Linear Dynamical Systems
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Tutorial on Controllability: Part-II
Lecture - 26
Tutorial - 4

So, now we will start with the second part of the tutorial on controllability.

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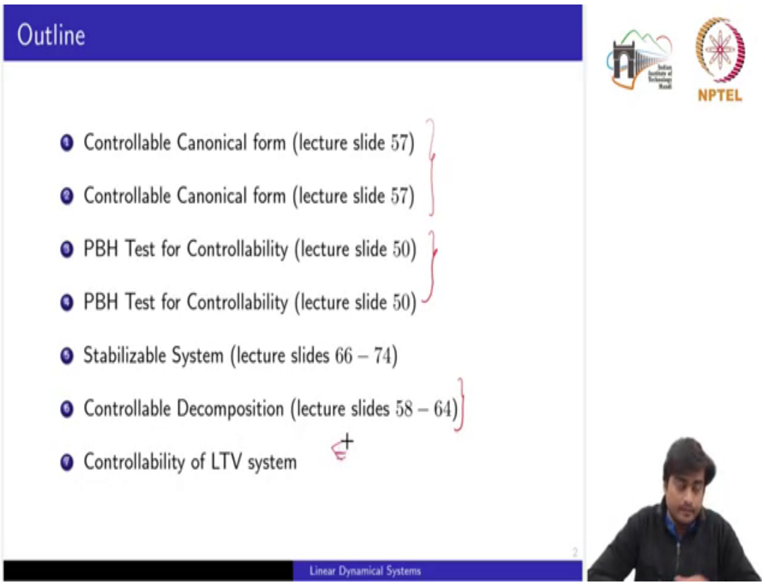

So, these are the topics which we would cover in this part. So, starting from the controllable canonical form, there are two problems this particular topic is the idea here is to compute the similarity transformation matrix, so that, so here we would discuss two ways of computing the similarity transformation matrix which would convert the normal or the given A B matrices into the canonical forms.

The third way of computing the similarity transformation matrix, we would see when we would discuss about the controllable decomposition. There are two problems on the PBH test, two different problems on which is basically the matrix or the rank test. The fifth problem deals with the stabilizable systems that is if my system is not controllable, then whether we can say about the stabilizability of the system. The last problem deals with controllability of the LTV systems.

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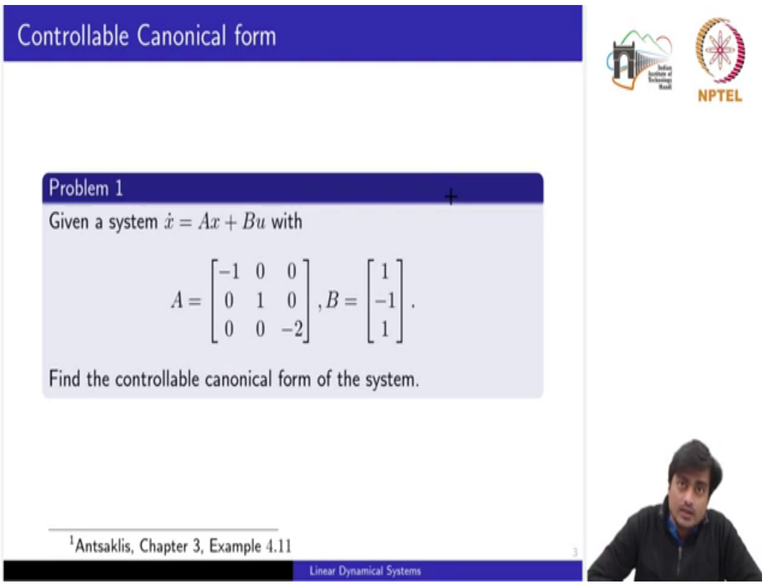

So, the problem 1 speaks about the given this A B matrices which is the linear time invariant system. We need to find the controllable canonical form for this given A B pair. So, this problem was taken from this book Antsaklis from chapter 3, which is an example 4.11. So, if you recall the canonical form of this pair A B, you can directly write the canonical form also by computing the polynomial or the characteristic polynomial. So, once you have obtained the characteristic polynomial, then looking at their coefficients you can directly write the controllable canonical form of this pair A comma B.

Now, there are other ways say for example, if we are able to obtain the similarity transformation matrix and using the concepts we have studied during the theoretical lectures, instead of directly or instead of computing the polynomial, we could directly compute the controllable canonical form.

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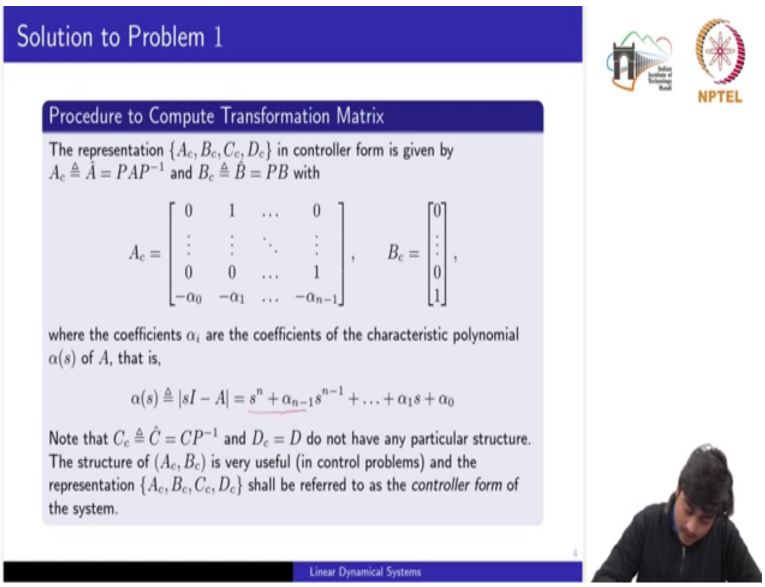

So, first we will see one of the procedures to compute the transformation matrix and then we will compute directly the controllable canonical form. So, given this pair A B, you can compute this controllable canonical form which are written by subscript c, A c and B c.

And if we I have this P matrix then by using this equation P A P inverse and P B we can directly obtain the canonical forms of this pair which would appear as this. So, the last row of this transformed A matrix is basically obtained from the characteristic polynomial. So, either for the given A matrix, you first of all write the characteristic polynomial and then obtain the controllable pair right.

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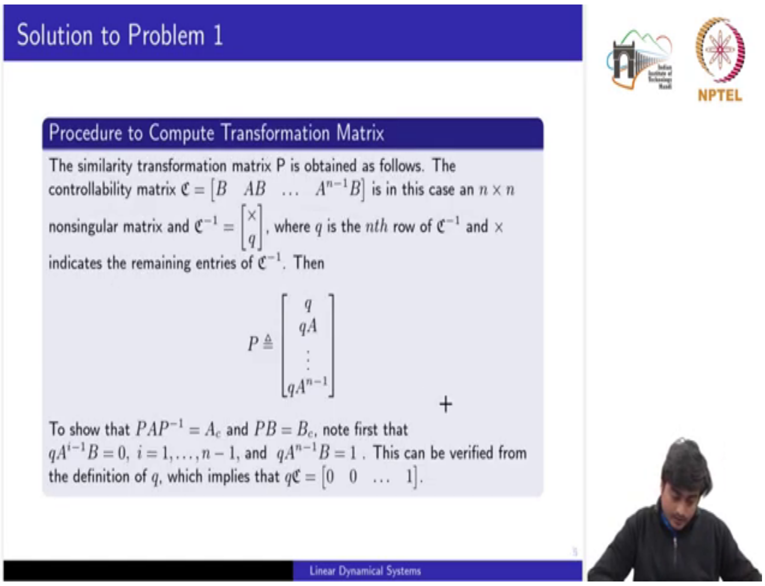

So, the procedure starts here that we can write the controllability matrix which is in the standard form B A B up to A to the power n minus 1 B of an n-dimensional system. So, after computing the controllability matrix, we would compute the inverse of that matrix and we will extract or we will take only the nth row of this of this inverse matrix. The upper elements we are not concerned with those elements.

Once we have extracted the last row of this inverse matrix, then we form A P matrix which is defined by this q, q A, up to q into A to the power n minus 1. So, this P would results into this A c and B c, which could be in the standard form we achieved in the last slide.

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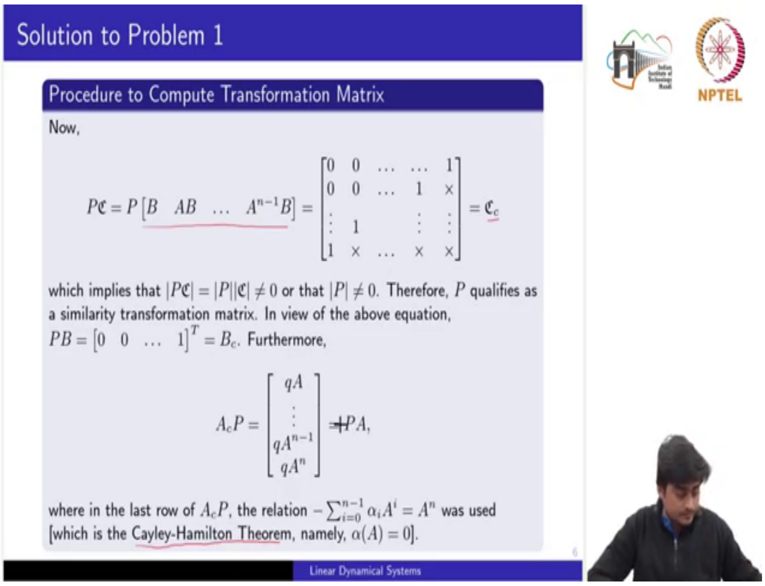

Now, if we multiply this P matrix which we have obtained in the controllability matrix you would see this P and this controllability matrix which could be arranged into this form. And finally, you would can say that this is the controllability matrix of the canonical form. And we have also seen during one of the lectures that the controllability matrices are also related by the similarity transformation matrix which is given by here.

Now, you would notice that whatever the P matrix we have obtained, it is also a non-singular matrix, because the determinant of P into C, we can write as individual determinants of P and the controllability matrix which is not at all equal to 0, because this is the controllability matrix of the canonical form. So, it cannot be equal to 0. If this determinant is not equal to 0, meaning it implies that the determinant of the P matrix itself is not equal to 0. So, P definitely qualifies to be a similarity transformation matrix which is non-singular and can be obtained from the controllability matrix.

So, if we pay attention to the multiplication of the P and B matrix, we would obtain this which is nothing but the B c. And the multiplication of A c into P would result in P a. So, here if you want to go a bit into the detail of the proof of this equation, you need to use some concept resulting from the Cayley-Hamilton theorem, and the application of that would result that the A c is nothing but P A P inverse.

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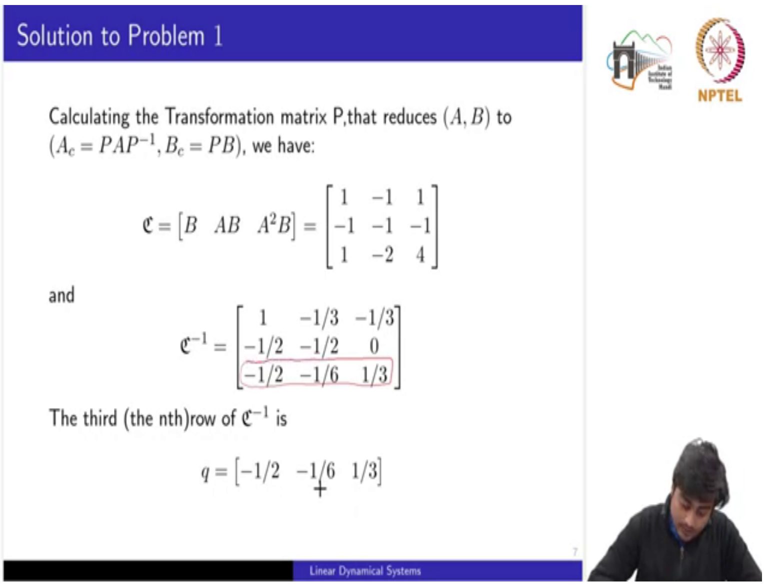

So, coming back to the numerical solution of the problem, we have this A B matrix, from there we can compute the controllability matrix and the inverse of the that matrix is given by this one. Now, since it is a three-dimensional system, we extract the third row which is this one, and assign it is the q vector.

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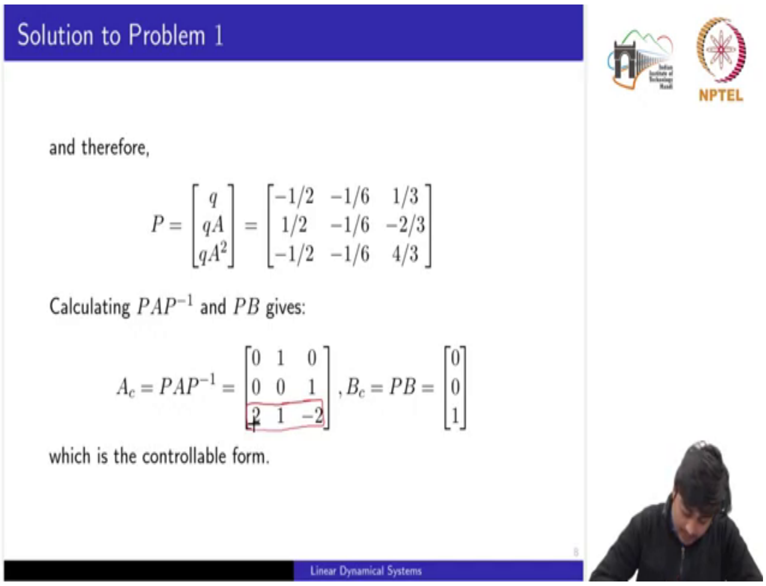

Now, once you have obtained the q vector and recalling the definition of the P matrix, it would be q, q into A, and q into A square. So, this would result into this P matrix and directly applying the equations for computing A c and B c, we will obtain the controllable pair. You can also verify that this last row of this A c are basically the coefficients of the characteristic polynomial of the original A B pair.

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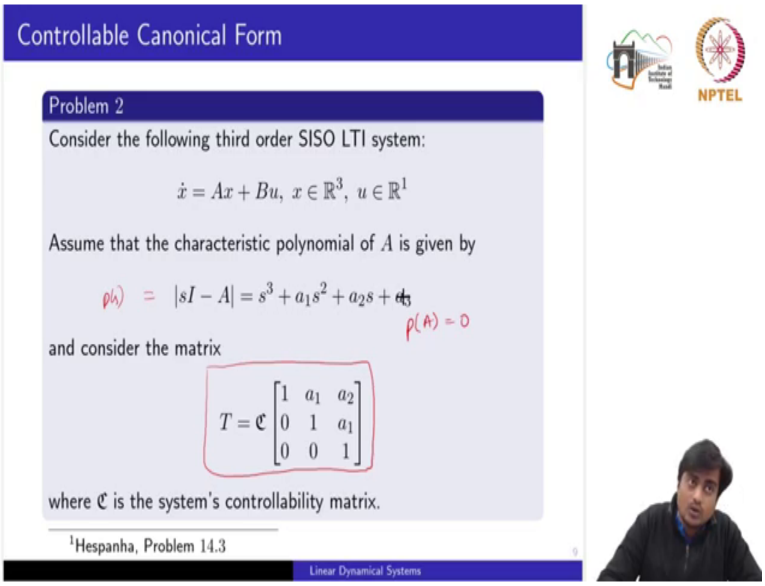

In problem 2, we are given with again A B pair where x is a three-dimensional and u is scalar. So, assume that the characteristic polynomial of A is given by this one. And we define another matrix which is T is equal to the controllability matrix, and another matrix which is composed of the elements or some of the elements of the characteristic polynomial right. So, this matrix is important here.

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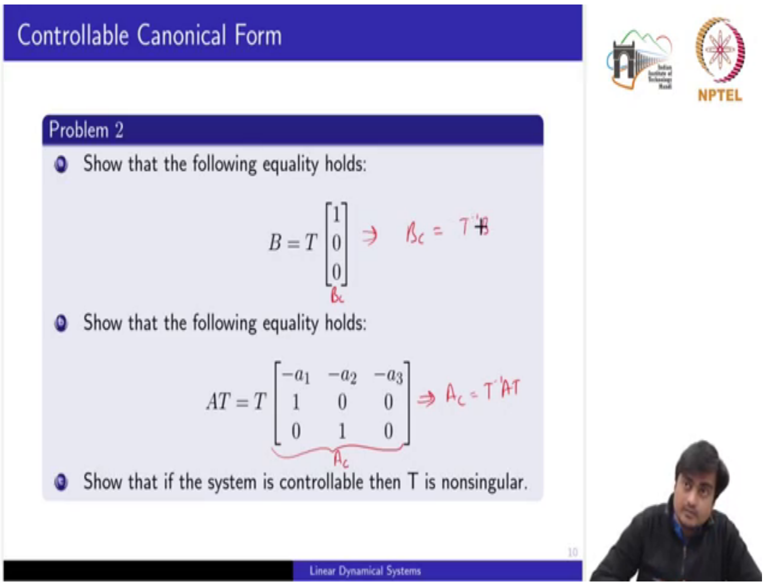

Now, the first part of the problem where we need to show that B is equal to T into this vector the second part that A into T is equal to T into this matrix. So, this matrix if you pay close attention that this matrix is nothing but the controllable form of the A matrix which is which has been obtained after some rearrangement of the rows. The C part is that if the system is controllable then the matrix T what we have specified in the problem is non-singular.

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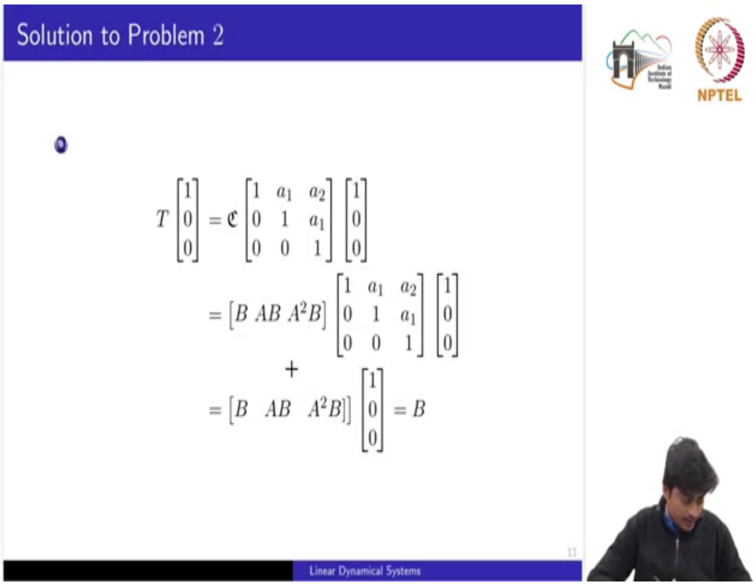

So, one implication of this problem is that this T matrix which we need to show that it is first of all non-singular if the other system is controllable. Now, if we see this equation, I can write let us call this matrix A c ok. So, I can write A c is T inverse A T which is nothing but the same equation what we had used to compute the controllable form. So, the first we are computing the transformation matrix is we had seen in the problem 1 by computing the q vector, and then determining the transformation matrix.

Another method is that you again using the controllability matrix and using some coefficients of the characteristic polynomial, we can define a T matrix which happens to give us the controllable A matrix, and at the same time it would be non-singular if the system is controllable. Also before even solving this, you would see that from here if this happens to be a controllable A matrix, so this could be your controllable B matrix which I can write B c is T inverse B ok, so which would be a straightforward to verify that T into this vector would definitely be equal to B.

So, we need to substitute this T matrix which was defined in the problem this controllability matrix, this matrix which was pre defined. And after doing some simplification, you would obtain which is equal to in fact which is also one of the implication of the problem we had seen earlier.

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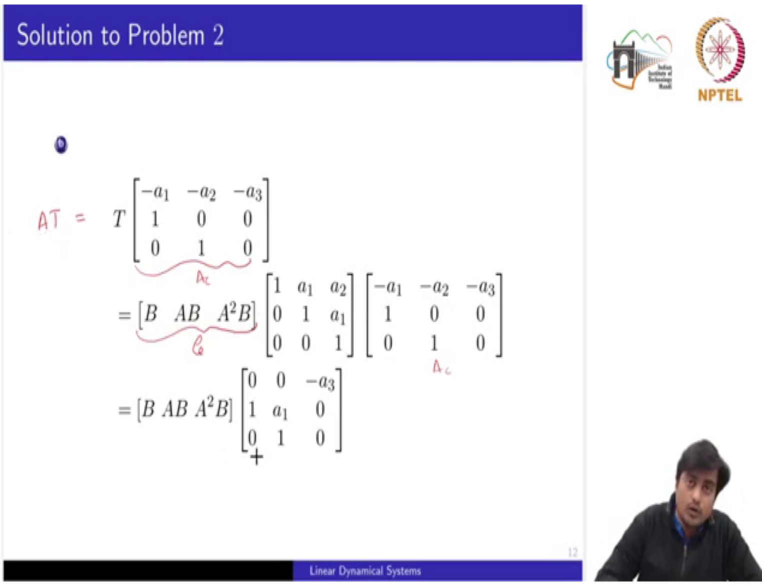

The part B requires to prove this if we call this matrix A c again that A T should be equal to T into A c. So, first we take the right hand side of that equation, and again writing the t matrix which is then this one is the controllability matrix, this matrix was defined for defining the T and this is the matrix A c. So, again if you multiply these two matrices, we would obtain this by keeping the same controllability matrix. Now, we will go onto the left hand side, and see if we are able to simplify the left hand side up to this point.

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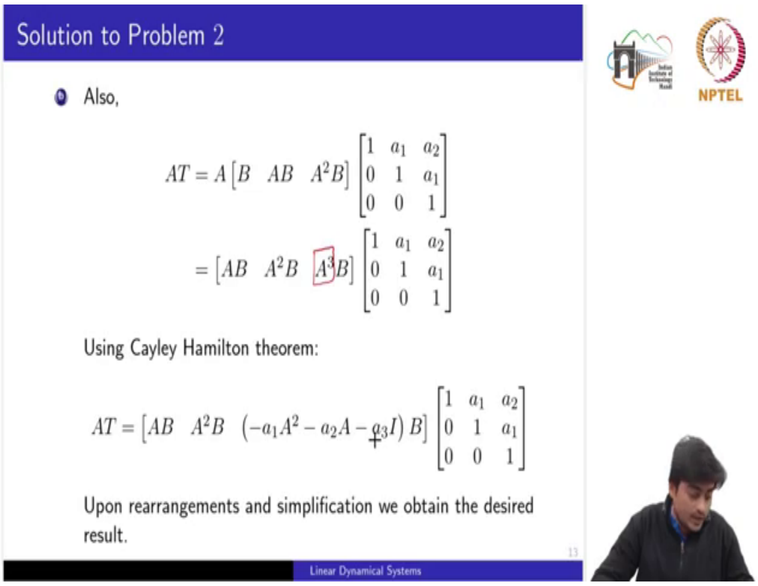

So, A into T we can write by writing T explicitly into this equation. And this A matrix would be would multiply with each and every element of the controllability matrix. Now, note here that this A q, I again using this Cayley-Hamilton theorem, I can represent this a q B A or I you could substitute A q by the lower order forms of the A matrix which you can obtain directly from this part let us call this polynomial P of s.

Now, the Cayley-Hamilton theorem says that P of A would be equal to 0. Now, P of A is equal to 0. So, the higher highest order of the A matrix, I can substitute in terms of the lower order of the A matrix. So, this is what has been written here. So, at if we simply this part, you would I think you should try by yourself as well that you would see that it is will come equal to up to this point where we had simplified the right hand side ok.

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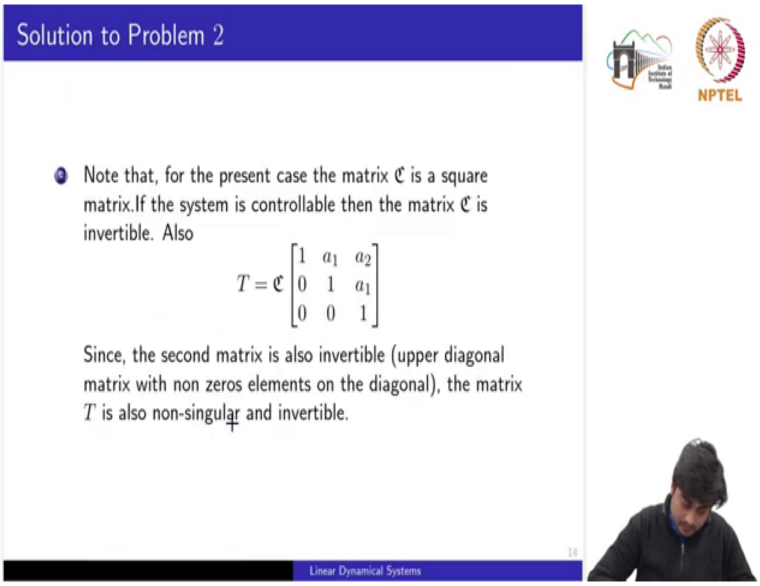

About the non-singularity of the T matrix, so one of the concepts we had seen in the part one tutorial and also during the lecture slides that the pre-multiplication and the post-multiplication by a non-singular matrix does not change the rank of the matrix. So, since T has been defined as this. So, if I want to compute the rank of this T matrix, so this c if it is full-rank or if the system is controllable and this matrix beings the upper triangular matrix would be non-singular as well.

So, definitely our T matrix would be a non-singular matrix if the system is controllable and if the system is not controllable then we cannot ensure the T is non-singular matrix ok. And particularly for this problem since we have defined u as a scalar, so for computing this the c matrix, the c matrix would be a square matrix because in because our B would be a vector.

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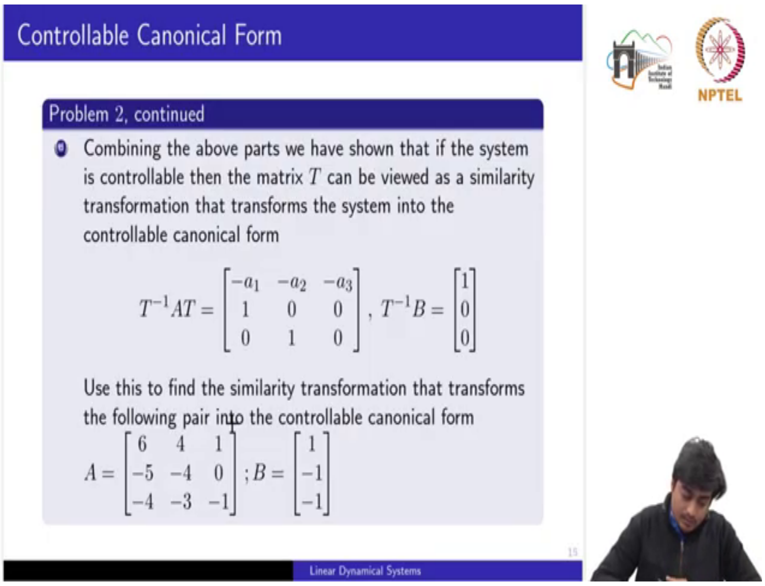

Now, there is an additional part to the same problem which was also one of the implications of the problem that if the system is controllable, then the matrix T can be viewed as a similarity transformation, the transformed system into the controllable canonical form. So, this T matrix what we have defined in this problem also transforms the normal A matrix into the canonical form. So, if we aim towards to compute the numerical values, let us take this a matrix as defined by this and the B vector as this.

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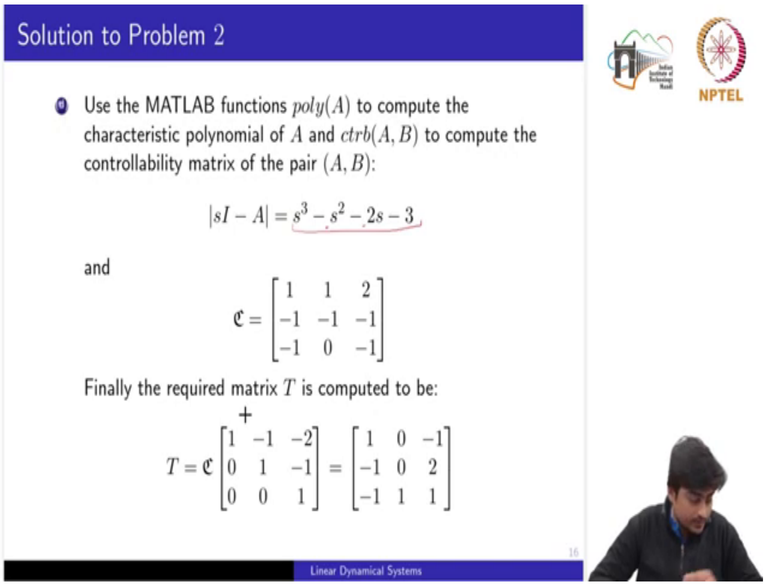

So, you can also use some MATLAB functions to compute this polynomial in the controllability matrix which we have also used here. So, if you compute the poly of A, this would result into this polynomial. Now, using the elements this element and this element, we would form the matrix the post matrix which needs to be multiplied by the C matrix to finally compute the T matrix which is similarity transformation matrix ok. So, this happens to this one right.

So, there are two ways we had discussed so far. One, so the in the first problem this thing should be noted that we have computed the similarity transformation matrix by taking the inverse of the controllability matrix. Now, if u is not scalar function, then your C matrix would not be a square matrix. So, you cannot use the first method to compute the similarity transformation matrix. But you can possibly use this method to compute the transformation matrix because here we no inverse of the controllability matrix is involved ok.

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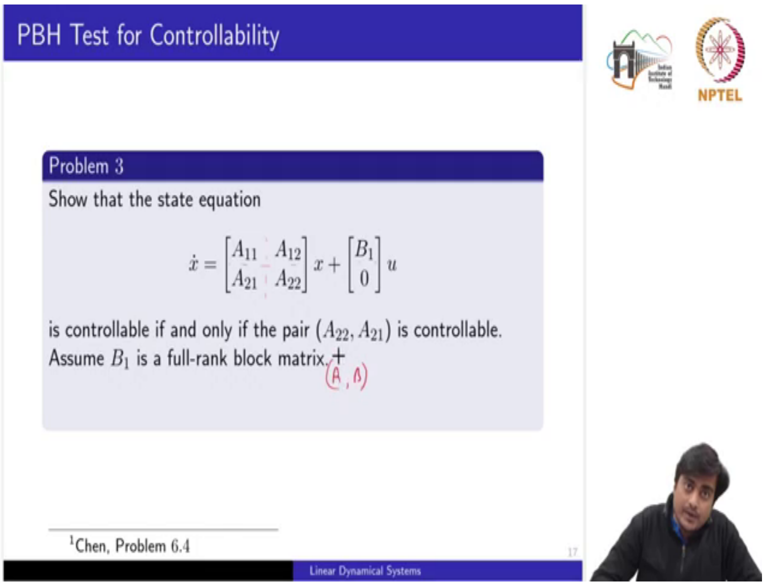

The problem 3 is on the PBH test for controllability where we need to show that the state equation which has been already partitioned into the block matrices is controllable if and only if the pair A 2 2 and A 2 1 is controllable. Assuming that B 1 is a full-rank block matrix. So, this problem was taken from one of the book Chen, it is an unsolved problem number 6.4.

So, one point we need to recall here that whenever we had discussed about the controllability, we have always discussed the if you want to visualize the controllability our A matrix is the straight matrix and B matrix is the controlled distribution matrix. But if we want to compute just the rank of the controllability matrix, they are just an pair AB pair.

Now, both these block matrices A 2 2, A 2 1 are not or any of the element of this pair does not belong to the controlled distribution matrix. So, here we are concerned only with the controllability of this pair meaning to say that the same formula we can use by replacing by using A as A 2 2, and B as A 2 1 ok. So, we need to show that whatever the controllability matrix, we would form using these two pairs or using this pair, it should be a full-rank.

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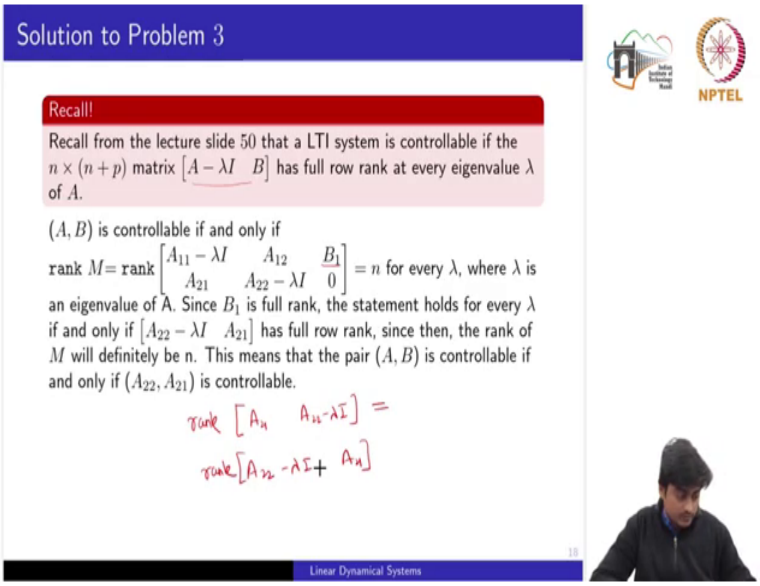

So, recall the from the lecture slide number 15, there we have given the PBH test the necessary and sufficient condition for the controllability of the pair A comma B that the rank of this matrix which has formed by A minus lambda I and B where lambdas are the eigen values of this a matrix has full row rank at every eigen value ok. Now, if we substitute our partition A matrix and the partition B matrix which is given in the problem let us call that matrix m, so that matrix we would obtain as this one.

Now, here one important thing to notice that B 1 is already assumed to be a full-rank matrix. Now, if B one is a full-rank the rank of this matrix would be a full-rank which definitely would be less than n here right. Now, for this pair let us write this pair explicitly. So, it could be A 21 and A 22 minus lambda I ok. The second row of this block matrix i can write this matrix as A 22 minus lambda I, and A 21 right. The rank of this matrix would definitely be equal to the rank of this matrix. um.

Now, if you see the similarity between the rank condition of this one and the rank condition of this one, it says that if there this A 22 and A 21 pair is controllable, in this sense in the sense of PBH test, then we could say the rank of this overall matrix would definitely be equal to n right. So, this is the becomes the necessary and sufficient condition for the controllability of the given A B matrices if and only if this A 22 comma A 21 is controllable ok.

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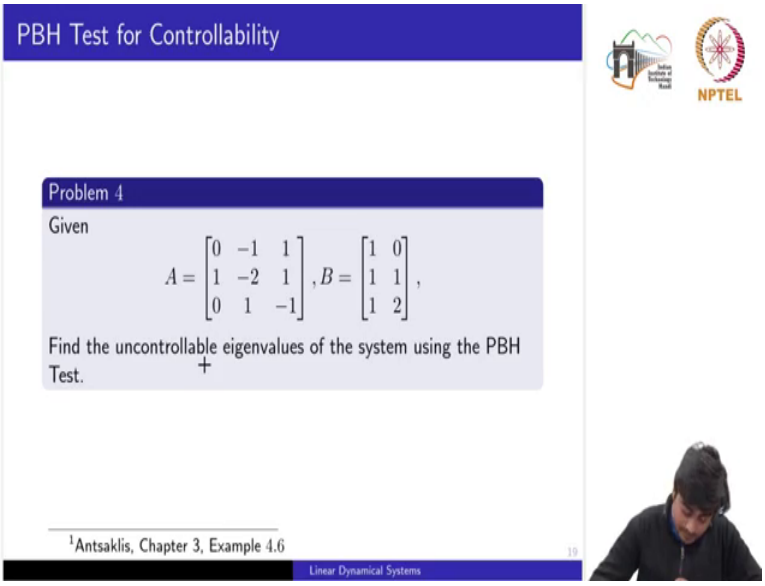

Ah Now, in this problem, we are given the numerical values of the A B matrices and we need to compute the uncontrollable eigen values of the system using the PBH test. So, we have studied that if the system is controllable, the rank of the controllability matrix would be a full-rank in would be full-rank, in this case it would be equal to 3 yeah.

Now, if the rank is not equal to 3, we can find that for what eigen values, the rank is or the matrix is or the pair is controllable deficient let us say. So, we want to compute those eigen values for which the system is uncontrollable in the sense that if we follow the PBH test, and since the PBH test in includes the or includes the eigen values, so we could compute explicitly that for what eigen values our system is controllable and what are the uncontrollable eigen values.

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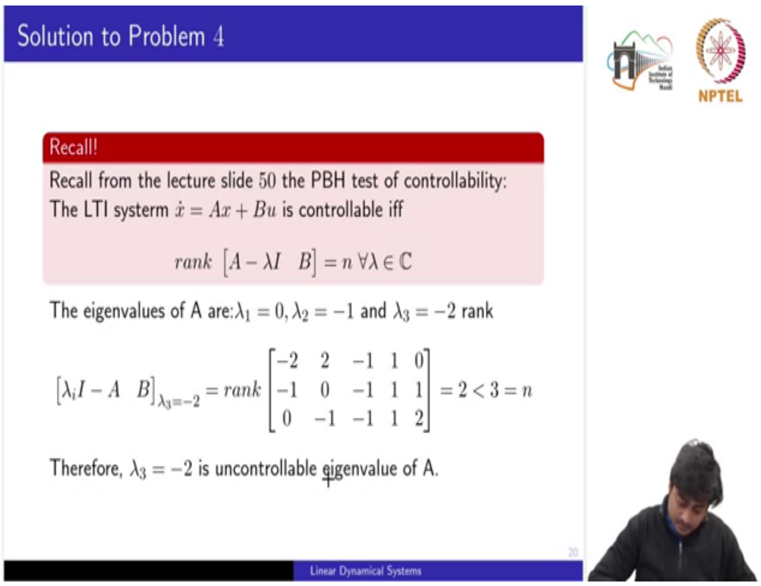

So, proceeding in the similar way, so first of all we would compute all the eigen values of the matrices which are given at 0, minus 1 and minus 2, and for all the eigen values we start, we will start computing the rank of this matrix. And for the eigen values for which the rank of this matrix becomes less than 3, it automatically becomes the uncontrollable eigen values ok.

Now, another implication of this, you can see that since lambda is equal to minus 2 is an uncontrollable eigen value. Now, if we recall our stabilizability concept that this eigen value is a stable eigen value though the system is not controllable or let us say not completely controllable, but it is stills at least stabilizable ok. Because the stability, its stabilizability test speaks only for all the eigen values which are on the right hand side ok.

Now, here neither or for the eigen values we require for which it comes uncontrollable it is on the left hand side ok. So, without even testing for the doing for the test, we can say that this system is at least stabilizable.

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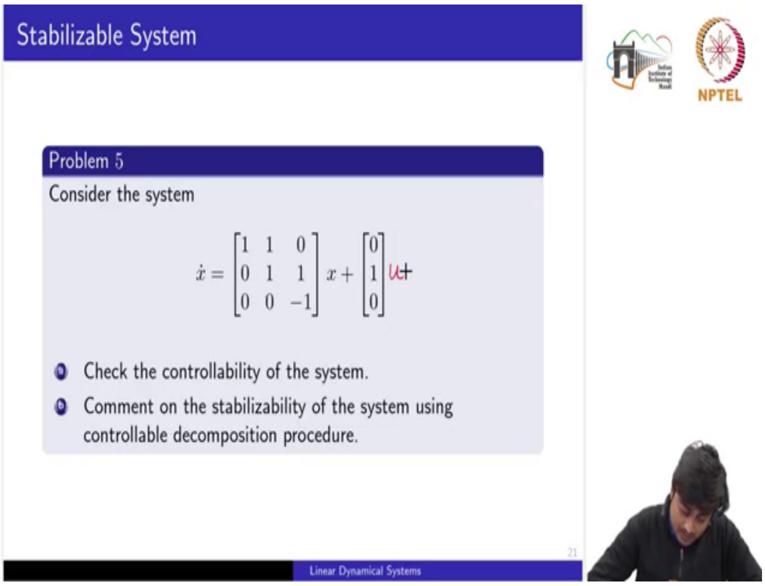

Problem 5 here we need to check the controllability of the system. And we need to comment on the stabilizability using controllability decomposition procedure. So, so far we have been doing different ways of compute or determining whether the system is controllability or not. So, here so the or forgot to write this u. So, using this A B pair, either you can cont compute the controllability matrix or you can compute the eigen values, and then you could see for what eigen values the system becomes uncontrollable.

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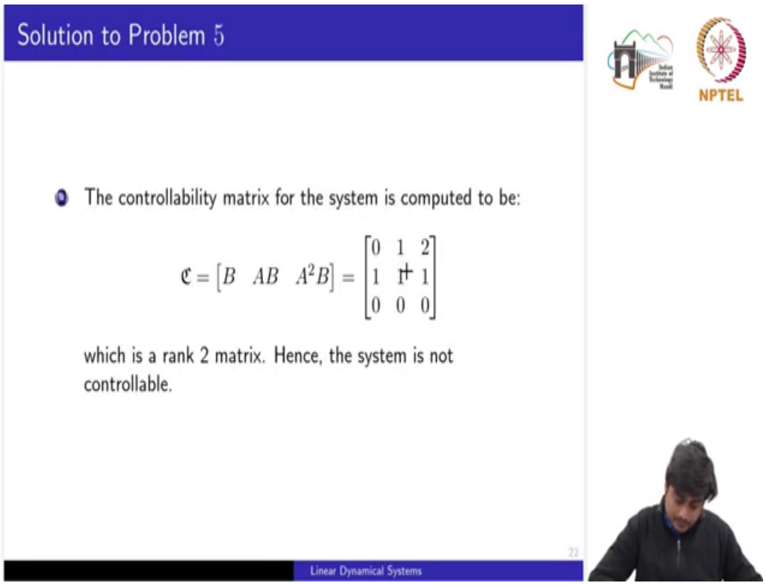

So, we will go directly with the controllability matrix and you would see that the rank is 2 and is not equal to 3. So, the system is not completely controllable.

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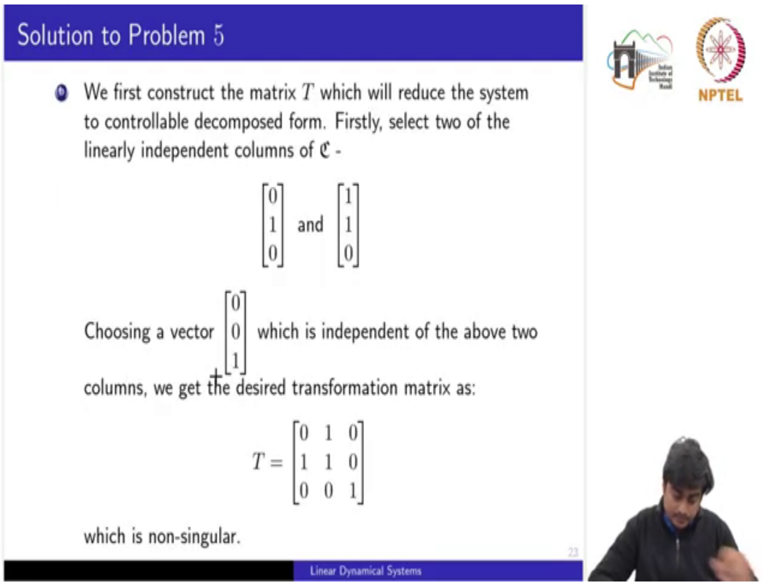

So, in this problem, you would see another way of computing the similarity transformation matrix and which would convert not into canonical form, but it will give us a decomposition. And there are different ways of doing the decomposition, in fact, when your are doing the or computing the canonical forms it is also one sort of the composition. So, here since the rank is 2, it means that there are at least or there are at most two independent linear linearly independent vectors.

So, see looking at the controllability matrix, we can pick these two vectors, the first vector and the second vector as the linearly independent vectors. And we can select another vector which is again so that all these three vectors become linearly independent. So, that third vector is 0 0 1, and then writing all these three vectors would give us a T matrix which is non-singular, because all the vectors are linearly independent ok.

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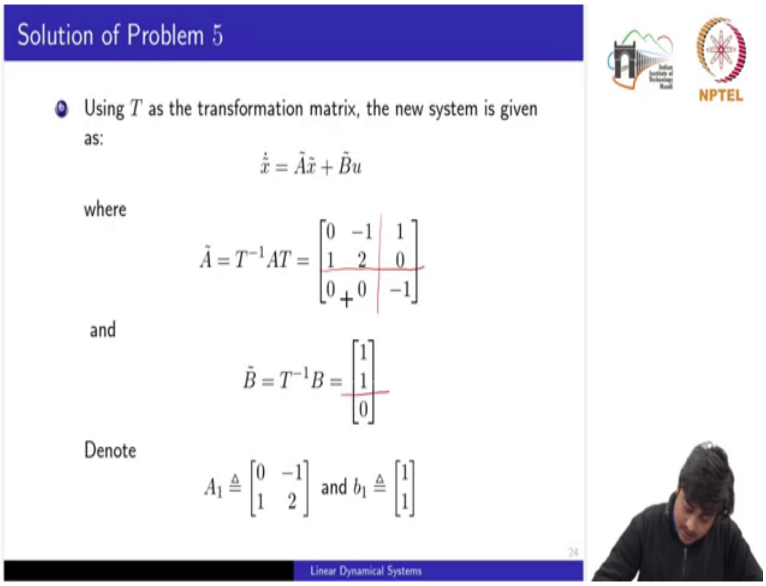

Now, using this T matrix and using the same formulas for the transformation, we can compute these two matrices A tilde and B tilde, there you would notice that this A tilde and B tilde are not exactly in the canonical forms ok. So, here from here we can extract the controllable and the uncontrollable part. So, this top row which is 0 minus 1 1 2, and another the block or let us say the vector 1 1. So, this pair A 1 B 1 is controllable pair, and the uncontrollable pair is this one.

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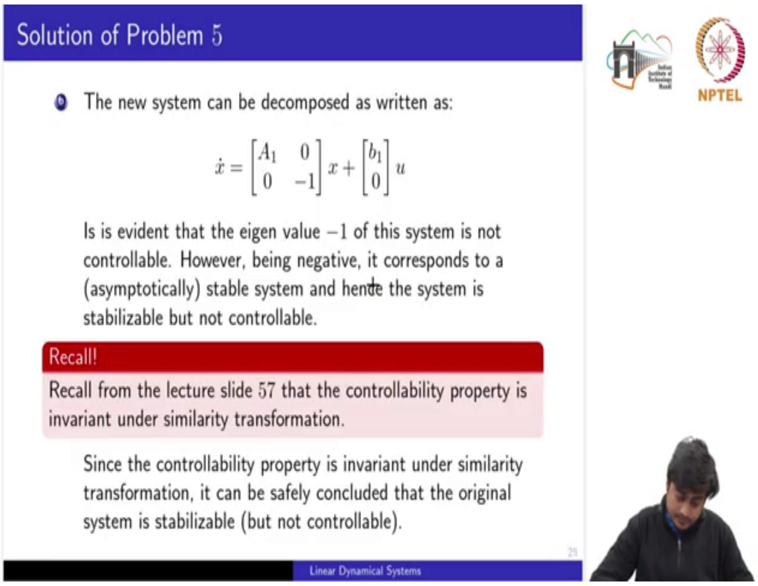

So, we can express the overall system into this decomposed form there A 1 A 1 b 1 becomes a controllable part of the system, and since the eigen value of the uncontrollable system is at minus 1. So, the system is at least stabilizable ok. For again whatever the concepts we had studied in the I am solving the problem number 4, you can use those concepts that is first of all compute all the eigen values, and then see for what eigen values and the system here becomes uncontrollable. Now, if that eigen values is on the left hand side then the system is stabilizable; and if that eigen value is on the right hand side, then the system is not stabilizable.

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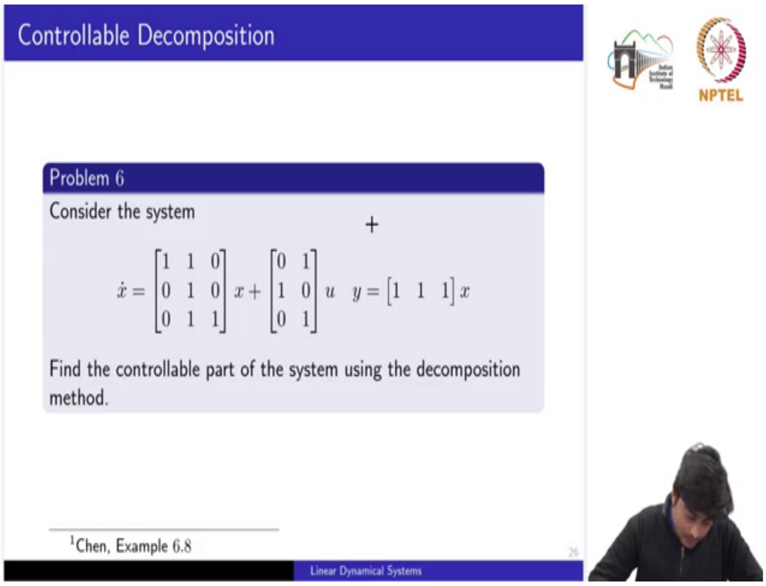

So, control, so this problem is also on the controllable decomposition the numerical computation of the controllable part using the decomposition method, where B matrix is now two cross 3 cross 2 matrix. So, meaning to say we have two control inputs and one output.

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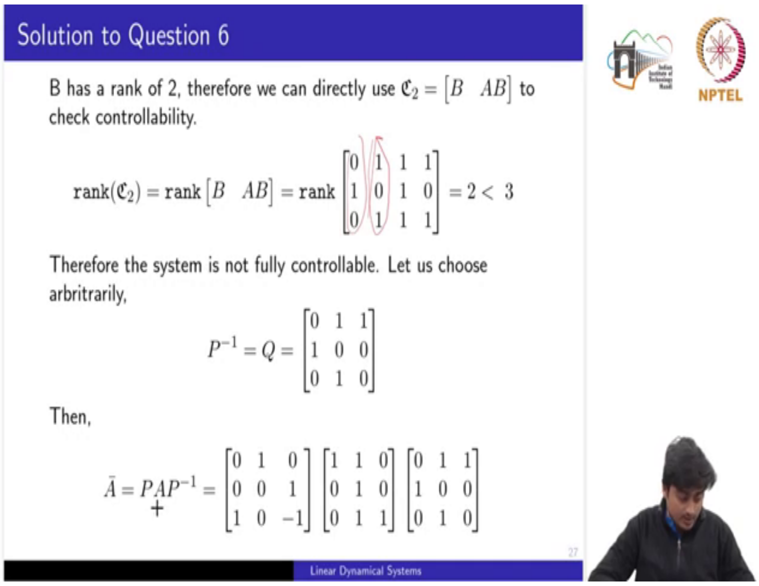

So, checking the rank of the controllability matrix, we see that again it is rank deficient, so meaning to say that system is not fully controllable. So, here we use the same method again that the we can find at most two linearly independent vectors which we can extract from the this c matrix, and they are happen to be the first two columns of the transformation matrix. And again the third vector has been chosen such that the Q matrix becomes a non-singular matrix.

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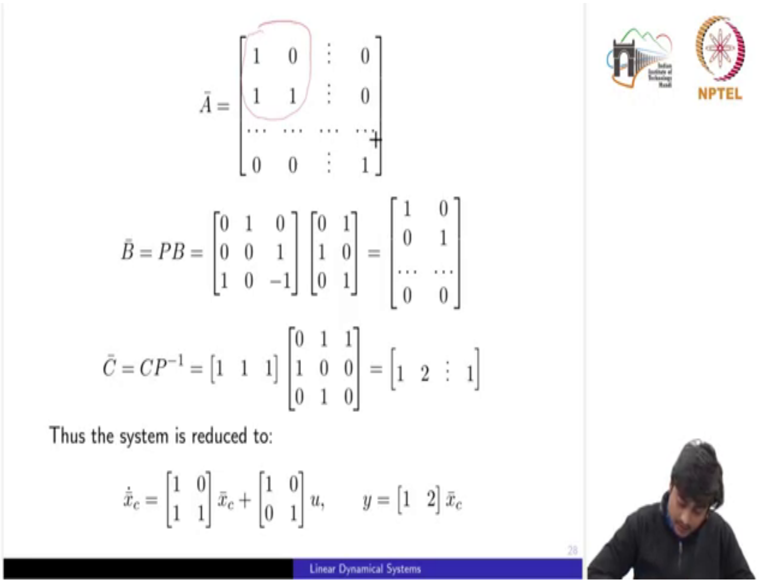

So, again computing the that canonical forms or by the transformations by using this P or Q matrix, we could obtain this A bar matrix B bar matrix, and from here this part would becomes a straight matrix of the controllable part of the system ok. And the rest of the part would become goes for the uncontrollable system.

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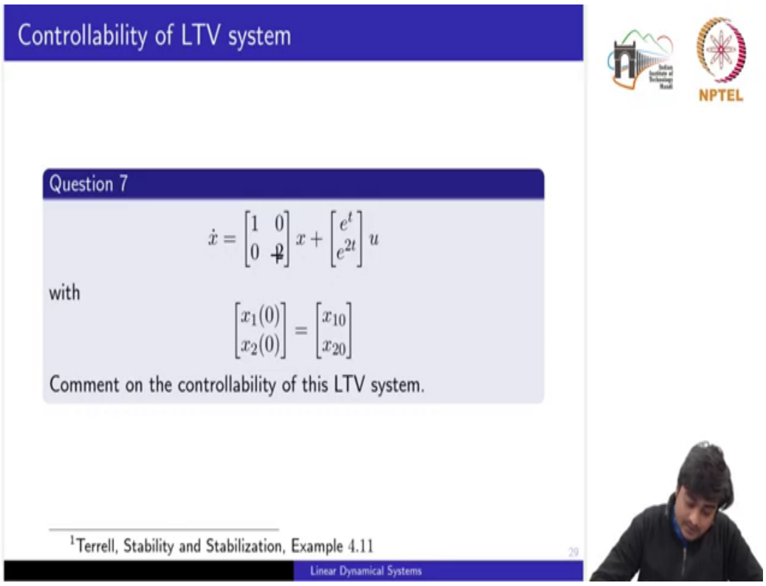

Now, this is the last problem where we discussed about the controllability of the LTV system. Here we have taken A as a constant matrix, but B is a time varying matrix. And this problem is taken from another book which is not mentioned in the references, but this is you can also take a look at it, the book by William Terell by Stability and Stabilization, and we have taken this example 4.11

So, we need to comment on the controllability of this LTV system. So, this problem is quite interesting in the sense there for the LTV system, we can compute if we want to concretely answer the controllability of an LTV system, we need to compute the Gramian the controllability Gramian matrix. So, this was one way.

Now, we had if you recall that for computing the Gramian matrix, we need to compute the state transition matrix and a state transition matrix for an LTV system is pretty much complex. Now, fortunately here the a matrix is a constant matrix. So, the state transition matrix would definitely be an exponential matrix. So, you can compute it here.

But if the A matrix is also a time dependent matrix of not that specific form which we had discussed during the lecture slides, then could be difficult to compute the state transition matrix. So, to circumvent that issue, we compute another matrix which was the M matrix.

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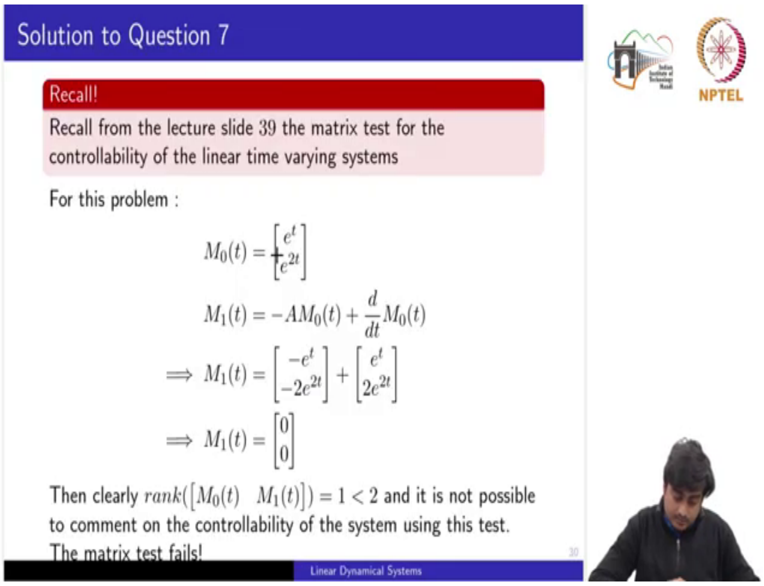

If you go to the lecture slide number 39, so there we had given basically the sufficient conditions only for the test of controllability, because once you have computed the controllability Gramian, then it is another neither seen in sufficient condition if the rank of this controllability Gramian is of full-rank for all time T naught and T.

Now, one the way we have compute this M matrix is detailed into the lecture slides now if the rank of this matrix M naught and M 1 is happens to be to 2, then it would definitely be controllable. Now, the rank of this matrix is less than 2 which is 1, and being the only the sufficient condition, we cannot guarantee that whether our system is completely controllable or not ok.

So, you can also compute by yourself the controllability Gramian, and you would notice that the rank of the controllability Gramian would also fail or would also be less than 2, meaning to say that the system is not completely controllable. But what we are interested in that whether if we could comment more into detail about the controllability under some condition.

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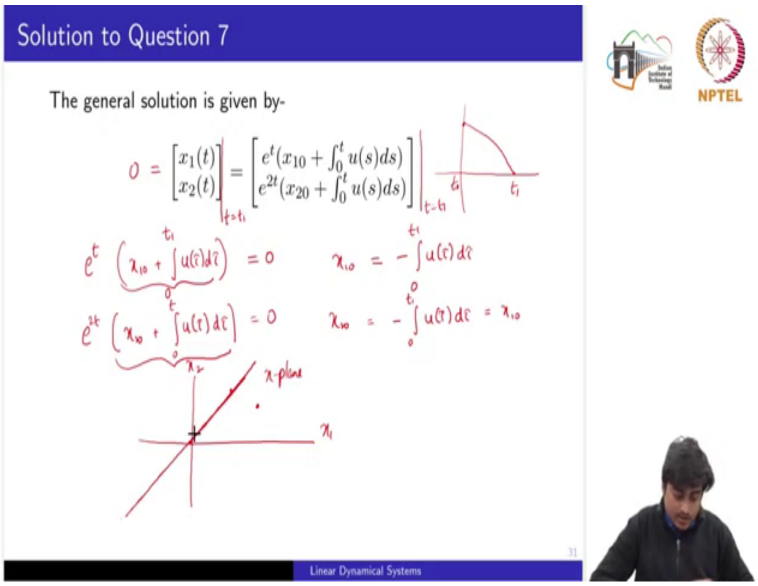

So, for that, we would investigate the solution of this system. So, if we so the general solution for this system where the A matrix is a constant matrix and the time varying terms are only in the B terms, so we using the variation of constants formula, we can write the solution for x 1 and the solution for x 2 ok.

Now, if you recall the definition of the controllability, we the controllability says that from some let us say this time is T naught and this time is T that from some nonzero value, we need to reach to the origin in time T meaning to say or let us say we call this at time t 1. So, if I comp[ute], so I need to reach a t is equal to t 1 is equal to 0 from some nonzero initial condition that are x 1 0 and x 2 0 ok.

So, if I write these two equations explicitly, I would have x 1 0 plus 0 to t 1, because now I replaced t is equal to t 1 both sides, now is equal to 0 equal to 0. Now, if you pay attention to this exponentials, these exponential would always be nonzero. So, these two equations can become 0 only if these are equal to 0 right meaning to say that x 1 0 would become in that case equal to d tau, and my x 2 0 becomes equal to the same as x 1 0 right meaning to say that if we see in x the plane, let us call this x 1, let us call this x 2.

So, if my initial conditions are along this line if my initial conditions are along this line, then in that case I would reach towards the origin, which speaks about the content which is incompletely in coherence with the definition of the controllability. I starting from any point which lies on this line this should be 0, starting from any initial condition lying in this point the system is controllable.

Now, if you choose any other point which does not fall onto this line, you can never reach to the origin in finite amount of time ok. So, the so we under this condition that the both the initial conditions must be same, we can say our system is controllable, but if the both the initial conditions are not same then the system is not controllable.

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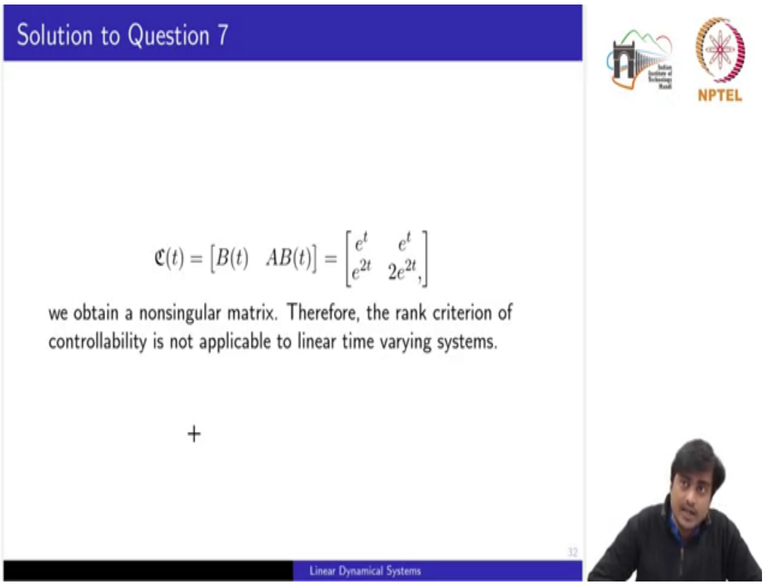

Now, just for the validation, we you can see that whether the rank test of the controllability matrix is also applicable to LTV system is it was a necessary and sufficient condition for the LTI system. So, if we compute this controllability matrix by taking this at time varying terms A and B, we will obtain this controllability matrix. And this matrix is a non-singular matrix which says that it is controllable. And this is also one of the conclusion that the test what we had seen for the LTI is not applicable for the LTV systems ok.