Linear Dynamical Systems
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Week - 05
State Feedback Controller Design
Lecture – 29

Lyapunov Method of State Feedback Design

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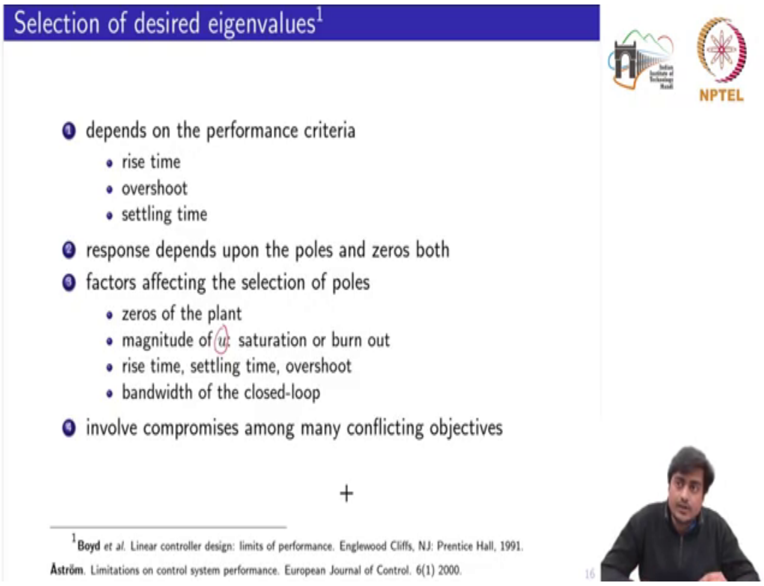

So, in the last lecture we have seen one of the method for computing the State Feed backing so, that method was known as the eigenvalue assignment. So, we stressed on to one fact that if the system is controllable then given a set of eigenvalues we can place the eigenvalues of the closed loop system. Now, another question arises that how do we get the information about the set of desired eigenvalues.

So, there are different ways of computing the set of desired eigenvalues. So, you have studied in your, possibly you have studied in your ug control course some methods from the root locus diagram or from the bode plot by taking into account the transient and the steady state characteristics.

So, first of all it depends on the performance criteria that what performance criteria we are targeting, meaning to say what are our control objective, it could be the rise time and the overshoot or the settling time if we speak these criterion in terms of the time domain. In terms of the frequency domain also the responses not only depend on the poles, but it also depends on the 0s. So, so far we have focused on to the location of the poles and also placing the location of the poles by using a state feedback, but the overall response is also affected by the location of 0s.

The factors which affects the selection of the poles first of all is the zeros, so in the last lecture we had seen that if by using the state feedback you are able to place or you might happen to place the poles. At those location, where the zeros of the plant are already there, then it may lead to the state of unobservability which we will discuss in the coming weeks.

So, we need to take care that first of all whatever the poles we are we want to place for the closed loop it should not overlap with other zeros. Second point here is the magnitude of u which takes into accounts the saturation or burn out. So, this point is more from a practical point of view; say for example, if you have certain plant then those plants are basically actuated by some actuators and we take some measurements with the help of some census.

Now depending on some actuator limits we need to put some constraints on the signal u. So, it might happen that you have design a controller in such a way that the control signal always leads to the situation or it leads to the maximum value to what the actuator is capable of. So, we need to take care of that because this is not very desired condition or scenario.

The rise time settling time overshoot is pretty much obvious, the most importantly is the bandwidth of the closed loop, which takes into account the frequency domain characteristics and there is a very strong correlation between the time domain characteristics and the frequency domain characteristics which possibly you have studied during your ug control course.

So, on and only it involves some compromises among many conflicting objectives. So, in the sense that when we started this week. We put different, two different objectives to synthesize the control log, meaning to say when we discuss about the open loop minimum energy control. So, the first objective was of the controllability the second was of the and there was the second objective also. Now, there are two good references which I would like to highlight here.

So, the first one is by Stephen Boyd on linear controller design limits of performance. So, this is about so many chapters of this book discusses about with what performances or what are the limits, on the performance of the closed loop system you can achieve. The second is paper by Cal Astrom on limitations on control system performance which was published in European journal of control in 2000.

So, this paper is also quiet good which discusses an overall overview of the performance of the control system. So, we would not go in to the details of computing the set of desired eigenvalues, but we will see some guidelines that how one can specify the set of desired eigenvalues.

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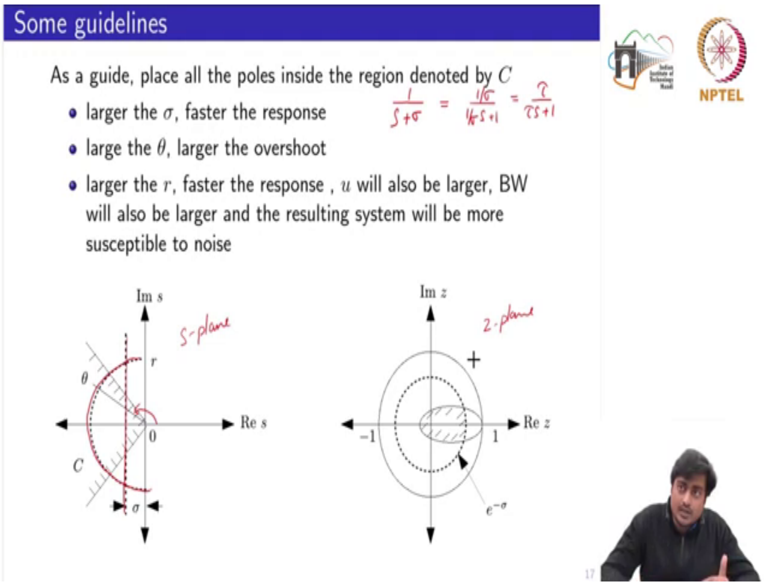

So, just some guidelines say for example, on the left hand side this one is basically the s plane in the continuous time domain and this one is the z-plane in the discrete time domain. So, first of all, the first objective that all the eigenvalues of the closed loop system should be on to the left side.

Now, here we are specifying an additional control objective that all the eigenvalues should be inside this region C which is defined by this region. So, when you see this region in the s plane by drawing the damping coefficient line, you can process some of the parameters let us say the sigma which is the real part of the s plane; the theta the angle which it makes from this axis and also the radius of this semi-circle.

So, if we have larger the sigma the response of the closed loop system would be faster if we have closed loop poles happens to be having the larger value of sigma. So, this implication you could see pretty much straight forward; say for example, if you say the larger the sigma let us say in terms of the transfer function. Let us say we have a transfer function some s plus sigma, ok. So, s plus sigma into the sense I can also write this is equivalent to 1 by sigma and 1 by sigma s plus 1 and replacing 1 by sigma by tau so, I can write this tau by tau s plus 1 where tau is the time constants.

So, if we have larger the sigma the value of the time constant tau would be smaller meaning to say the response would automatically, will be faster. If we have larger the theta, larger the overshoot, if we have larger the r the response of the system would be faster. With the faster response as we have noted in some tutorial problems when we if you recall some of the problems we discussed with the controllability as a part of the tutorial on controllability.

So, if we want to reach to some desired state value in t even amount of time the lesser the time t 1, the faster is the response and faster is the energy. So, we; so this is obvious larger the r faster the response would be and u would also be larger in that particular case. In the same time bandwidth will also be larger and the resulting system will be more susceptible to noises, ok. So, this is in the s plane equivalently is drawn in the z plane.

So, most of the time in this week we would discuss the design of the controllers in the continuous time domain, all the results whatever we discuss we directly applicable on the discrete time systems as well, right.

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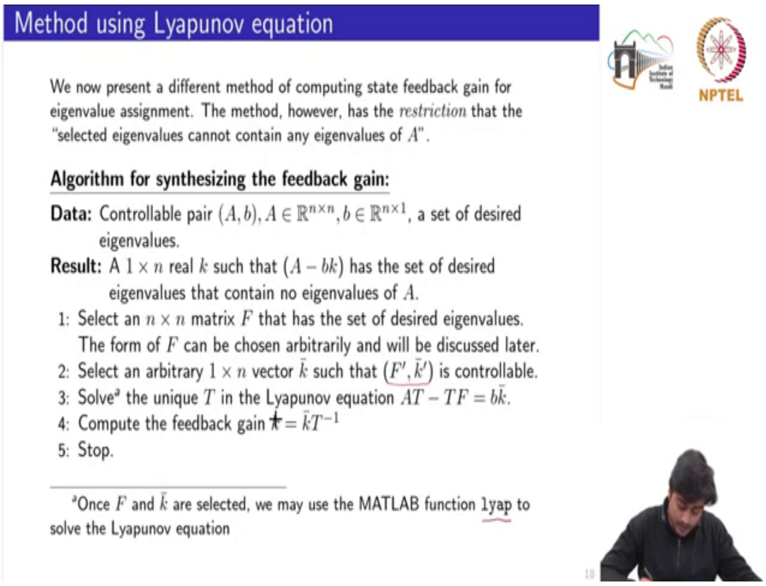

So, we will now discuss another method of computing the state feedback gain for the eigenvalue assignment problem. So, in the last lecture we studied one way of computing the state feedback gain using the controllability matrix. So, this method has some restriction that the selected eigenvalues cannot contain any eigenvalues of the plant itself or of the A.

So, here first we will discuss the algorithm for computing the switch feedback gain and then we will try to justify that algorithm by computing certain conditions which are necessary insufficient for this algorithm to work. So, the data which we supply to this algorithm is the pair A comma b which supposed to be a controllable pair where A is n-dimensional matrix and for the moment we are taking u as a scalar.

So, in that case we would be having n cross 1 dimension vector and a set of desired eigenvalues. So, based on these three information the result we want to compute, 1cross n real vector k such that the eigenvalues of this matrix of the closed loop matrix A minus b k has the set of desired eigenvalues that contain no eigenvalues of A.

So, the first step is we select an n cross a square matrix F of dimension n that has the set of desired eigenvalues. Now, the form of F can be chosen arbitrarily and we will be discussing it after the justification that what F matrix you can select. The second step is select an arbitrary 1 cross n vector k bar such that the pair F transpose and k bar transpose is controllable.

Now, if you recall in the last algorithm based on the information of the coefficient of the characteristic polynomial of the A matrix and of the set of desired eigenvalues we pre specify this k bar. Now, here we got some freedom that once we have a set of desired eigenvalues first of all the form of F matrix could be anything such that it is having the those eigenvalues.

Now, here we can select any arbitrary k bar vector, the condition which we need to satisfy is that the pair F bar F transpose and k bar transpose is controllable. Third, we solve for the unique matrix T in the Lyapunov equation given by AT minus T times F is equal to bk bar. The final step is once we have obtained the T matrix we compute the feedback gain vector k as k bar into T inverse.

So, once we have selected a k bar matrix such that this pair is controllable then we can use the MATLAB Lyapunov function this lyap function to solve this Lyapunov equation. So, you could use this a function in a similar way what we can discuss during the stability week. And once this k bar matrix or k bar vector is selected we can solve this Lyapunov equation to solve for T and after putting T here we would obtain the state feedback vector.

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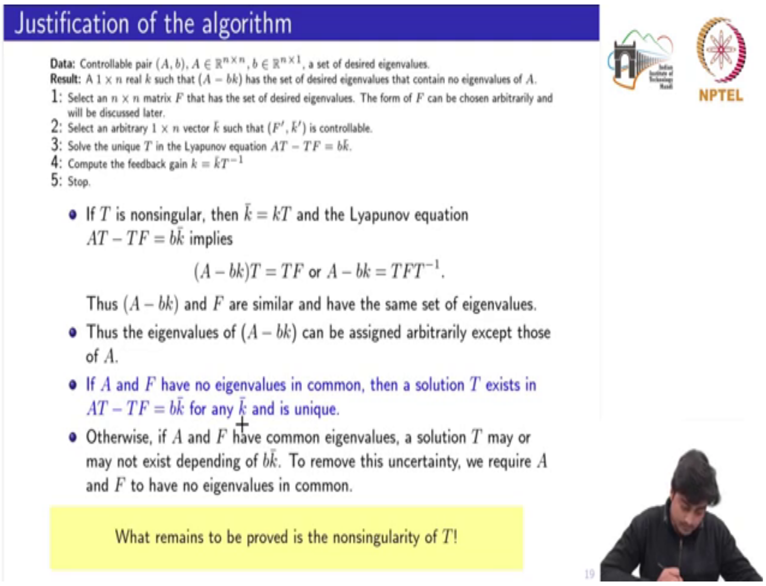

So, how do we make sure that once we that whatever the k we have computed were going to yield the set of desired eigenvalues. So, here we will see the justification of all these four steps. So, that we can ensure that once we plug in this case state feedback vector we would obtain the closed loop as this one which would contain the all the set of desired eigenvalues that contains no eigenvalues of A.

So, the first point is at if T is nonsingular, if the matrix T is nonsingular then I can write k bar is equal to k into time, k into T from here and the Lyapunov equation can be replaced in which the k bar can be replaced by kTs. Say for example, if I put k bar is equal to kT here and they take it on the left hand side I would have A times T minus b k times T, and after taking the T matrix common the rest of the thing is A minus bk and we take this part on the right hand side so, it would gave me TF. Now again taking this T matrix on to the right hand side I would have A minus b k is equal to TFT inverse.

Now, if you pay close attention to this equation, the same equation we have used when we discuss about the simulate transformation. Where T could be any nonsingular matrix and here T happens to be transformation matrix, meaning to say that and we know that under the transformation the eigenvalues do not change. So, whatever the eigenvalues we would specify in the matrix F, it would definitely be of the matrix A minus b k, and this is what we expect the result of this algorithm.

Now, thus the eigenvalues of this matrix A minus b k can be assigned arbitrarily except those of A as well. Now, if A and F have no eigenvalues in common that is the plant and the closed loop system have no eigenvalues in common, then a solution matrix T exist in this Lyapunov equation for any k bar and would be unique.

So, this is the most important point here which we would formulate into a result and we will see the proof of this equation as well. Because the most important thing in all these four steps, that T because here we solve for the unique T. So, when we should know first as a preliminary step that whether that T exist or not, because F is the user defined or is completely based on the objectives, k bar is also use a define based on this condition which can be satisfied.

Now, using this k bar and F we solve for this T; so, we need to ensure that whether that such T exist or not first of all. On the other hand if A and F matrix have common eigenvalues a solution T may or may not exist depending on bk bar. So, in order to remove this uncertainty we required A and F to have no eigenvalues in common. So, this particular point in the first statement we have seen this while if you recall the result on the Lyapunov stability for linear time systems; so, this was one of the time which we have highlighted.

So, what remains to be proved is the non singularity of T so, at the same time if T happens to be a nonsingular meaning to say that it could be unique, ok.

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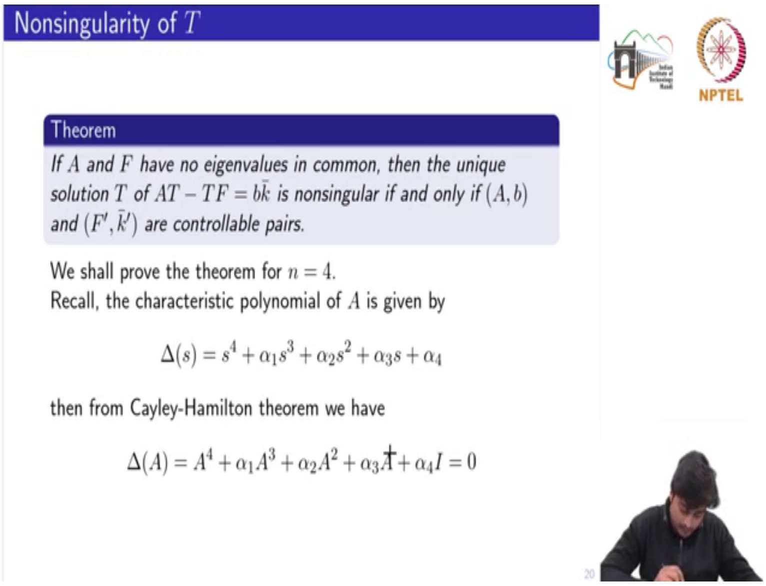

So, this is one of the important result that if A, if these matrices A and F have no eigenvalues in common then the unique solution T of this Lyapunov equation is nonsingular, if and only if the pair A comma b and F transpose and k bar transpose are controllable pairs. So, we need the controllability of the pair A comma b and also of this one which is an additional condition to solve for this Lyapunov equation.

So, we will see a detailed proof of this result so, that by doing the proof we want to recall many of the basic concepts so that it would also help you to carry out your own proof of your own results. So, we shall prove the theorem for n is equal to 4, but it would also be applicable for any n-dimensional system. So, recall that the characteristic polynomial of the plant or of a matrix is given by this polynomial for n is equal to 4, their alpha 1 to alpha 4 coefficients are already known, ok. We also known from Cayley Hamilton theorem that if I replaced s by the A matrix this would also be equal to 0.

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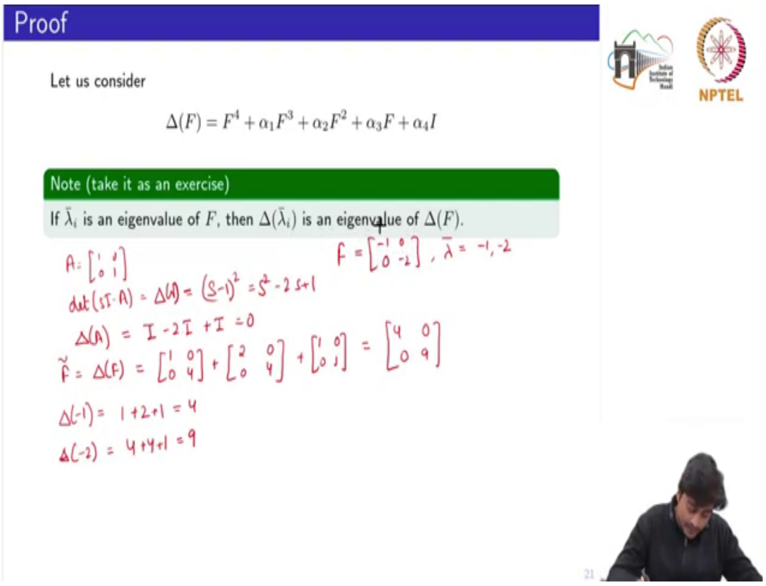

So, let us see the proof. So, here we define a matrix which is delta of F given by this one. Now, this is a matrix we can say let us say F tilde, and we know delta of a would be equal to 0, but delta of F would never be equal to 0, ok. So, we define one matrix by replacing s by F in the characteristic polynomial. So, if lambda i bar is an eigenvalue of this matrix G, then delta of lambda i bar is an eigenvalue of this matrix.

So, this you can take it as an exercise for a generate n dimensional system, but we can see through one example that how this statement is true say for example, we take A as an identity matrix, if A is an identity matrix then we know that system is an unstable system, ok.

The next step is to compute the determinant of S I minus A which is delta of S and it could be given by S minus 1 of square or S square minus 2 S plus 1. Just for the confirmation we can verify that delta of A would definitely be equal to 0 so, if I substitute S by A matrix we would have A square. So, A square again would be identity minus 2 it would be twice into I plus an identity so, it would definitely be 0.

So, now define another matrix let us say we want to place the eigenvalues of the closed loop matrix to let us say minus 1 and minus 2. So, I form another matrix let us call it F tilde which is delta of F and is given by the square of this matrix would be 1 0 0 4 minus so, I write plus and it could become 2 4 plus identity, ok. So, this matrix happens to be F tilde.

Now, it is says if lambda i bar is an eigenvalue of F, so lambda i lambda bar are minus 1 and minus 2 of this matrix. So, if we need to compute and verify that delta of lambda i bar is an eigenvalue of this matrix F tilde. So, delta of, so once we compute for delta of minus 1 delta of minus 1 would be equal 4, and again delta of minus 2, delta of minus 2 would be 9.

So, it is clear now that delta of lambda bar i are 4 and 9 which happens to be the eigenvalue of this F tilde matrix. So, here we are taken one example to verify this statement, but you can try take it as an exercise to show in generic for an n dimensional system that this statement would all true, ok.

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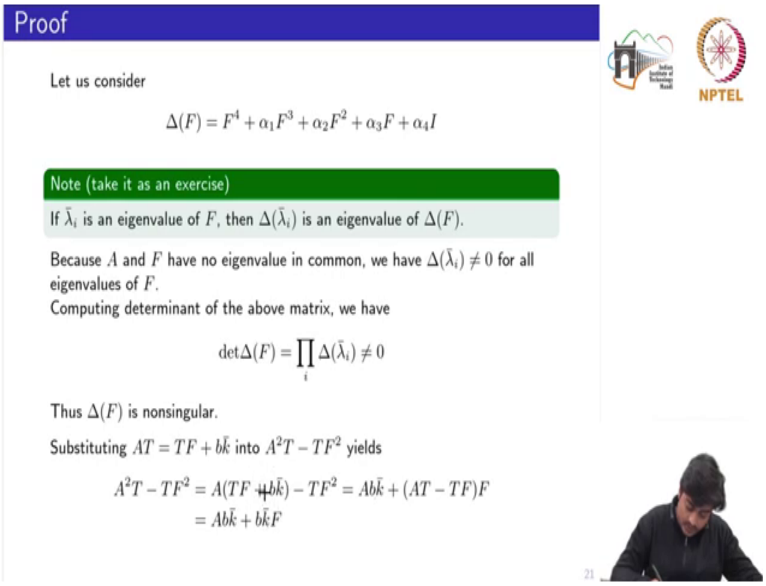

Now, taking this statement further because A and F matrix have no eigenvalue in common we would have delta of lambda i bar is not equal to 0 for all eigenvalues of F. Then we compute the determinant of this matrix we have so we know that determinant of any matrix is the multiplication of all its eigenvalues. The eigenvalues of this delta F is delta i lambda delta of i lambda and the product of this delta of lambda bar i; over all i would also not be equal to 0 because this is not equal to 0. So, this delta F is a nonsingular matrix.

Let us go further then we substitute taking from Lyapunov equation A T is equal to T F plus bk bar into another matrix which we are defining by us A square T minus TF square, ok. So, this matrix we are defining and want to substitute the Lyapunov equation into this equation which yields that A square T minus TF square is equal to.

So, here we would have I can take A matrix common so, the rest matrix would be AT and replace TF plus b k bar minus TF square. And, then clubbing this ATF matrix with this matrix and taking F matrix comma I would have AT minus TF and Abk bar the rest of the part. And this AT minus TF is nothing, but bk bar, so I can write a square T minus TF square as equal to Ab k bar plus bk bar F, ok. The relevance of doing this procedure is would be clear in the next slide.

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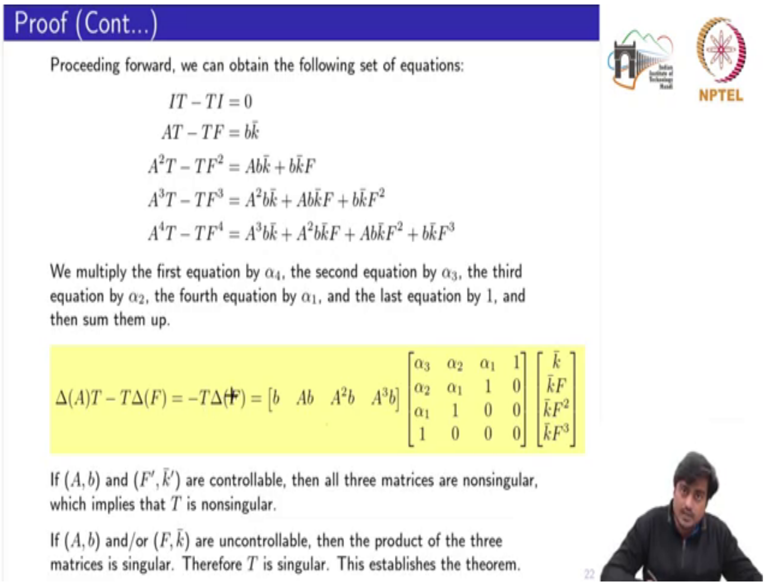

So, preceding forward we can obtain the following set of equation so we start with I into T minus T into I would definitely be equal to 0. Then AT minus TF we know it is already Lyapunov equation, further A square T minus TF square this is what we have computed. Now, going forward for power 3 and power 4 we would you can also simplify by yourself, the right hand side to whether they are coming equal to this or not.

Now, we multiply the first equation by alpha 4; so, we multiply by this alpha 4 multiply this equation by alpha 3 both sides this is by alpha 2, this is by alpha 1 and this remains as it is or by 1 or by I and we will sum them up. So, you would notice that T matrix we can take it common and the rest of the elements are or in the bracket would be alpha 4 plus A times alpha 3 plus A square alpha 2, A cube alpha 1 and A 4, ok.

So, we would obtain this delta of A into T minus T into delta of F would be equal to minus T of into delta of F. Why because from the Cayley Hamilton theorem this part would reduces to 0 because delta of A is nothing, but equal to 0. Now, looking at the right hand side I can rearrange the right hand side by this matrix in times this matrix, times this matrix, it is just a simplified version of taking all these form by multiplying each and every equation by alpha Is, ok.

Now, here this is the most important thing. So, delta F we have already shown that it could be a non-singular matrix this matrix being upper triangular matrix it would always be a nonsingular matrix. This matrix would be nonsingular if and only if A comma B pair is controllable, and this matrix is nonsingular if and only if F transpose and k bar transpose is controllable.

So, in that case all this matrices this, this, this and this matrix are nonsingular matrices so T would definitely be a nonsingular matrix, and this only happens if the matrix A and the matrix F has have no common eigenvalues. Now, if either of the matrix is uncontrollable if this matrix or this matrix or either of the pair is uncontrollable, then your T matrix would be a singular matrix. So, this is establishes the, if and only f condition, ok.

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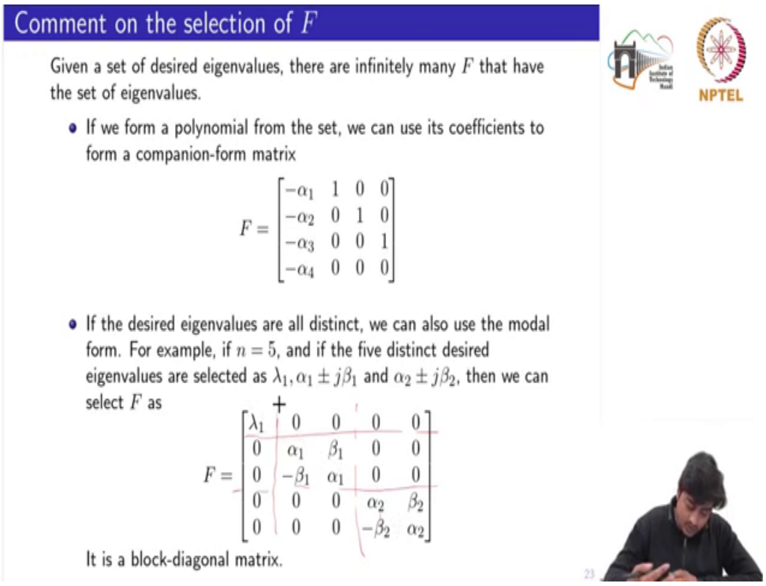

Now, commenting on the selection of F given a set of desired eigenvalues there are infinitely many F that have the set of the eigenvalues. So, we can take two forms, the first form could be if the form of polynomial which we had seen earlier from the set we can use its coefficient to form a companion form matrix.

So, that all the coefficients I can arrange into this way where all these coefficients are of the F matrix, if I write the characteristic polynomial of the F matrix this would be the coefficients. And, arranging all these coefficients into this form would yield me a companion form matrix whose eigenvalues would definitely be the eigenvalues of the required eigenvalues.

Now, this particular F matrix we have also seen in some of the tutorial problems. Let us say if there are some complex eigenvalues and we want to specify a real matrix then we can write the complex eigenvalues as this block. And the complex eigenvalues should definitely occur in it is in its conjugate pair, ok.

So, here it contains 5 eigenvalues and it is a block diagonal matrix; so, one block is this one, then another block is this one, another block is this one, ok. So, two eigenvalues are here, two are here and one is here. So, there are different F matrix which you can form so that it could contain the required set of eigenvalues.