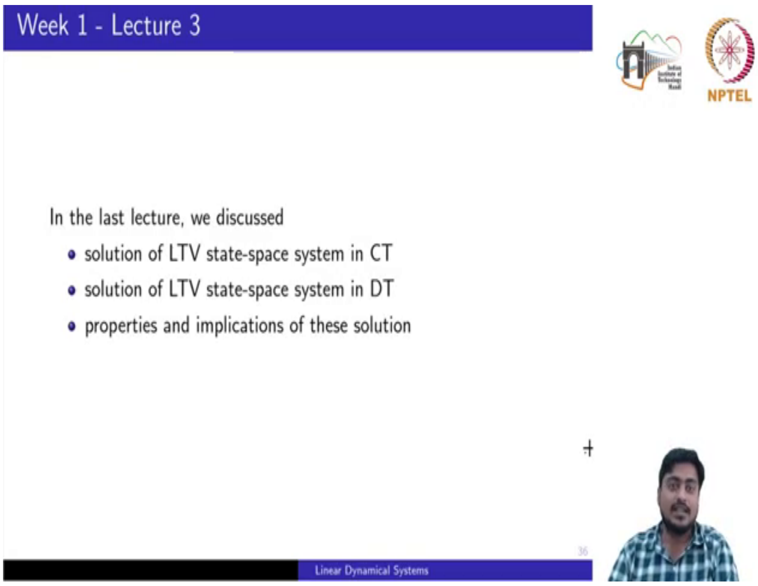
Linear Dynamical Systems
Prof. Tushar Jain
Department of Electrical Engineering
Indian Institute of Technology, Mandi

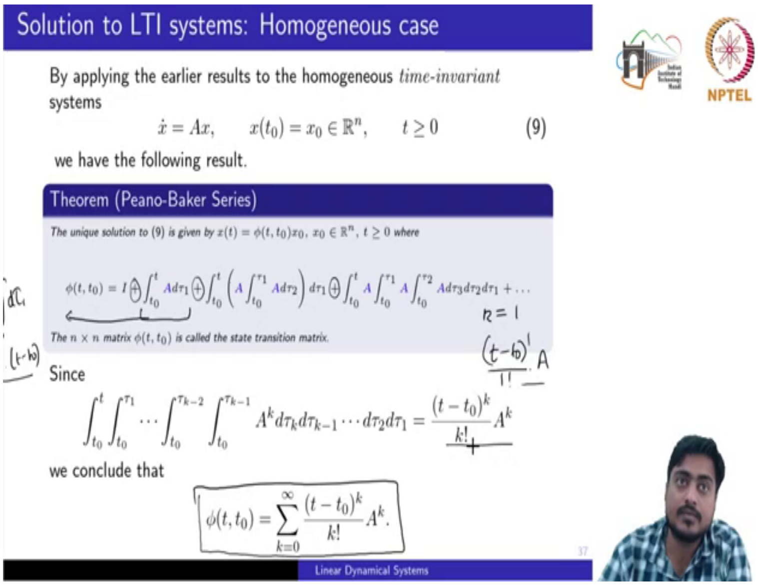
Week - 01
State-space solutions and realizations
Lecture - 03
Solution of LTI systems

(Refer Slide Time: 00:13)



So, hello everyone; today we will be starting with the lecture 3 of the first week, where we will discuss the solution of the LTI systems. So, in the last lecture, we saw the solution of the linear time varying state space system, both in the continuous time domain and in the discrete time domain. We also saw some properties and implication of this solution in terms of some properties of the state transition matrix and its relationship between the state space parameters and the impulse response.

(Refer Slide Time: 00:45)



So, proceeding with the LTI systems in a similar way what we had done for the time varying systems, first we considered the homogeneous case. So, the homogeneous case, where we said explicitly that u is equal to 0. Now, we are considering only this part; the only difference between the time varying case and the time in varying case that here the state matrix this A is now long is a constant matrix ok.

So, recalling the Peano-Baker series, what we had introduced for the time varying case because the time varying case is basically a generic case. Now, the idea is whatever the solutions we had seen for the time varying case, we would specialize those solution for the time invariant case right. So, the unique solution to this equation is given by to this part right, where phi t comma t naught is the state transition matrix of an n times n and this in this Peano-Baker series all these A matrices are now the constant matrices ok.

Now, let us consider that k’th term of this matrix, of this series. The k’th element of the series. So, I would have. So, this is the zeroth element because if this the zeroth, oneth, second and the third element. So, if we consider the kth element, I would have the outer integral from t naught to t. This subsequent integral from t naught to t 1 and the last integral would be t naught to tau k minus 1, similar to if we see the third third element it is from t naught to tau 2 ok.

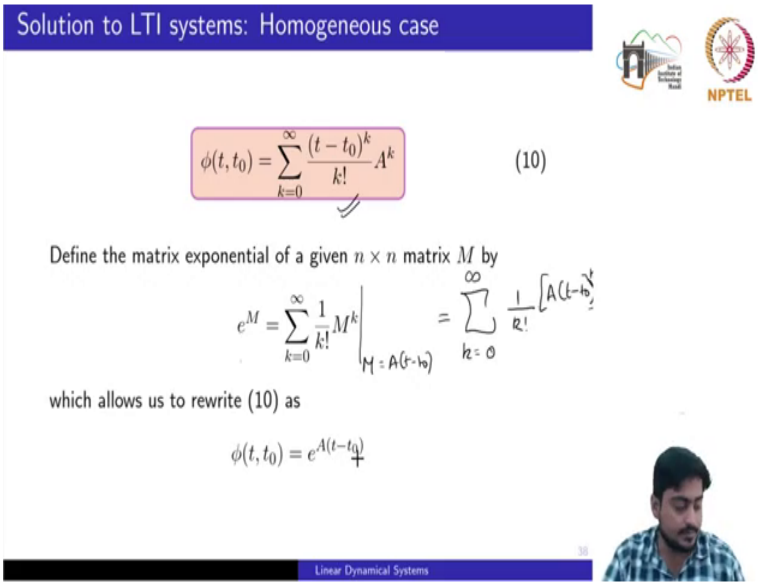
So, this is k minus one’th part. So, here in the third element we have A to the power 3. So, for the k’th element, I would have A to the power k and similarly, all these d tau k d tau k minus 1 up to d tau 1 right. Now, this k’th element I can write as the ratio of t minus t naught to the power k and k factorial into A k the reason is that all these A k are constants first of all.

So, I can take this A k outside this integral and the all the inner integral are basically the integrals of the of the zeroth power of t ok. So, it would turn into t 1 minus t naught to the power k divided by k factorial right. So, we can take one specific case let us say we want to consider for k is equal to 1 right, just to better understand the k’th term let us consider k is equal to 1. So, k is equal to 1 basically its these term ok. If I put k is equal to 1, the solution would be a t minus t naught 1 over 1 factorial into a ok.

Now, notice here ok, this part where since A is a constant matrix, I can take its outside the integral it would be t naught to t d tau 1 right and this one is basically t minus t naught and this part and this part is basically equal just for the better visualization, we you can also consider for k is equal to 2, k is equal to 3 just to understand that the k’th term or the solution of the k’th term would be given by this one right.

So, I get if I sum all these k’th term starting from k is equal to 0 to infinity because these are all the summations because of this summation if I sum all these k’th elements starting from 0 to infinity, I would actually get the closed form solution of the state transition matrix which is equal to the summation over k is equal to 0 to up to infinity of this part ok.

(Refer Slide Time: 05:01)

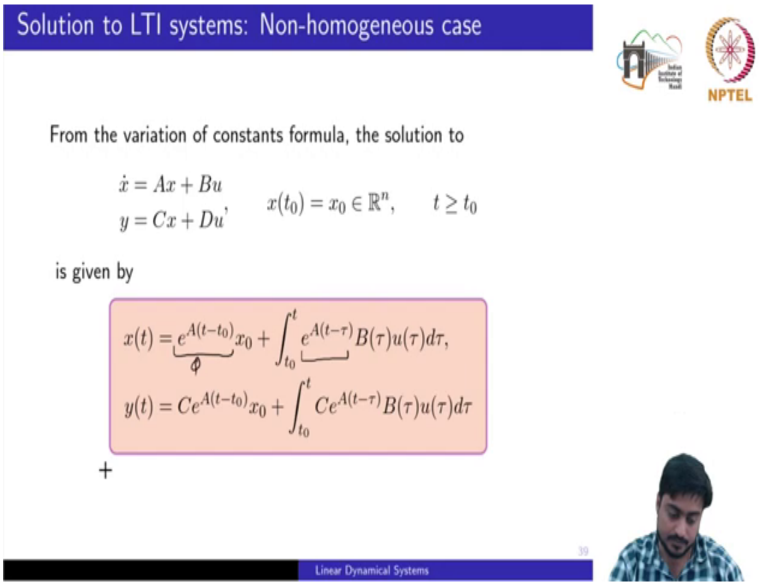


Now, let us see some interesting thing of this state transition matrix ok. So, first of all we will define the matrix exponential of a given n cross n matrix M which is given by e to the power M is equal to the summation of basically this is the series listen this is exponential it would be summation over k is equal to 0 to infinity 1 by k factorial into n M to the power k.

Now, using this exponential, if I try to rewrite this part I could write phi t comma t naught is basically the exponential of A t minus t naught ok. So, if I replace M by A into t minus t naught here, let us say replace M by A t minus t naught, I would have summation k is equal to 0 to infinity 1 over k factorial and M is basically A t minus t naught to the power k right, which I can write a to the power k into t minus t naught you know to the power k which is equal to this one.

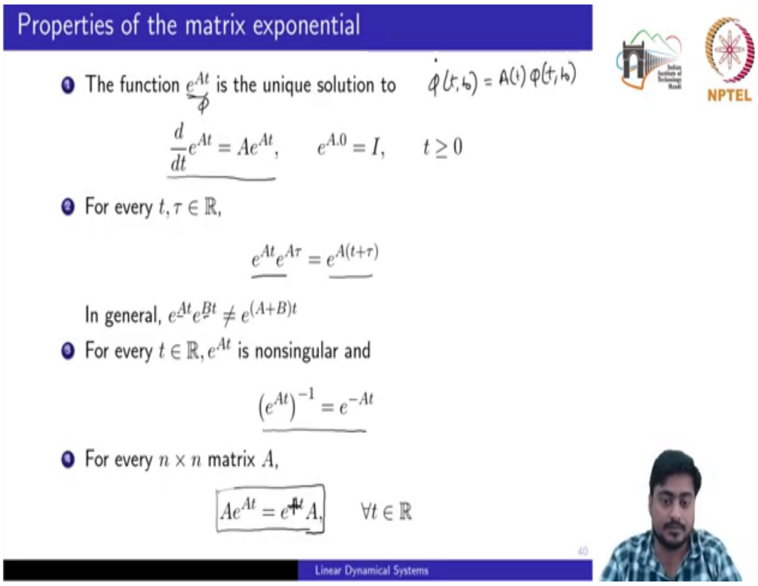
So, my state transition matrix and the linear time invariant case is basically the matrix exponential, where t naught is the initial time, where I am taking the initial value of the state ok.

(Refer Slide Time: 06:43)



Coming onto the non homogeneous case, where we now using u also. So, this u would have the similar effect what we had seen in the linear time varying case. This is the state transition matrix fine. This is also the state transition matrix into B into u. So, it will B if we go through the solution of the LTI system and if I replace those state transition matrix by this matrix exponential, I would get the same solution for the x and for the y as well ok.

(Refer Slide Time: 07:21)



Now, there are certain interesting properties of this matrix exponentially in a similar way for the LTV case that if e to the power A t is the state transition matrix, we know that the state transition matrix is the solution of this part. If you recall from the LTV case that phi dot of t comma t naught is basically given by A t phi t minus t naught for the LTV case.

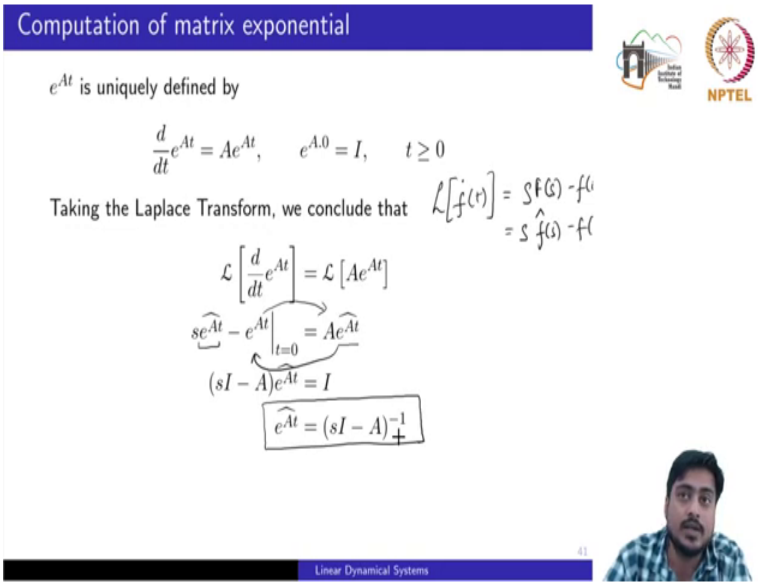
Now, phi dot t comma t naught is basically e to the power A t in the matrix exponential the derivative of this matrix exponential. In this case we have A as a constant matrix and e to the power A t is again the state transition matrix. So, the first property remains similar to the LTV case, where we just replaced that time varying state transition matrix right.

The matrix exponential again for every t comma tau belongs to R, we can write the multiplication of matrix exponential at t and at tau by the matrix exponential at t plus tau ok. We should also remember that if I am computing the multiplication of two matrix exponentials at the same time, but with different matrices ok, this is not equal to the summation of their matrix exponential ok.

This is possible or I would say this is equal to this one only if A B are scalars right. Here we have A B as the matrices. Now, for the third property says that for every t belongs to the set of real numbers, u to the power at is non singular and it is satisfies this part that the inverse of the matrix exponential is given by the negative power of the matrix exponential. The fourth property that for every n cross n matrix A, the multiplication of A that I can commute I can compute this multiplication of the matrix A and the exponential matrix with matrix A ok.

The proof of this part you can you can do by yourself; the hint is that you can first of all use the series of the exponential series and then, multiply with the pre multiply in the post multiply with the A matrix.

(Refer Slide Time: 10:01)

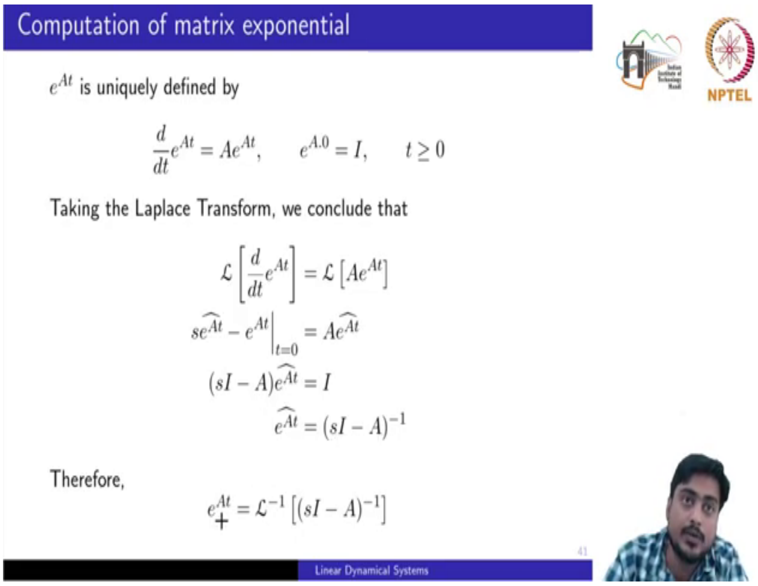


Now, the computation of the matrix exponential; so, we know that e of A t is uniquely defined by this relation by using the first property of the matrix exponential, we can compute the function ok. So, taking the Laplace transform both sides, we would have the Laplace transform on this part and the Laplace transform on this part of this part, we denote s into the Laplace transform of this function. So, this is denoted by the hat minus the value of that function A t t is equal to 0 ok.

Basically, the idea here is if I have a function f of t and I want to take the Laplace of f dot of t, it is given by s of f s sorry excuse me s of f s minus f of 0 and this f of s, we have been representing as f by a hat. Just to denote that it is the frequency domain counterpart of the time domain function. This is what we have represented; this notation what we have used here ok.

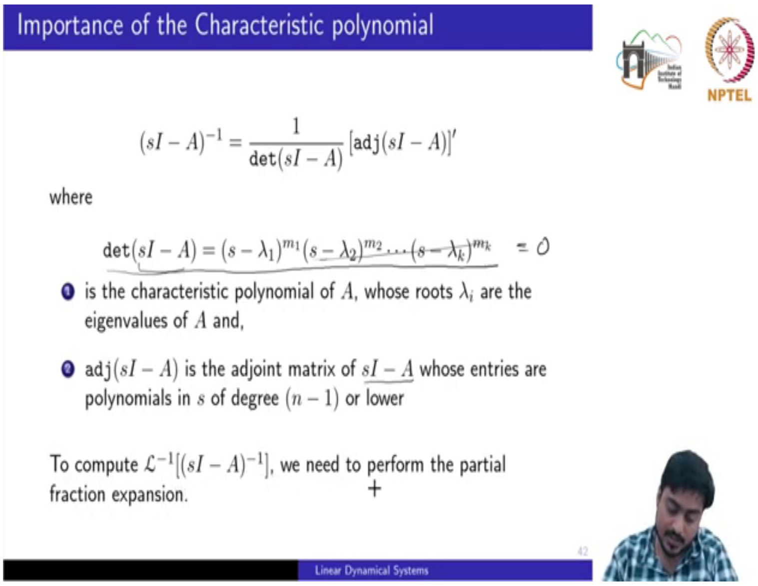
Similarly, here since A is a constant matrix, I can take it outside of the Laplace transform A into the Laplace transform of this function. Taking this part onto the left hand side and this 1 onto the right hand side I can write this part as s I minus A into the Laplace transform of e to the A t is equal to I and finally, I can write the Laplace transform of the matrix exponential by s I minus A inverse ok.

(Refer Slide Time: 11:47)



So, simply by taking the Laplace inverse of this part, I can compute that time domain matrix exponentiation ok. Now, there are some interesting properties of this s I minus A, which reveals some internal structure of the state space system.

(Refer Slide Time: 12:07)



So, this s I minus A, the inverse of s I minus A I can write as the ratio of the transpose of the adjoint of s I minus A divided by the determinant of s I minus A. Now, this determinant of s I minus A it can be given by this one, where are all these powers; so, what I have done here this s I minus A, if I take the determinant, first of all it would be a polynomial right.

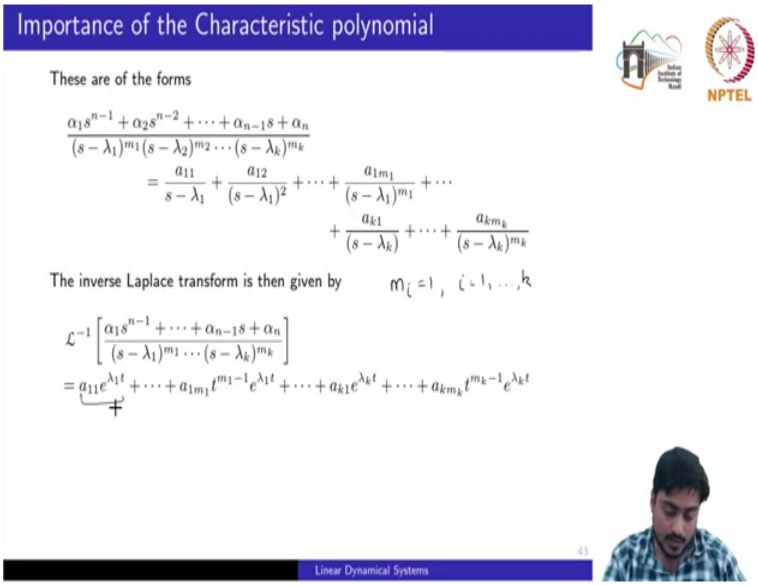
Now, when it is a polynomial, I can factorize into the lower order polynomials. Here we are taking the first order polynomials which could have their multiplicity greater than equal to 1 greater than 1. So, here we are taking m 1, m 2 up to m k ok.

So, this denotes the characteristic polynomial of A, and all these roots if I put it equal to 0, then I would all these roots lambda 1 up to lambda k or the lambda i are all defines the eigen values of A ok. Now, the idea of denoting this m 1 and m 2 meaning to say that there could be 2 eigen values at the same location.

Let us say all this part is equal to 0; oh sorry all this part is equal to 1 and here we have m 1 is equal to 2. It means that we have 2 eigen values at lambda 1. This is the reason we are defining this multiplicity ok. Now, this adjoint s I minus A is the adjoin matrix of this matrix s I minus A whose entries are polynomials in s of degree n minus 1 or lower.

So, we know that if our a matrix is the n cross n matrix, then this polynomial which is the determinant of this part, it would be having the degree n ok. But when we take the adjoint of s I minus A say it would always have the degree up to n minus 1 hm.

(Refer Slide Time: 14:33)

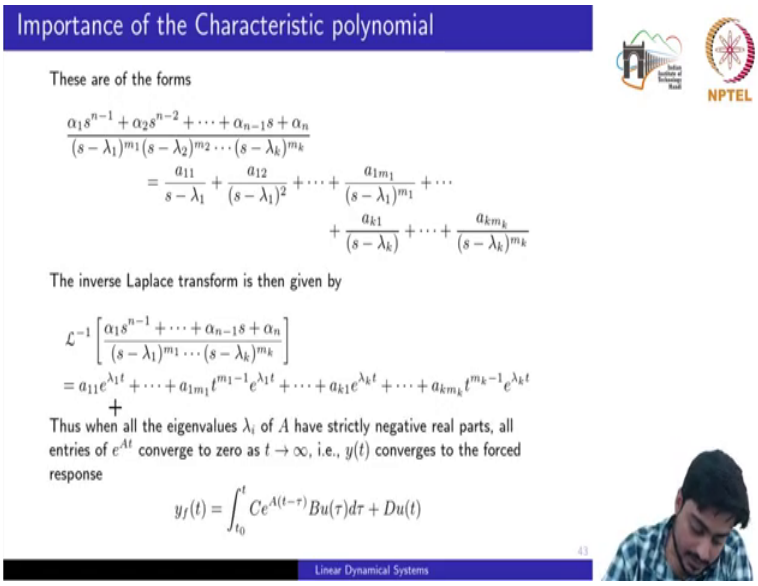


So, to compute the Laplace inverse or the inverse of s I minus A, we do the partial fraction which would take this form possibly that here we write the determinant the characteristic polynomial right and here, the we take the order up to n minus 1 depending on all these coefficients, this order would be lower than n minus 1 right.

Now, if I partial do the partial fraction for only this part s minus lambda 1 or a power m 1, this would be the partial fraction with all the coefficients which are need to be identified fine. Similarly, all these lower degree polynomials of multiplicity m I, I have individual parts.

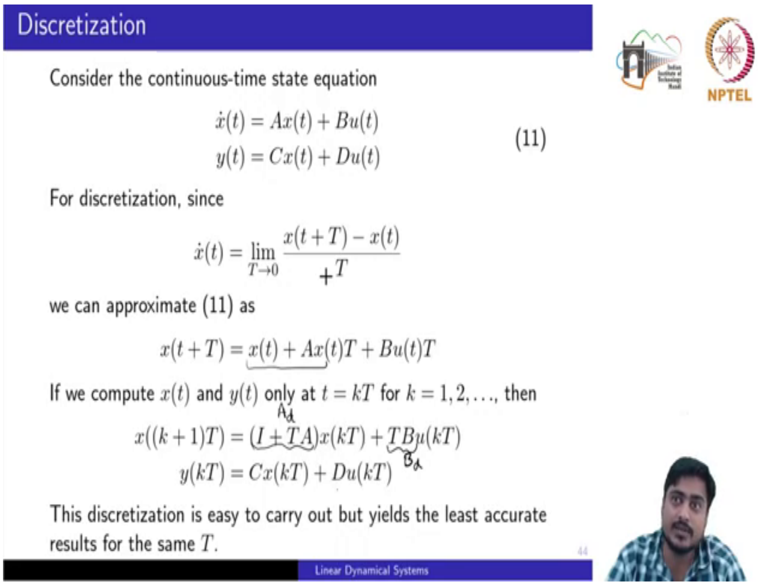
Now, if I take the Laplace inverse, it would have a 1 1 e to the power lambda 1 t plus this a 1 m 1 t to the power their multiplicity minus 1 into e to the power lambda 1 t. Similarly, for all these lambda i up to k starting from 1 to k ok. Now, suppose if the all the all the m's all the mi's are equal to 1 or i is equal to 1 up to k ok. We will not be having this term because if I put mi equal to 1, it would be 0. So, the only remaining term is this one.

(Refer Slide Time: 16:09)



Now, see that when all the eigen values of lambda i of A have strictly negative real parts and all entries of e to the power A t converge to 0 as t tends to infinity that is the response y of t would converge to the forced response given by this equation ok.

(Refer Slide Time: 16:37)



Now, talking about the discrete time system; so, there are some couple of interesting page. So, we know that most of the real world systems are continuous time system and we saw in the time varying case that the solution of their discrete time system is basically easier to compute than the, then what we do in the continuous time systems because we can compute the state transition matrix recursively right.

So, the first approach is either we discretize the continuous time system or we directly use the discrete time system. So, let us say given a continuous time system, what are the different methods or first of all we would see how we can discretize that continuous same system.

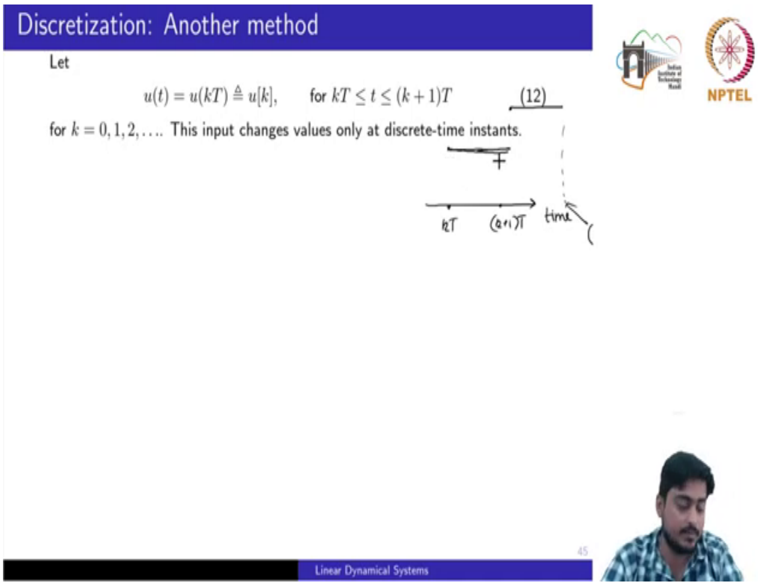
So, say for example, given the continuous time system by this part the straight forward approach for discretization is to approximate this x dot t by this part, where T is the sample time. Ok x t. So, we are approximating this derivative by x t plus capital T minus x t divided by capital T. So, whenever this capital T approach to 0, this part would become the x dot t ok.

So, now if I put this part this equation into this equation, I would have x t plus T equal x t plus A x t capital T plus B u t capital T. I just put this part into this equation and further for simplicity, we can write that if we are computing the x t and y t only at t is equal to k T, where T is my sample name and k is the sampling instance ok.

Starting from 1 to t up to infinity, then I could write x of k plus 1 into T is equal to if I combine these two part, it could give me I plus T A into x of K T plus T B into u of k T; similarly this one. So, this matrix is defined as the discrete time A matrix of this continuous time A matrix ok. This is B d. Let us denote the d as a discrete time d matrix ok. There won't be any effect on the C and D matrices of the discrete time which are I mean very similar to the continuous time matrices ok.

But this discretization since its we saw that it is quite easy to carry out, but yields the least accurate results for the same thing. You can verify this in simulations also by taking different values of t and taking any scale of system you can take any scale of continuous time system and by using this approximation, try to compute the discrete time values and see whether the values at discrete time instance matches with the continuous time values ok.

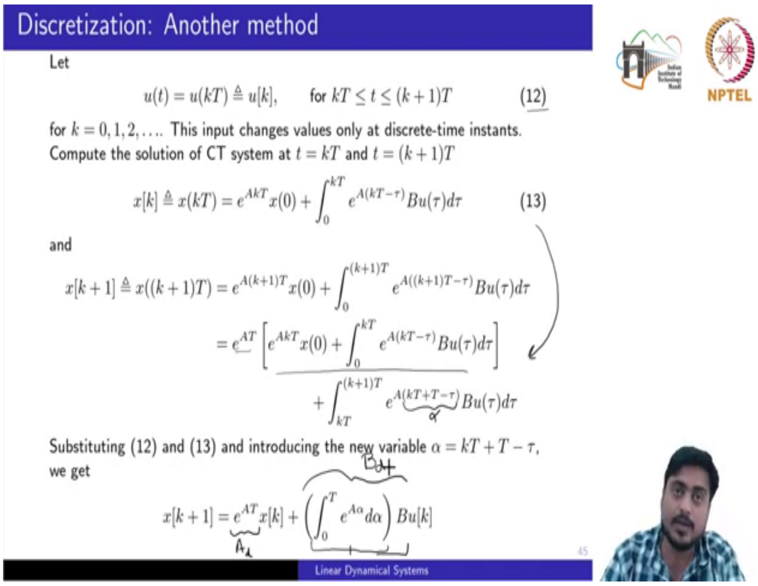
(Refer Slide Time: 19:59)



So, there is another method which is much more efficient of the discretization, where we suppose that our u of t is basically is equal to u of k T; for T lying between K T to k plus 1 T ok. This means that I am considering u is a piecewise constant signal. Say for example, if let us consider this time axis; time axis. This instant is my K T and this instant would be my k plus 1 T, where T is my sample time in the k plus 1. So, whatever the value I am having here of this u, I am keeping it constant up to the next time impulse.

Now, if I move towards to the next sampling instant, let us say it is having this value. So, I will keep it as a constant value up to the. So, this part would define the k plus 2 times T ok. Now, since this is a continuous signal between the 2 sampling instance. So, I can use the continuous time solution to write the solution of the or between the to write the solution between this two sampling instants right ok.

(Refer Slide Time: 21:21)



So, let us see. So, I compute the solution at t is equal to K T and also at t is equal to k plus 1 T. So, this solution is given by I am using this denotations in the square brackets just to explicitly denote that we are dealing in the discrete time ok. So, we are having this x of k which is equivalent to x of k T, we can remove the sampling time once it is fixed for all the computations.

So, here I replaced all T by kT's right. e to the power K T x naught starting from 0 to T be replaced by K T and similarly this part. So, this is the solution of the continuous time state equation at T is equal to K T and at T is equal to k plus 1 T. Ok, now let us see.

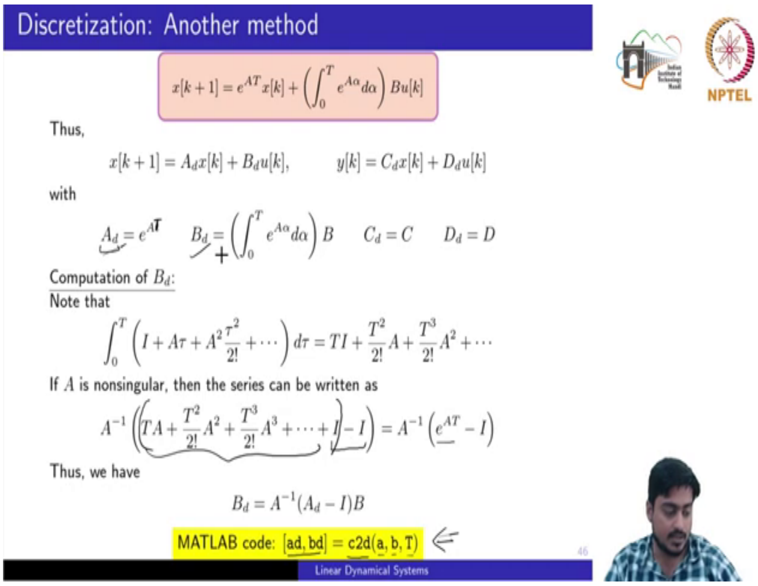
So, first of all, I break this integral into 2 parts. This integral starting from 0 to K T which is this 1 0 to K T and from K T to k plus 1 T ok. Now, from the break integral from 0 to k T, I take this part matrix exponential as a common from this one and from the break integral, which I can write e to the power A K T because if I take e to the power A T common, the only remaining part is this one into x naught plus this part because this part would also I can take it outside the integral. But since this is a constant for this integral because this integral, I am taking over tau.

And the break other break integral from K T to k plus 1 T remain as it is this is k plus 1 into t minus tau ok. Now, pay attention here that this part it is nothing but equal to this one right. we have e to the power K T x naught, e to the power K T x naught and this term also is equal to this term. So, I can replace this whole term by x of K T ok.

So, I substitute this 12 and 13 into this equation to finally, write x of k plus 1 is equal to e to the power A T into x of k which is this one plus this integral 0 to T e to the power A alpha d alpha into B u k ok, because I replace this part by alpha. So, I would have e to the power A alpha d alpha because B, since B is a constant matrix and then, rearranging these the lower limit of the integral and the upper limit of the integral, we finally got this expression ok.

Now, let us see. So, this part I can define now that discrete time A matrix of the continuous time system and this is and this whole part basically this whole part is the B d matrix. Let us see in the next slide.

(Refer Slide Time: 24:53)



So, we have x k plus 1 is equal to e to the power A T x k plus the this one. So, with I can write into the discrete time state equation is A d and B d, C d and D d; where A d is now equal e to the power A T and B d is this whole expression; the C and D matrices do not change.

Now, basically it should be e to the power A into capital T right because we are telling e to the power AT, given the sample time, we can compute this matrix exponential ok. Here for the computation of B d, we need to compute this integral right; but we also have a solution for that that instead of computing the solution, we can write the closed form solution of this one.

So, first of all seeing the computation of B d note that if I take given this series right, given this series and I take the integral of this series from 0 to T, I can write this one; the first one would be T right.

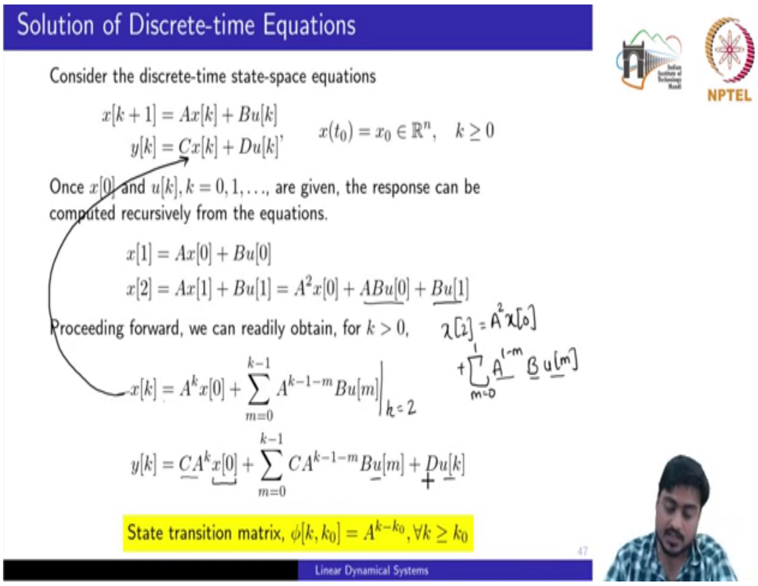
It would be A T squared by 2 factorial; similarly this one ok. Now, if A is non singular, then the series, this series can be written as I multiplied by this A inverse ok, multiplying by this A inverse would give me this part, since T is a scalar. So, I can compute right. Now, at the end I do the addition and the subtraction of the identity matrix of appropriate direction not dimension ok. Now, see this part; now this part is basically equal to the matrix exponential e to the power A T and this e to the power.

So, this part would become e to the power A T minus I pre multiplied by the inverse of the A matrix and this e to the power A T is basically my A d. So, I can write my the discrete time d matrix as A inverse multiplied by A d minus A into B. So, once I have computed the matrix exponential to compute the discrete time A matrix, I can directly compute the B d matrix instead of computing the exponential, the integral of that exponential.

Now, in the same discretization you generally see in the MATLAB function, when you do the continuous two discrete time, this is the MATLAB code. You can take any A B matrices given a sample time, capital D these AB matrices are the continuous time matrices. Now, if I use this function to convert the or to transform the continuous to discrete, it would give me A d and B d.

If you compute for the given AB matrices using this equation and this equation, you would get the same result ok. So, this is the most accurate discretization.

(Refer Slide Time: 28:25)



After seeing the discretization, let us come on to the solution of the discrete time equations. Say for example, if we are directly given the discrete time systems, we know that this both these equations are the algebraic equation and one direct way of computing the solution is to do recursively. Once x naught and u of k are given for k is equal to 0 to up to the current time like even the response can be computed recursively from these equations.

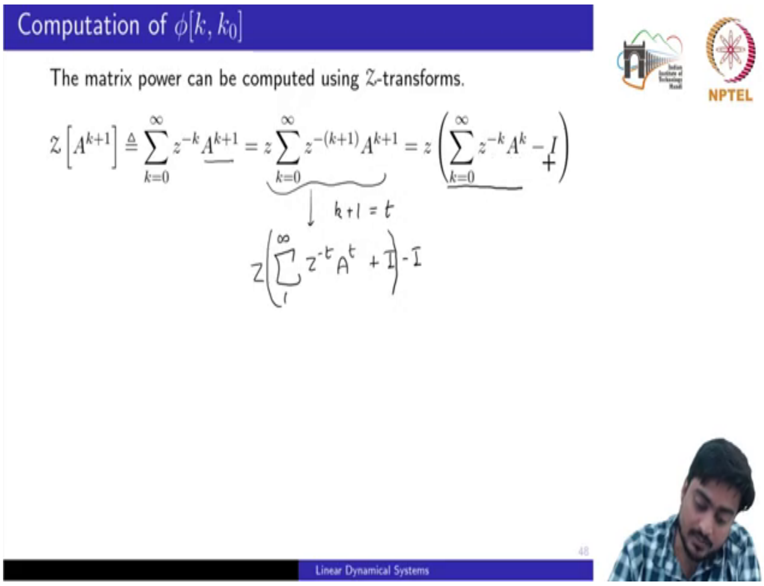
Now, the interesting point here is that now we see the explicit significance of the initial condition and all the us starting from k is equal to 0 up to the infinity. Say for example, if I put k is equal to 0, k is equal to 0 in this equation, I would get x 1 is equal to A x 0 plus B u 0. Let us put X is k is equal to 1, I would have x of 2 is equal to A into x of 1 plus Bb into u of 1.

I can write x 1. I can replace this x 1 by this x 1 to finally, have A square into x naught plus AB into u of 0 and B times u of 1. So, I can parameterize all the solution, the solution of the state equation in terms of the initial conditions and the u of k for all case starting from 0 to up to the current time ok. Now, proceeding forward we can readily obtain for k taken as 0 is this part and the state transition matrix phi of k comma k naught is given by the k minus k naught power of A for all k greater than equal to k naught.

This you can once you start doing for the k'th time, you will finally approach towards this solution. Say for example, if I put k is equal to 2 just for the better visualization in this part, let us see if I put k is equal to 2 because we already have the solution for k is equal to 2. So, we would have x of 2 is equal to 2 hm. If I put k is equal to 2 here and this part would be summation m equals 0 to 1 k minus 11 and then A 1 minus m B u of m. Now, here in the summation, I would have only two terms for m is equal to 0 and m is equal to 1.

So, if I put m is equal to 0, I would have A into B into u of 0 which is this term. If I put m is equal to 1, this would be identity B into u of m which is this term. So, you can validate also for other values of k. Now, just putting this x k again back to here, we get the response which is basically the summation of the two individual responses; one because of the initial step and another because of the input hm. This is the zero-input response and this is the zero-state response with the state transition matrix is given by this one.

(Refer Slide Time: 31:59)



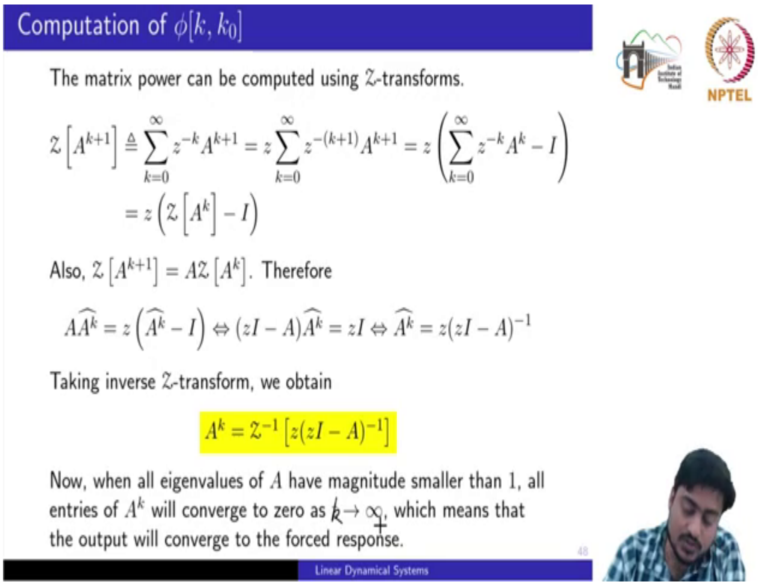
Now, coming onto the computation of the state transition matrix; so, here the matrix power can be computed using z transforms. So, for the continuous time, we use the Laplace transforms; for the discrete time, we would use the z transforms. So, we need to compute the z transform of A to the power k plus 1 ok. This I can write by this expression z to the power minus k into A k plus 1, where k is from 0 to infinity ok.

Now, see here if I multiply and divide this expression by z, I can take z outside of the outside or inside of the summation because, I am taking the summation over k not over z. So, z is a constant for this summation. I multiplied by z and divide by z, I would get z to the power minus k plus 1 into the rest of the part that is A the power k plus 1 ok. Now, see interesting thing here let us say I represent k plus 1, I replace k plus 1 by some variable t ok. Now, this one would be given by z; no excuse me.

So, at k is equal to 0, I would have t is equal to 1, the lower limit and the upper limit would be infinity right. Here I would have z to the power minus t and A to the power t ok. Now, here I need to include this k is equal to 0. So, let us say the value of this summation at t is equal to 0, the value of this at t equal to 0 would be what? Identity.

So, I add plus I and subtract I the identity matrix of appropriate direction. Now, this part I can write as this one, where I have changed the lower limit of the summation from 1 to 0 ok. So, this in this summation is k is equal to 0 to infinity the rest of the part would remain the same minus I into Z ok.

(Refer Slide Time: 34:27)



So, I can write this expression as z times the z transform of this A to the power of k minus I and also we know that the Z transform of A to the power of k plus 1, I can write this one because a is a constant. So, I can take A outside of the transform ok.

Now, in a similar way if we proceed, this A is this one and this Z transform of A k, I represent by a hat. So, the hat the a to the power k over hat is basically the Z transform of A k. So, after simplifying all this part I finally, get this part that the Z transform of A to the power k is given by z times the inverse of z I minus A ok.

So, taking inverse of Z transform, we obtain the a of k as the z inverse of this whole part ok. Now, similarly if all the eigen values of a have magnitude smaller than 1 because all the eigen value should lie inside the unit circle; then, all entries of A to the power of k will converge to 0 as basically it should be k tends to infinity which means that the output will converge to the forced response right.