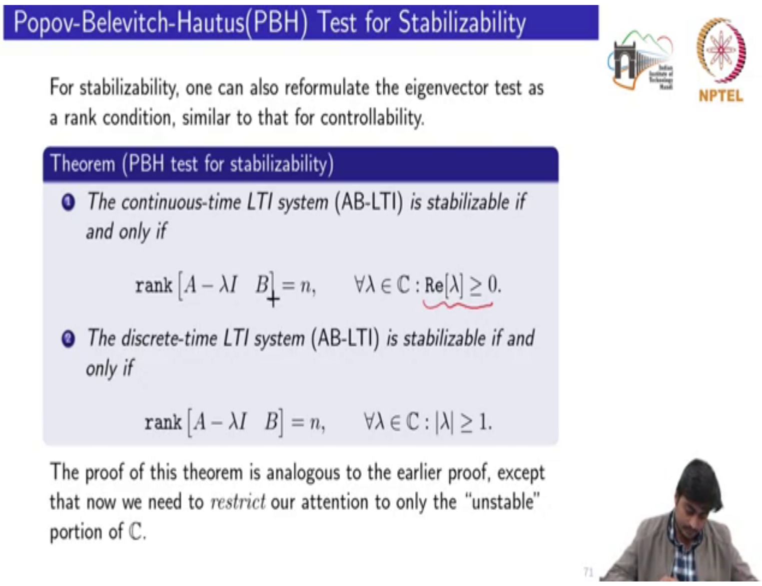
Linear Dynamical Systems
Prof. Tushar Jain
Department of Electrical Engineering
Indian Institute of Technology, Mandi

Week - 3 and 4
Controllability and State Feedback
Lecture - 25
Test for Stabilizability

(Refer Slide Time: 00:13)

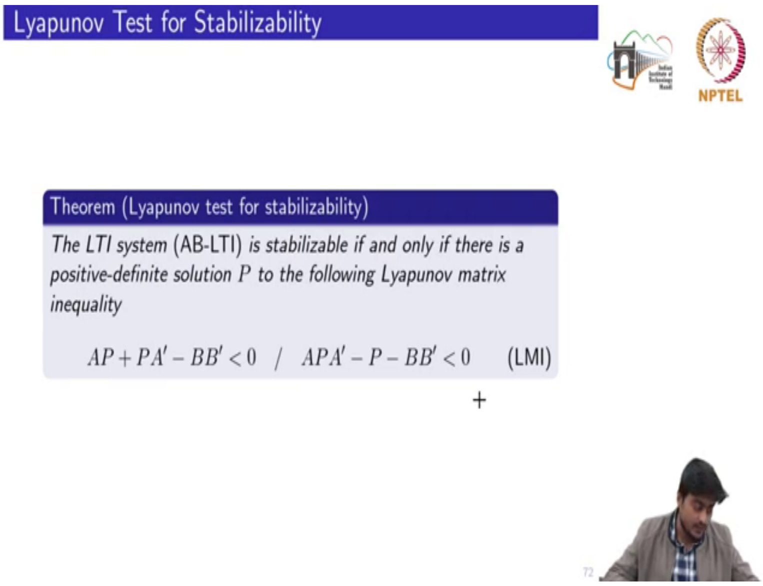


So, the next test is the PBH test which is Popov Belevitch and Hautus Test for the stabilizability. So, for stabilizability one can also reformulate the in fact, this is what we have defined the PBH test as an elegant restatement of the eigenvector test. That the continuous time system is stabilizable if and only if this rank condition is satisfied for all lambdas belonging to the set of complex number such that the real part of those lambda is greater than equal to 0, right.

So, if you recall the result for the controllability this condition was not there while testing the rank condition, but for stabilizability as we had seen in the eigenvector test that we are only concerned with the eigenvalues which are on the right hand side. So, again for stabilizability we are concerned with the eigenvalues which are onto right hand side, ok. So, for the discrete time system we both have the same result, but those eigenvalues should be outside the unit circle or on the boundary of that unit circle.

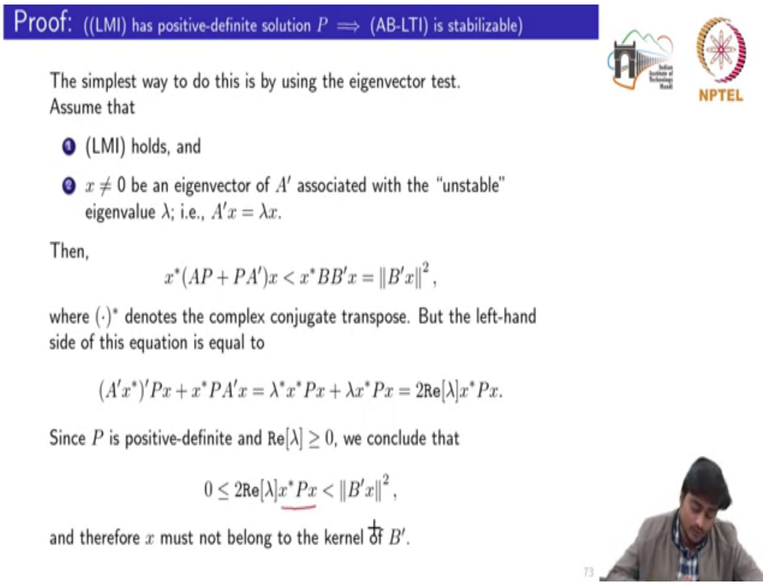
So, the proof of this theorem is analogous to the earlier proof that is the controllability proof except that now, we need to restrict our attention to only the unstable portion of the set of complex numbers, where the eigenvalues lies, ok. So, we would not be going through the proof this PBH test. But, if we want to determine the stabilizability directly from the given AB pair first of all we will compute the eigenvalues of the A matrix, and then only for the eigenvalues which are on the right hand side we will carry out this test, ok.

(Refer Slide Time: 02:17)



The last test is the Lyapunov test for stabilizability. It says that the system is stabilizable if and only if there is a positive definiteness solution P to the following Lyapunov matrix inequality, ok. So, this is in the continuous time and this is in the discrete time domain. If you are typing this pair AP coming from the discrete time system and we term this as an LMI which is a linear matrix inequality.

(Refer Slide Time: 02:51)



So, let us see the quick proof for this result. So, the first implication in that the LMI has positive definite solution matrix P which implies that the pair AB is stabilizable. So, again we would be using this eigenvector test to prove this part. So, there are two assumptions here which the first one is already given here that the LMI has a positive definite solution P, it means that disc that LMI holds. Ok, the second part is that there is nonzero vector x which happens to be the eigenvector of A transpose associated with the unstable eigenvalue lambda that is A transpose x is equal to lambda x, ok.

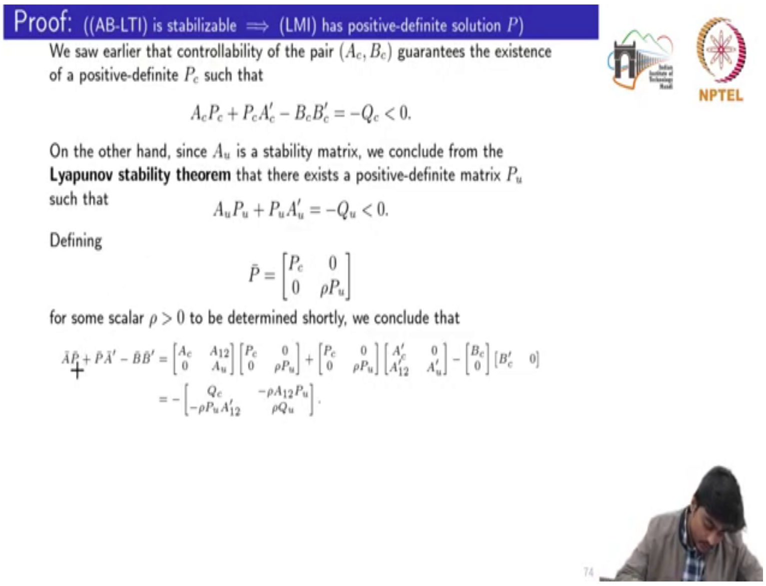
So, forming the quadratic form of the left hand side of the Lyapunov equation so, this was we were having on the left hand side and then using that eigenvector. We form this x star this matrix into x which should be less than or in fact, is less than x star BB transpose x and using the property of the norms I can write this as the squared norm of b transpose x, ok. So, we have starred we already know that it denotes the complex conjugate transpose.

So, see now proceeding in a similar way what we had done for the Lyapunov test in the controllability, this right hand side part we have represented by A star, A transpose x star transpose which would be nothing, but your x star and AP x ok. And on this side since A transpose x star is would be equal to lambda star x star. Similarly, here we would have A transpose x is equal to lambda x and since lambda being scalar I can taken him onto the left to the I can compute them which finally, I would have the twice of real part of lambda into x star P x.

So, see this part which is pretty much important. So, since P is positive definite and we know that the real part of lambda is either greater than or equal to 0. We can conclude that this twice of real part of lambda x star P x would definitely be less than of squared norm because squared norm is positive; P is a positive definite already.

So, this part would be positive definite, but since lambda we are having onto the right hand side which is also positive, but from this inequality. This linear matrix inequality we would have this less than part and which would always be greater than or equal to 0, ok. And therefore, x must not belong to the kernel of B transpose, if x happens to belong to the kernel of B transpose we would have b transpose x equal to 0 ok, but it is not equal to 0.

(Refer Slide Time: 06:19)



The other direction of this implication that the AB is stabilizable implies that LMI has a positive definite solution. So, we already know that the here AB is stabilizable and if AB is stabilizable we can decompose it into the controllable part and the uncontrollable part. And stabilizable meaning to say that all the eigenvalues are on the left hand side.

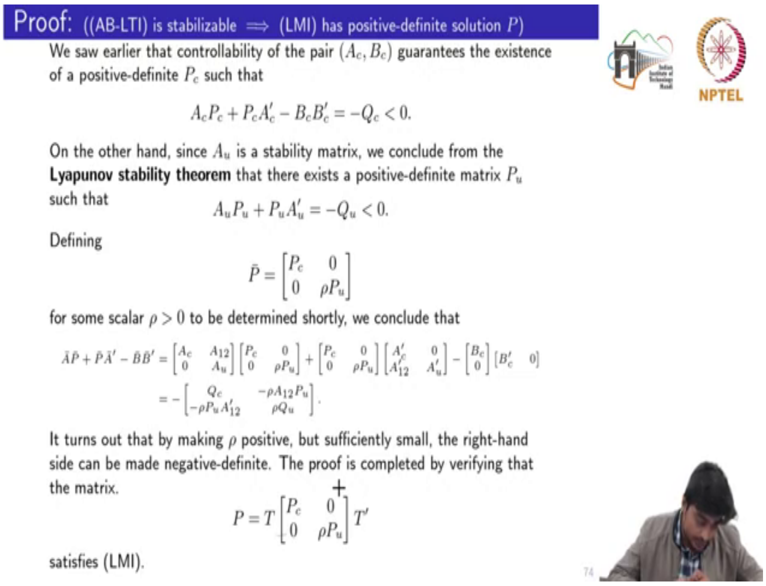
So, we will go step by step. So, first of all we from the controllability of the pair Ac and Bc. We have this the Lyapunov equation in the context of the controllability this is one of the results where Qc, where we have this Bc Bc transpose and PC is a positive definite solution satisfying this equation.

Now, on the other hand since A u is a stability matrix. So, it is already assumed in in this implication that the pair AB is stabilizable. So, all the eigenvalues are already onto the left hand side and Au is a stability matrix.

Now, using the Lyapunov stability theorem we can this write this simply we can write this LMI, ok in terms of A u and P u and P u is a positive definite solution of this equation. So, this equation we basically have written from the stability result and this equation we have result from the controllability result with different A c P c and A u P u pair, ok.

So, now we defined another matrix P bar which is given by the elements P c and row P u on the diagonals and 0 on the half diagonal so, where row is some positive scalar or parameter. So, let us see this part for the complete equation which is in the result. In fact, ok

(Refer Slide Time: 08:29)



So, this is in the result. So, we need to show that this element has a positive definite solution P, ok. So, let us take the left hand side of that equation and start putting those matrices what we had obtained A bar is basically this one if I write all these matrices explicitly, P bar we are introducing we are defining this matrix P bar.

And similarly B bar I can write this B c. Now simplifying them I would have A c into P c plus P c into A c transpose minus B c into B c transpose which is nothing, but minus of Q c. So, here I would have Q c and I taken the minus sign outside of this matrix, ok.

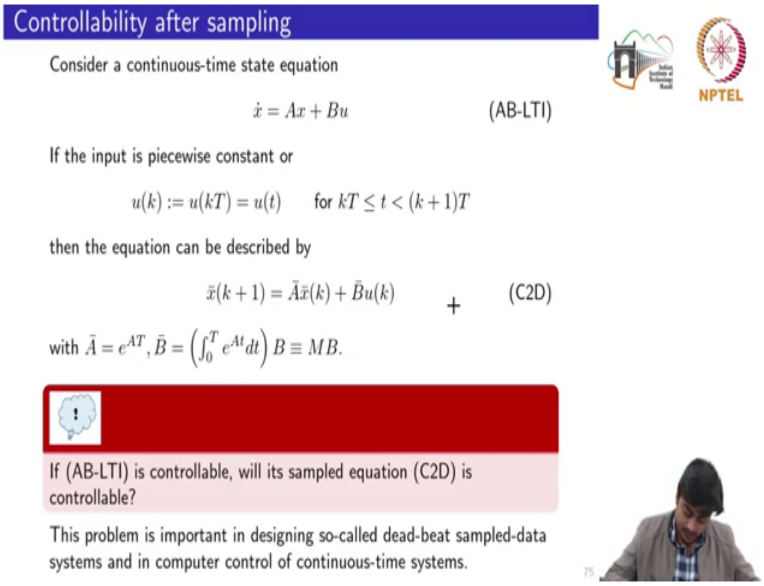
So, similarly see this part we would have A 1 2 row P u here and the rest would be 0 and with the negative sign because we have taken the negative sign outside. Another half diagonal element we would have A u rho into P sorry, this one. So, here we would have this would be 0 and this would be rho P u into A transpose 1 2, ok and seeing the last element we would have rho is a scalar. So, I can write A u P u plus P u into A u transpose which is nothing, but minus Q u. So, I obtained this matrix with rho multiplication.

Now in order to see this matrix, this matrix first of all is a symmetric matrix because the off diagonal elements are equal or let us say in fact, if I take the transpose of this matrix I would obtain in fact, this equivalent, ok.

Now, the second part Q c we have defined already is a positive definite matrix, ok. Q u which we have obtained from the uncontrollable part is a positive definite matrix, ok. Now, I can choose rho u a very small value which is already a positive such that the diagonal elements would be positive. If it is a positive and it is symmetric then the entire matrix would be a positive definite matrix, and this positive definite matrix can be written which was given in the main result being this matrix a positive definite matrix we would have the right hand side the negative definite, ok.

Now, since this P bar is the transformed matrix of the transformed system. So, in order to obtain the P matrix of the original pair A B, again I using this T transformation matrix I could obtain this P and you can verify that this P matrix satisfy that linear matrix inequality which is given in the result, ok. So, this completes the proof.

(Refer Slide Time: 12:07)



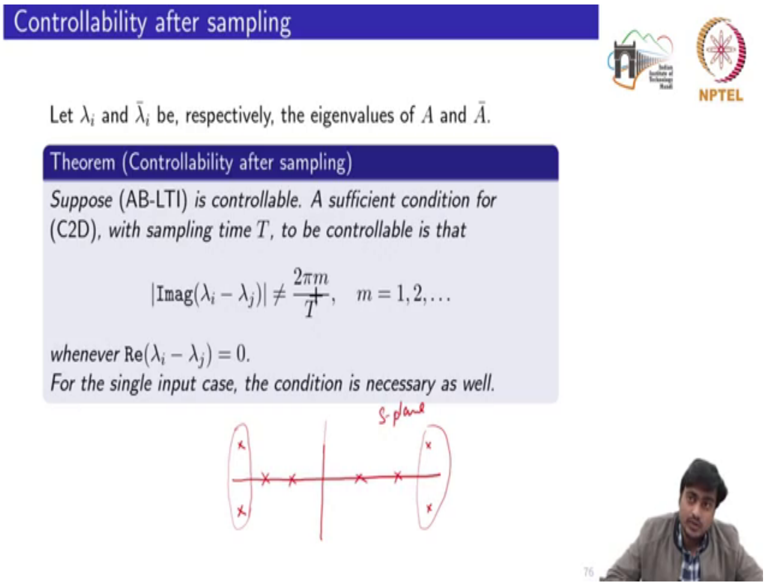
So, additional result we have on the controllability. That so far we have; so, in the first week we discussed the continuous to discrete time transformation given a continuous time system. So, we need to sample our system sampling time T to finally, obtain this discrete time system if the discrete time system is already given to us, ok. Then we had studied different test to carry out whether the system is controllable or whether the system is stabilizable.

Now, we had separate test for the continuous time system also, but now the here the problem is if I use the continuous to discrete transformation given a continuous time system. So, what can I comment on to it controllability, ok. So, if you recall from the first week. So, we had studied two ways of doing the discretization; the first one is using the Euler method and the second one by assuming that u k is some piecewise constant signal between this time interval, ok.

So, but when we assume that u as piecewise constant given by this, one we obtain this discrete time representation where A bar and B bar. We have explicitly computing using this A bar is given by e to the power A into the sampling time and B bar is given by M into B, where m is the this part, ok.

So, now the problem here is which we want to address that if the pair AB is controllable will its sampled equation which is given by after doing the C2D continuous discrete conversion is controllable, ok. So, we will see the answer to this problem. In fact, this problem is quite important in designing so called dead-beat sampled-data systems in the computer control of continuous-time systems.

(Refer Slide Time: 14:23)



So, let lambda i and lambda i bar be respectively the eigenvalues of A and A bar. So, here note that when. So, this A bar I should have written this A d which is the discrete time matrix of the continuous time matrix. So, here A bar is not the transform matrix algebraically equivalent transform matrix. So, whenever I am speaking of this A bar and the context of the controllability after sampling, this A bar is the discrete time counter part of the continuous time matrix A. So, lambda i is the eigenvalue of this and lambda i bar is the eigen value of A bar.

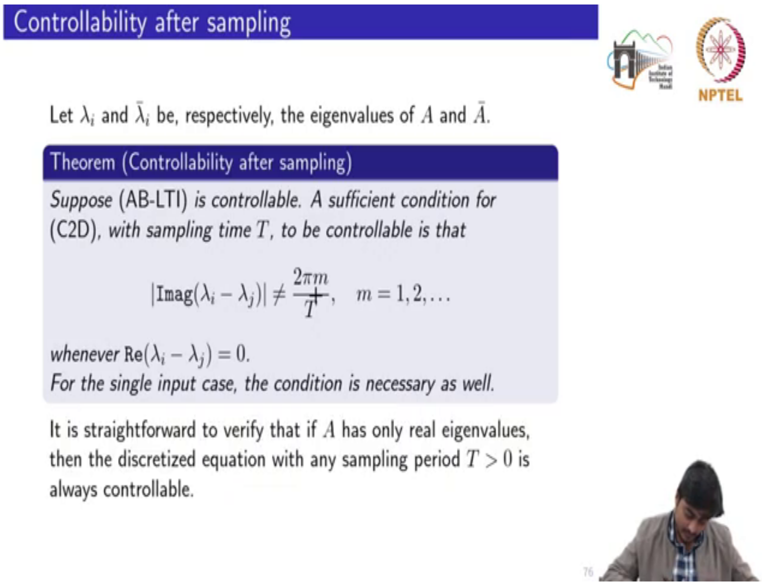
So, this is the result like suppose that the continuous time pair AB is controllable. So, a sufficient condition for the discretized equation with sampling time T to be controllable is that, that the imaginary part of the difference of two eigenvalues lambda i. And, lambda j of the continuous time system is not equal to 2 pi into m by T where m is a positive integer, whenever the real part of the difference of two eigenvalues is 0, ok.

So, for the single input case this condition is necessary as well, but for the multiple input case this condition is only a sufficient condition, ok. So, we would not be going through the proof of this result, but we will see various implication of this result.

So, first of all notice here that if the eigenvalues of the A matrix are only on the lets say what would be the possible cases. So, this is my S plane, now if all the eigenvalues are on to the real axis, ok. Let us say we have the eigenvalues here or it we have the eigenvalues anywhere, ok. If we have the eigenvalues here then it means that the system would always be controllable right because there would not be any binary part, ok.

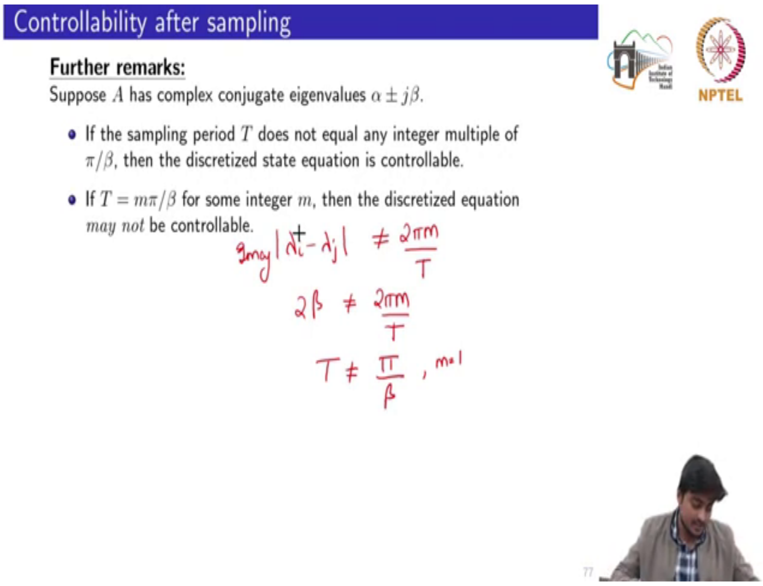
Now, this imaginary condition is for those eigenvalues which have a complex conjugate let us say I have one eigenvalue here and one eigenvalue here or one eigenvalue here and another eigenvalue here, right. Because only in this and in this case we could have the difference, the real part of the difference of two eigenvalues equal to 0, right. And in only those cases we would check the imaginary part of that difference, if that imaginary part of the difference is not equal to 2 pi m by T where T is a sampling time then we say that the system is the discrete time system is also controllable, ok.

(Refer Slide Time: 17:43)



So, it is straightforward to verify that if the matrix A has only real eigenvalues then the discretized equation with any sampling period T greater than 0 is always controllable, right.

(Refer Slide Time: 18:01)

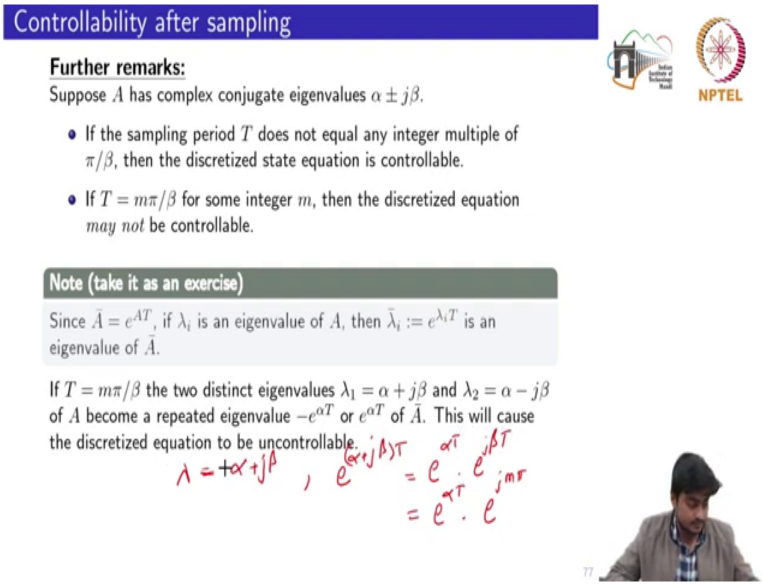


So, let us see we have complex conjugate pair of the matrix A. Now, if the sampling period T does not equal to any integer multiple of pi by beta. Let us recall which is the imaginary part of lambda i minus lambda j should not be equal to 2 pi m by T.

Now, the difference here the imaginary part we would have twice beta from here and this should not be equal to 2 pi m by T which is equivalent to same that t is not equal to pi by beta or for m is equal to 1, ok. So, if it is not equal to the integer multiple of pi by beta then the discretized state equation is controllable, ok. But if it becomes equal to pi by beta for some integer m then it only says that the, for the discretized equation may not be controllable. this is the because of that this is only a sufficient condition.

So, if it satisfies, if the results satisfy it means that the discrete time version is also controllable, if it does not satisfy it does not mean that the system is not controllable, ok; it means that the system may not be controllable or be controllable.

(Refer Slide Time: 19:35)

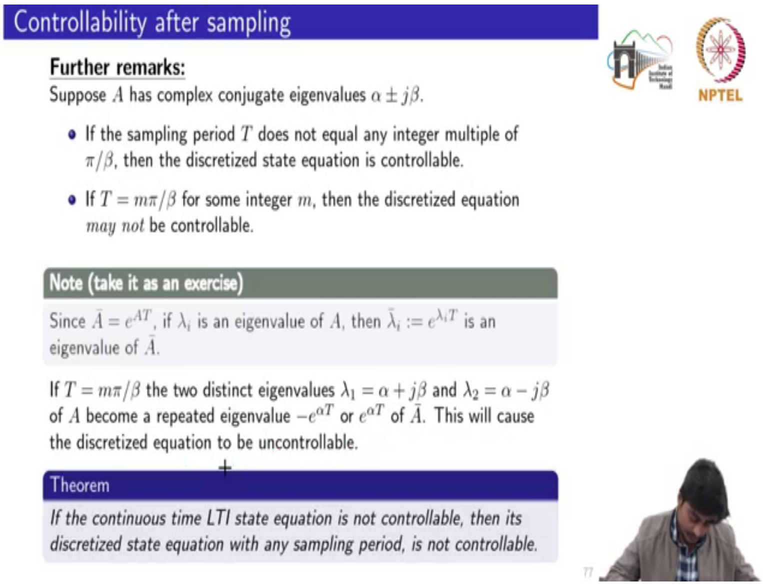


So, we can see the implication of the second point more precisely. So, first of all is note that since A bar the discrete time state matrix is equal to e to the power A into T where A is the continuous time matrix and T is the sample time. If lambda i is an eigenvalue of A then lambda i bar which is equal to e to the power lambda i into T e is an eigenvalue of A bar, ok. So, you can take this as an exercise if you have any difficulty in visualizing this because both the matrices A and A bar are related by this one. So, the lambda i, the eigen values are also written in a similar way.

So, the see if T becomes equal to m pi by beta for some integer m or and given these two distinct eigenvalues alpha plus j beta and alpha minus j beta, ok. In fact, become a repeated eigenvalue either of minus e to the power alpha T or e to the power alpha T of A bar, ok. So, we can quickly visualize this say for example, take lambda is equal to alpha plus j beta. So, the eigen value of the discrete time would be e to the power alpha plus j beta into the sample time.

So, this i can write as e to the power alpha T, sorry it should be j e to the power alpha T times e to the power j beta T, and T i already know is a multiple of m pi by pi by beta. So, this would become e to the power alpha T times j m pi and this part is nothing, but either it would be plus 1 or it would be minus 1. So, that is why we it becomes a repeated eigenvalue of the discrete time state matrix and this will cause the discrete time equation to be uncontrollable.

(Refer Slide Time: 22:09)



So, the last result we have that if the continuous time system is not controllable then its discrete state equation with any sampling period will never be controllable matrix, right. If the system the LTI system is controllable then we had a test or we, you could compute the sampling time also; so, that is your discrete time version of the state equation becomes controllable. Now, in the original pair is not controllable then for any sampling period your discrete time version would never be a controllable.