Linear Dynamical Systems
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Week - 05
State Feedback Controller Design

Lecture – 28

State Feedback – II

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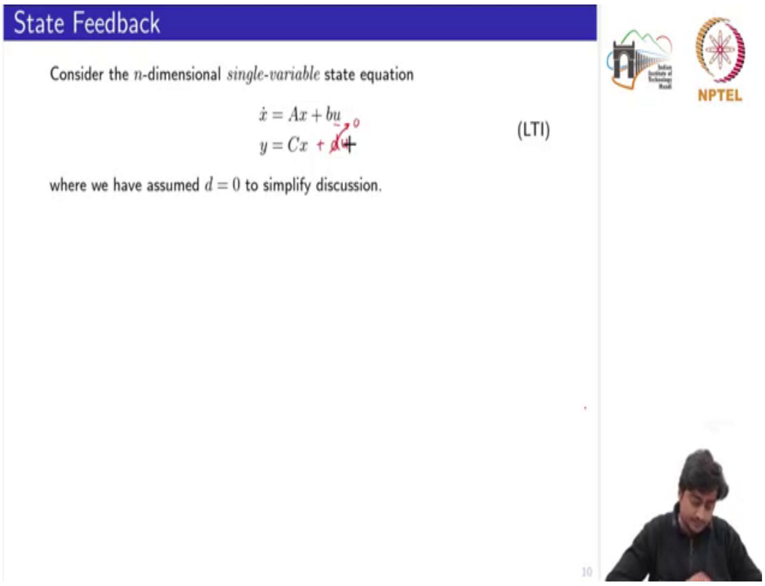

So, in the previous lecture we have seen the importance of the Feedback mechanism. Now, we will see the detailed treatment of the designing a straight feedback controller. So, one important point to note here that whatever the controller design techniques, we would see we will design only for the linear time when varying systems. So, we will not be discussing the controller design for the linear time varying systems because it would require more theoretical tools from the non-linear systems for designing the controller for LTI systems.

So, consider the n dimensional single variable state equation by single variable we actually mean by the input u, where we assumed d is equal to 0 to simplify the discussion. So, by d it is not the so it should be confused with the disturbance variable, but this d is basically the distribution matrix of the u, it is this one du in this d we have taken 0. So, it has nothing to do with the disturbance variable what we have specified in the in the previous lecture.

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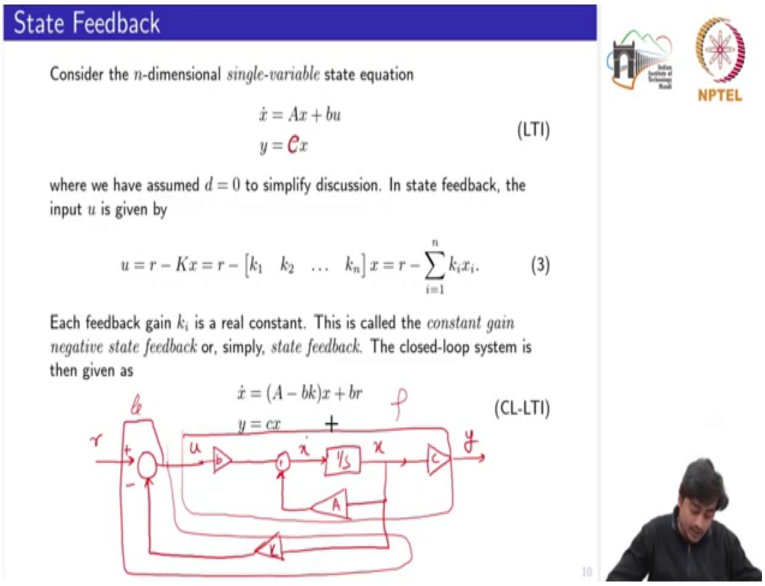

So, in the state feedback the input u is given by u is equal to r minus K times x, where r is the reference signal and k since x is a n dimensional system. So, it could be of one cross n, why because use a scalar variable. This whole part can also be representatant representative by the by this summation i is equal to one to n of k i x i ok.

So, each feedback gain k i is a real constant and this is called the constant gain negative state feedback or simply the state feedback. Now, if we connect this control signal or we a substitute this u into the moral of the plant we would get x dot is equal to A minus bk times x plus br and y is equal to cx. So, we have specified this is closed loop LTI system and this one is only LTI which is denotes the plant ok.

So, there are many important things which needs to be which needs to pay attention here that. For this plant the input is u and the outputs are x and y ok, if we have the excess to the state variable as well.

Now, when we substitute this u into this equation we obtained another state space equation where the A matrix would be is becomes A minus bk and the input to this plant becomes the reference trajectory r ok. So, what we want to implement let us say if we see the state space description that we have the integrative 1 by s this x dot and x dot is basically the summation of we are having and the gain matrix b or the gain vector b and here u. And if I put a gain c we would obtain y ok. So, this you should read as a small c because it is a single input single output system.

So, this is the LTI system we have now when we implement this u we have just k k from x some variable reference trajectory r which yields a single u ok. So, if you say u becomes r minus k times x and this is the LTI system we have and this part is basically our controller ok. Now, if I see the overall closed loop system, but the overall closed system the input becomes this reference trajectory r and the A matrix is changed to a minus bk because of the inclusion of this state feedback.

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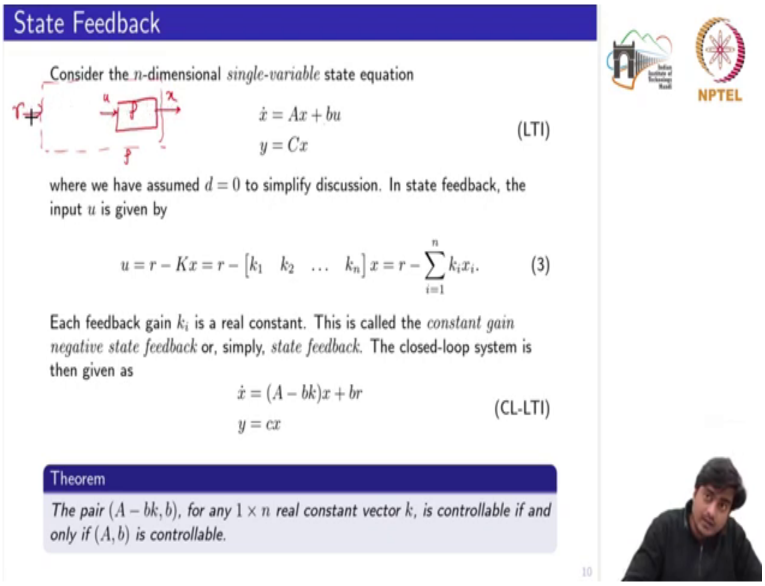

So, in the last week we have discussed many tools many implications of the controllability of the system. So, we have one important result here that the pair this pair A minus bk comma b for any one cross n real constant vector k, is controllable if and only if the pair A, b is controllable. If we mean to say if this pair A coma b is controllable, then this pair new pair where the A matrix has been changed would definitely be controllable and vice versa because this is necessary insufficient condition.

So, if we do not draw the controller in the picture what we have in a let us say this is the plant with u and let us say we are having x. So, we are assuming for the moment that we have x is to x and by using by having the x is to x we can compute y always.

Now, there is some control signal r sorry the reference trajectory r this is the plant the overall plant description, where the LTIS or the model has excess to u and x, but has no connection with the r. Now, if we recall this previous architectural description of the connection between the plant and the controller we have substitute only this part. So, here we have one controller which is taking some feedback from the states.

Now, we can show the proof of this important result only by through this structural description of the block diagram. Say for example; see that if how we have computed the signal u the u is being computed by this formula r minus kx. So, if u cannot steer the trajectory of x from let us say x of t 1 to x of t 1, then r can never be ok. So, this is one of in this way we can see the significance of this theorem, but we can see the formula proof of the basis statement.

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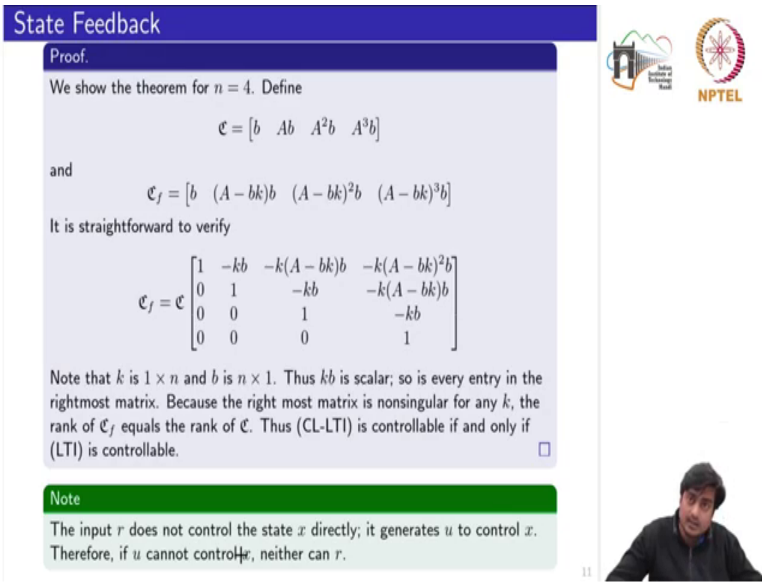

So, we will show this theorem by considering a 4 dimension system where n is equal to 4, we defined the controllability matrix b Ab A square b A cube b. So, since it is a single input single output system we know at the outside that the controllability matrix would definitely be invertible.

And the controllability matrix of the feedback system which we are denoted by this subscript f its b A minus with a new state matrix A minus bk in a similar way ok. So, it is straightforward to verify that that the controllability matrix of the feedback system can be expressed as the multiplication of the controllability matrix of the plant into an upper triangular matrix ok.

You can see some similarity between this matrix and some of the matrices which we have discussed in the tutorial problems part 2 and part 2 two of the controllability with. So, one important results we have discussed there that this matrix Cf would be would have the full rank if and only if this C matrix has full rank or it is nonsingular why because this is an upper triangular matrix and for all k's it would definitely be a nonsingular matrix.

So, note that k is 1 cross n, b is n cross 1 thus kb is a scalar definitely. So, is every entry in the rightmost matrix because the rightmost matrix is nonsingular for any k because being the upper triangular, the rank of Cf always equals the rank of the controllability matrix thus the closed loop LTI is controllable if and only if the system is if the plant itself is controllable ok.

So, this equals the same thing what we have discussed previously that the input r does not control the state x directly ok, but it generates u to control x. Therefore, if u cannot control x neither can r right. So, this is another way of seeing the of visualizing that important result.

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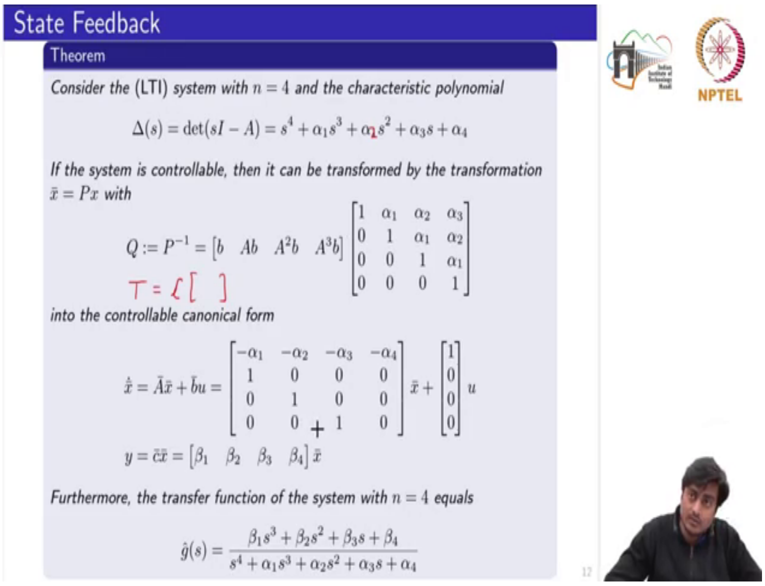

This is another results which we have discussed during one of the tutorial problems of the controllability v. Let consider the LTI system with n is equal to 4 there is a four dimensional system and the characteristics polynomial of that LTI system which defined by this by some coefficients alpha 1 it should be alpha 2, alpha 3 and alpha 4.

If the system is controllable then it can be transformed by the transformation x bar is equal to Px when the matrix Q is compared by this. So, if you recall in one of the tutorial problem we have define a matrix T which happens to be a transformation matrix equal to the controllability matrix times another matrix which is nothing, but this one which is in upper triangular matrix containing some coefficients of this characteristic polynomial. And then we have showed that T is in fact, transformation matrix which not only transformed the system from having a state x bar to x to x bar, but also it gives us the Controllable Canonical Form or the CCF form.

So, it this specific matrix Q or T what we extend the tutorial of part 1 and part 2 into the controllable canonical form into this form where all the coefficients of the characteristic polynomial were constitute the first row with a negative sign and the b matrix would transform into b. So, suppose if the c bar matrix is given by b 1 beta 1 to beta 4, then the transfer function can be express as this one ok.

So, we will not see the proof of this theorem because this is we have already discussed it during the tutorial because first of all we need to show that this q or p matrix is a non singular matrix and it would definitely transformed into this controllable canonical form which we had seen many times earlier.

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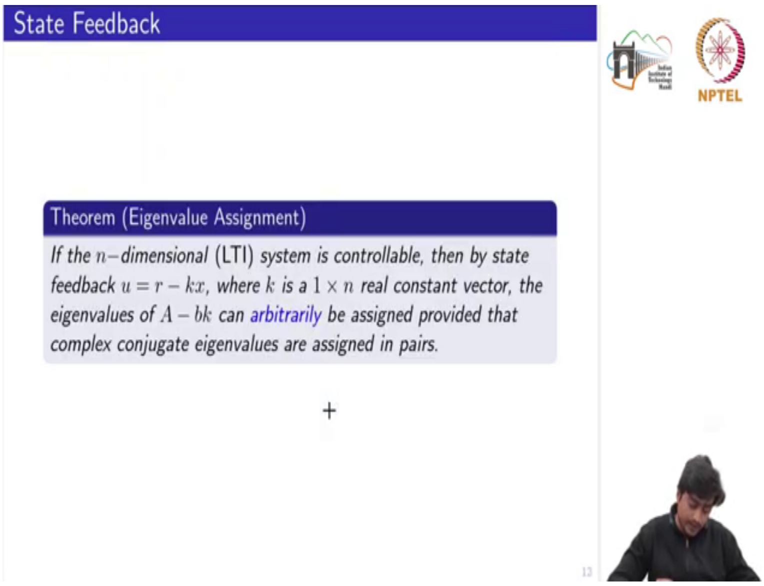

But based on that this is the most important result where we are actually design in a controller. So, it says that if the n dimensional LTI system is controllable, then by using this state feedback u is equal to r minus kx, where k is a 1 cross n real constant vector, the eigenvalues of A minus bk can arbitrarily be assigned provided that complex conjugate eigenvalues are assigned in pairs. So, there are many important things associated with this statement that from the earlier definitions of controllability we know that if the system is controllable, then we can there exist a control law ok.

Now, if we want to assign some eigenvalues of the closed loop in we know at the same time that the eigenvalues defines the transients in the steady state properties of the characteristics of the system. So, if we have those specifications and we have extracted that advert eigenvalues we want to place the eigenvalues of the closed loop system, then we can assign, we can take the plant to any eigenvalues by applying this state feedback or by using this state vector k feedback vectors k.

At the same time we need to show that if we are assigning any eigenvalues which is a complex eigenvalue, then those complex eigenvalues should occur in as a pair of conjugate pair. So, while seeing the proof of this eigenvalue assignment first of all we will. So, there are two things we will demonstrate here that if the pair Ab is controllable then we can assign the eigenvalues of the closed loop to anywhere in the left hand side or through the right hand side it does not matter with the using the feedback in k. Now, the second part is we will design that k also that for what the how the k can be designed.

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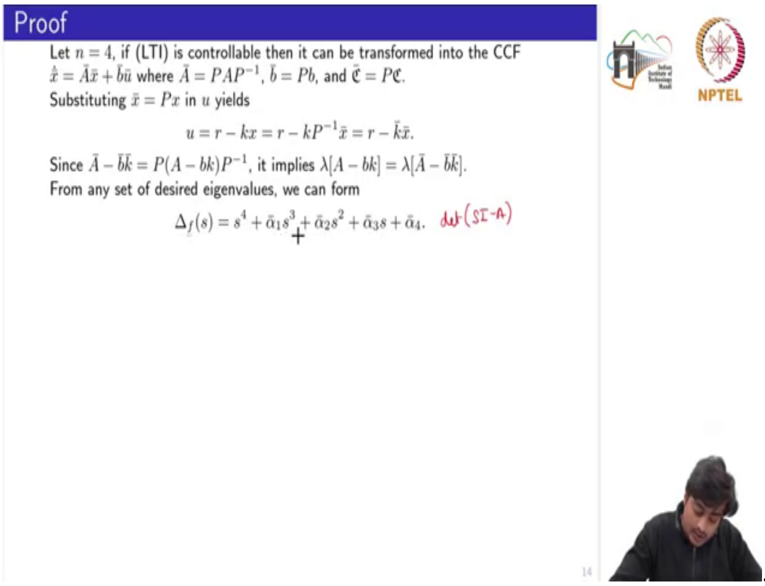

So, once again consider let n is equal to 4 if the plant is controllable, then it can be transformed into the controllable canonical form by using the previous results into having the state with x bar and so this u bar will not be there because the input would remain the same ok. And A bar we have seen many time that A bar can be computed by PAP inverse, where P is the transformation matrix and b bar can be computed as Pb and there is also a relationship between the controllability matrices. So, this with a bar sign represents the controllability matrix of this transform pair and this matrix is of the original LTI Ab pair and they are related by this P matrix as well ok.

So, first of all we substitute x bar is equal to Px in u. So, this was the original control law u is equal to r minus kx and from here we could write x as P inverse of x bar. So, we substitute here as P inverse x bar and we assign kP inverse as k bar. So, the control signal becomes u is equal to r minus k bar x bar, where x bar is the state with the transform system and we have defined k bar as kP inverse where P is also known to us.

Now, note that if we see this closed loop system with this control signal. So, if I apply this u to this transform signal, we would have the state matrix as A bar minus b bar k bar. Now, a bar is nothing but pap inverse b bar is Pb and k bar is kP inverse. So, if I substitute all these this A bar b bar in k bar into this we were obtain P times a P into A minus bk into P inverse which is nothing but is equivalent to this one which implies that the eigenvalues of this matrix would definitely be equal to the eigenvalues of this one because P is nothing but a transformation matrix and both the matrices are related by this transformation under P. It implies to lambda denotes the eigenvalues to the eigenvalues of this matrix would definitely be equal to the eigenvlaues of this matrix.

Now, say suppose we are given a set of desired eigenvalues and using those eigenvalues, we can determine the characteristic polynomial of the feedback system. So, we need to fix some eigenvalues of the closed loop feedback system that is why we have we have we have use in subscript f right.

So, using those eigenvalues we can compute this characteristic p olynomial which is nothing, but the determinant of SI minus A of that matrix hm. In since A is 4 dimensional we would have the order of this polynomial of the degree of this polynomial is 4 with some coefficients alpha 1 bar to alpha 4 bar right. So, we also have this information of this all these coefficients from the given set of desired eigenvalues.

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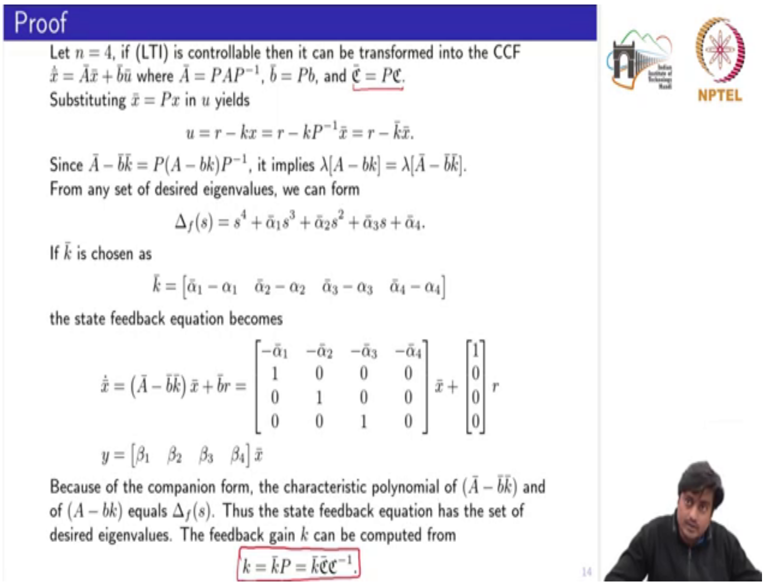

Now, if k bar is chosen as k bar is equal to alpha 1 bar minus alpha 1, alpha 2 bar minus alpha 2 and so on. So, note that here that alpha 1 bar is known to us from the set of given desired eigenvalues, these alpha are also given to us which are the coefficient of the characteristic polynomial of the plant itself. So, from there we can compute one vector and we assign that vector to k bar ok.

Now, we substitute this k bar here our state feedback equation will become this one. We have A bar already in the transformed field in the controllable canonical form, b bar we would at Pb and k bar we are using this one.

So, if I substitute all these vectors in matrices into this equation we would have this A matrix in this B matrix, the c matrix would remain as it is because beta 1 to beta 4 are already of the transform system. So, because of the companion form the characteristic polynomial of this A bar minus b bar k bar and of A minus bk equals this one hm. Under any transformation the eigenvalues does not change, if the eigenvalues do not change then the characteristic polynomial were also not change ok. So, thus the state feedback equation would definitely has the set of desired eigenvalues.

Now, the feedback gain can be computed from this equation where we have substituted kP inverse as k bar. P is known to us because it is a transformation matrix k bar we have defined it by using the information of the coefficients of the plant and of the desired feedback loop. So, from there we have directly computed k is equal to k bar into P and P is nothing you can compute p from here which is nothing, but C inverse into sorry C C bar into C inverse ok. Now, we can take the C inverse here because we know we are dealing with the single input output system. So, this controllability matrix would definitely be a square matrix, so we can take the inverse.

And now since it involves the inverse of the controllability matrix for the computation of k this controllability matrix should definitely be a non singular. So, if the system is not controllable then you cannot compute the k, because this inverse is already included in the definition of the or in the computation of the feedback k. So, this is one of the very important result.

So, since we are stressing on this fact that the controllability matrix is a square matrix and the inverse is possible, but if it is a multi variable system the controllability matrix would no longer be a square matrix. But still it would involve in some way the involvement of the full rank of the controllability matrix, ok.

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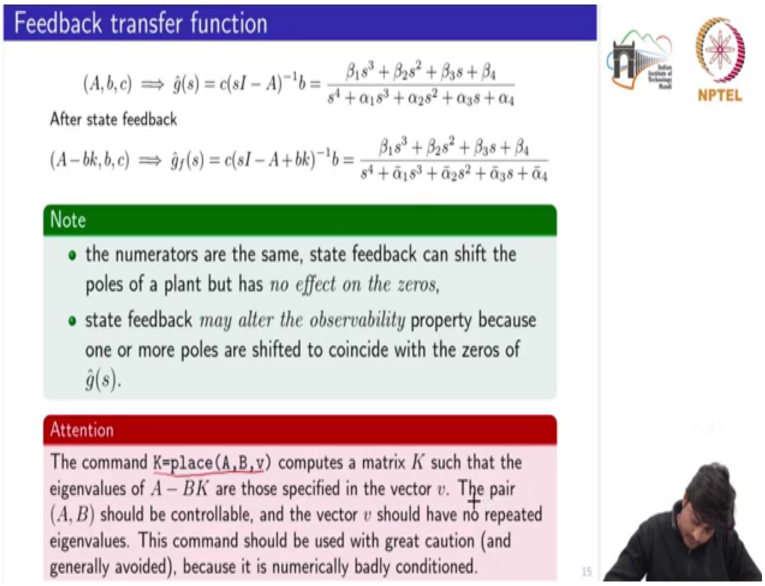

If we pay attention to the feedback transfer function that what happens? So, once we apply state feedback in the state page analysis, then what would happen to the transfer functions? Given this A, b, c pair we can compute the transfer function in g s c sI minus A inverse times b and this would be this one. So, this transfer function is already specified in one of the earlier results, where alpha i's are the coefficients of the characteristic polynomial of this plant.

Now, after the state feedback this pair would change into another a matrix, but the bc would remain as it is. So, this is the transfer function of the closed loop system which we can compute by this one and finally, we would have this transfer function where beta is would remain as it is and the numerator polynomial would is replaced by another polynomial represented in terms of alpha bar is ok.

So, there are many important things to note here. So, first all that the in both the transfer functions of the plant and of the state feedback of the closed loop system the numerators remains the same. If you compute the root of this numerator and of this numerator the roots would remain the same which are basically the 0s of the system.

So, the state feedback can shift the poles of a plant, but has no effect on the zeros. So, we cannot shift this zeros of the plant, but we have full control over shifting the eigenvalues of the or the poles of the eigenvalues the plant given that the system is controllable right.

Now, second the state feedback may alter the observability property, this is the most important part which we should note at this time and once we come on to once we come on to the week where we would discuss the physical significance of the observability. We will especially show that how the state feedback can alter the observability property.

So, now, at the moment you can visualize this thing in the sense say suppose there are some zeros of the closed loop system or of the plant and in the plant there is no overlapping of the zeros and the poles hm. Now, since we can shift the poles to anywhere with the inclusion of the state feedback, it may happen that some of the eigenvalues or some of the poles of the feedback system may overlap with the zeros of the closed loop system.

So, in that case those polynomials would cancel out and whenever that cancelation happen it means that the system has lost its observability ok. But we will see if this in the more detail, but the, at this time we can see that because one or more poles are shifted to coincide with the zeros of the g hat s ok.

Now, if you want to compute the gain K in MATLAB you can use this command K equal place, where A is the A matrix, B is the B matrix and v is the vector containing the eigenvalues ok. So, if the pair should be controllable while using this command place the A B pair should be controllable and the vector v should have no repeated eigenvalues ok. Now, this command should be used with great caution and generally avoid this command because it is numerically bad condition badly conditioned ok.