Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture - 24
Stabilizable Systems

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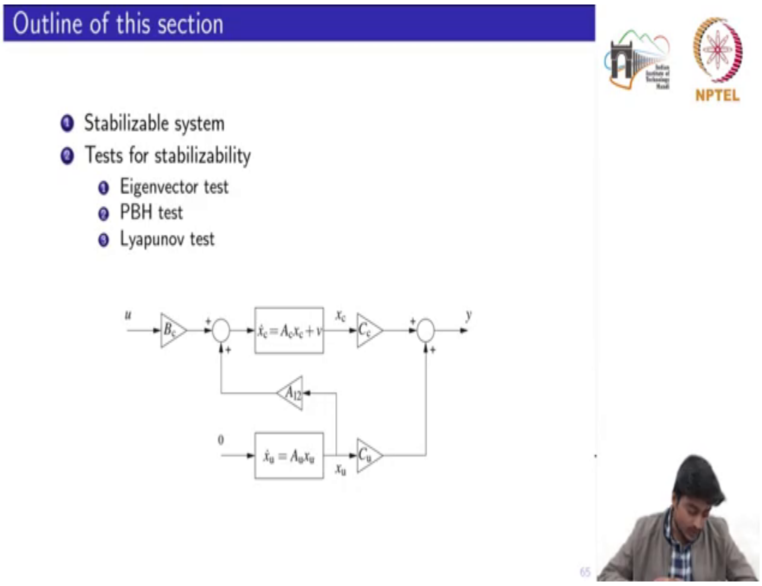

Today, we will start with the last topic of this Controllability week. This is called a Stabilizability. So, in this week we will study about the stabilizable system. So, one important thing with which we concluded in the last lecture of controllability that we have we did some decomposition of the overall system into controllable and uncontrollable ones.

So, now we want to know that ones we have obtained the decomposition and the rank of the controllability matrix is let us say n bar less than n, then we would like to know that whatever the uncontrollable states are whether they are at least stabilizable. Stabilizable is a weaker condition than controllability because we know that if the system is controllable or if the rank is if the rank is full rank then it means all the states can be taken from x t naught to x t 1, ok.

So, first we will introduce the definition of the stabilizable systems then we will see 3 different tests for stabilizability similar to what we had seen for the controllability; the eigenvector, the PBH test and the Lyapunov test. So, this figure demonstrate the decomposition the block diagram decomposition with which we have concluded. So, we see that u only has an impact on the controllable states and through any path u does not have any impact on the uncontrollable state. So, you cannot influence the x u states, ok.

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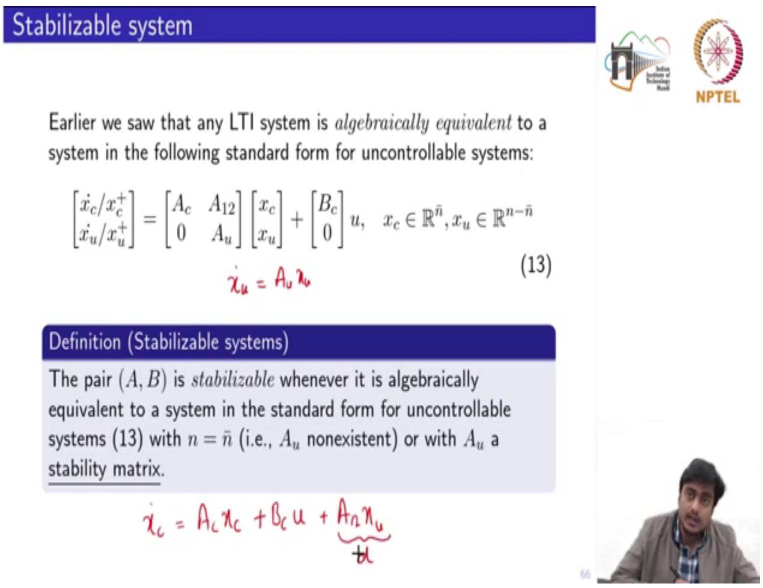

So, earlier we saw that any LTI system is algebraically equivalent to a system in the following standard form for uncontrollable systems. So, we are representing the continuous time and the discrete time by a derivative and by the advancement in time respectively.

So, this is the upper right hand side triangular matrix in which we have decomposed and similarly the B matrix by introducing non-singular matrix. So, if you want to recall the definition of algebraically equivalents it was discussed in detail in the first week, when we discussed about the states by solution in realization. So, the definition of stabilizable system says that the pair A B is stabilizable whenever it is algebraically equivalent to a system in the standard form for uncontrollable systems that is this one, with n is equal to n bar that is A u is nonexistent or with A u a stability matrix.

So, first of all try to visualize the implication of this definition in the sense that once we have obtained this canonical form or the standard form of the uncontrollable mode the dimension of this A c matrix is n bar cross n bar and this x c is the controllable states. Now, x u is the uncontrollable states. So, if I write the equation for the uncontrollable mode it would be x u dot is equal to A u, x u and this is nothing but your homogeneous system which is not at all influenced by any input.

Now, this A c or the state x c is controllable, so it does not matter whether the eigenvalues of the A c matrix are on the left hand side or on the right hand side. But whenever that eigenvalues of this A u matrix lie on the left hand side then we say that the systems are stabilizable, ok. If this if the eigenvalues of A u are on the right hand side then you cannot or the system is completely uncontrollable, because it does not even satisfy the weaker condition on stabilizability, ok.

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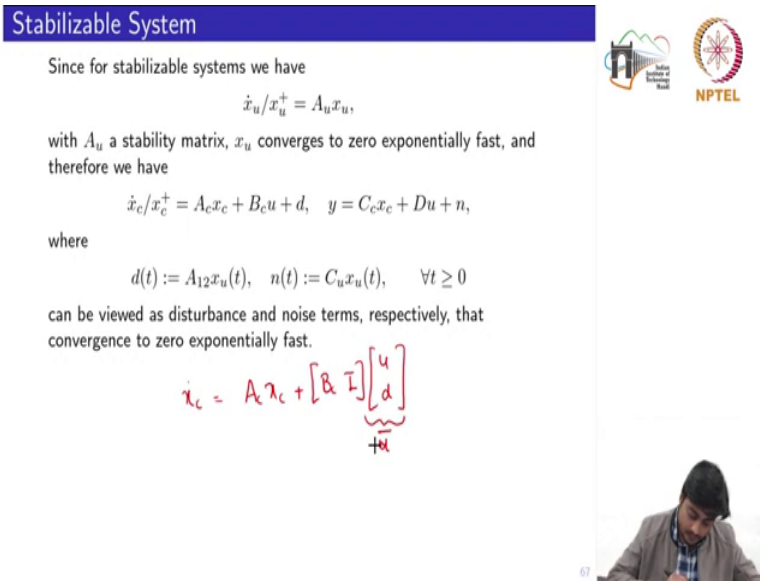

So, let us see. So, this is the homogeneous system in terms of the uncontrollable states, where A u is a stability matrix. So, we know that x u converges to zero exponentially fast and we can write the equation only for the controllable states, where this B is nothing but your A 12 x u. If you have any difficulty to visualize this one; so if I write only the first equation that is for the controllable state. So, I would have x c dot is equal to A c x c A 12 x u.

So, this part I have written as another signal d, and similarly I would have the similar representation of the x u in terms of another signal called n. So, both are in fact, time dependent. So, these two signals can be viewed as disturbance and noise term respectively that converges to zero exponentially fast, ok. So, in fact, you can represent the system in the form of let us say x c dot is equal to A c x c u comma d and here you can say that this is some input u bar, ok. So, either you use this representation or this representation. So, this is u bar, ok.

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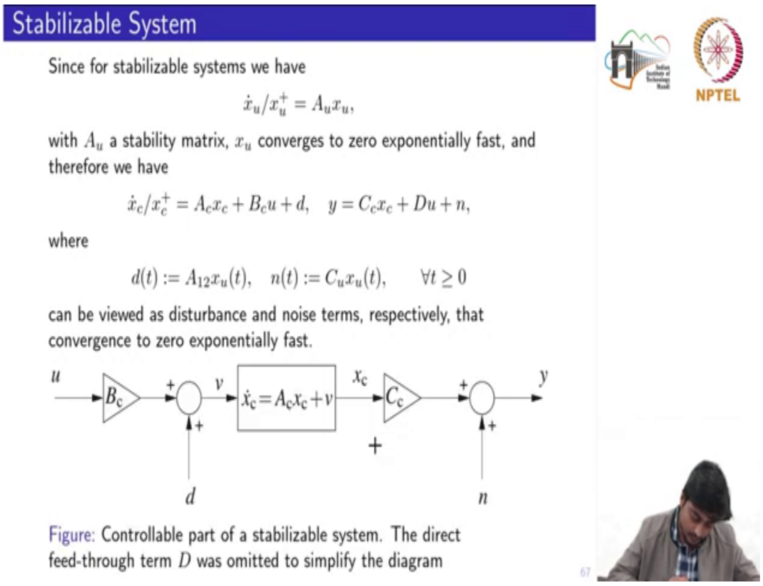

And this is the block diagram representation only of the controllable state, where this d is your uncontrollable state and n again the uncontrollable state with the output matrix, ok.

So, there are if we go, if we want to test for the or let us say if the controllability test has been failed and we want to check for the weaker condition of stabilizability, so there are two steps. First we do the decomposition and then we see that whether the uncontrollable states are stable or not, ok. So, there are two steps to go to verify whether my system is at least stabilizable.

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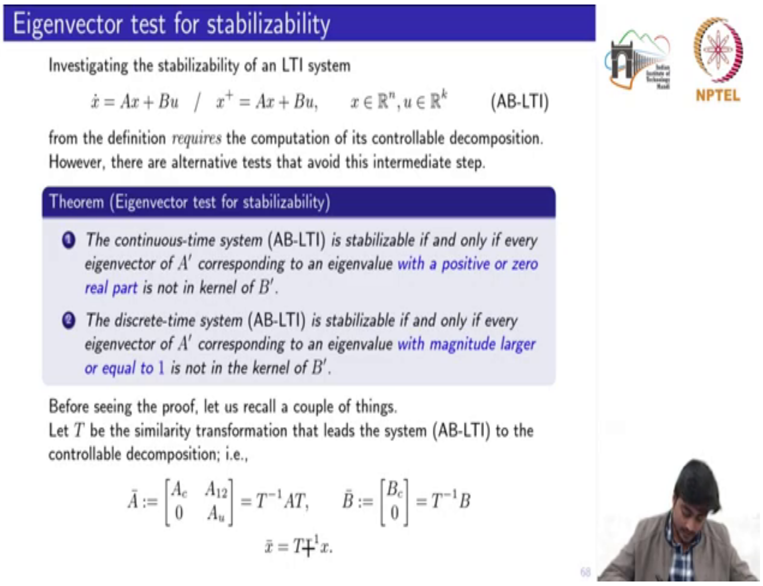

So, if we do not want to do these two steps we are now interested in computing directly whether the system is stabilizable or not, ok. So, these 3 tests which we spoke about in the outline, so those tests ensures that without even decomposition we can carry forward those test for the stabilizability. So, let us say we want to investigate the stabilizability of the LTI system either could be continuous or discrete from the definition which does not require the computation of the decomposition matrices, ok.

So, first one is the eigenvector test. So, this result says that the continuous time system or the pair AB of the LTI system is stabilizable if and only if every eigenvector of A transpose corresponding to an eigenvalue with a positive or 0 real part is not in the kernel of B transpose. So, if you recall the definition of the controllability or the eigenvector test for controllability it was without this blue letter blue colored words. So, it was that the eigenvector that every eigenvector of A transpose is not in the kernel of B transpose irrespective of whether the eigenvalue is on the left hand side or the eigenvalue is on the right hand side. But when we are testing for the stabilizability we need to ensure that the eigenvector corresponding to only unstable eigenvalues is not in the kernel of B transpose, ok.

Similarly, the for the discrete time system the condition of unstable eigenvalues would turn into the eigenvalue with magnitude larger or equal to 1, which are outside or on the boundary of the unit circle. So, first of all we will recall a couple of things and this these two, these 3 equations are pretty much clear to you that the A bar which was decomposed into this form by using a non-singular matrix t, similarly the B bar and this transformation matrix between the two vectors, x and x bar.

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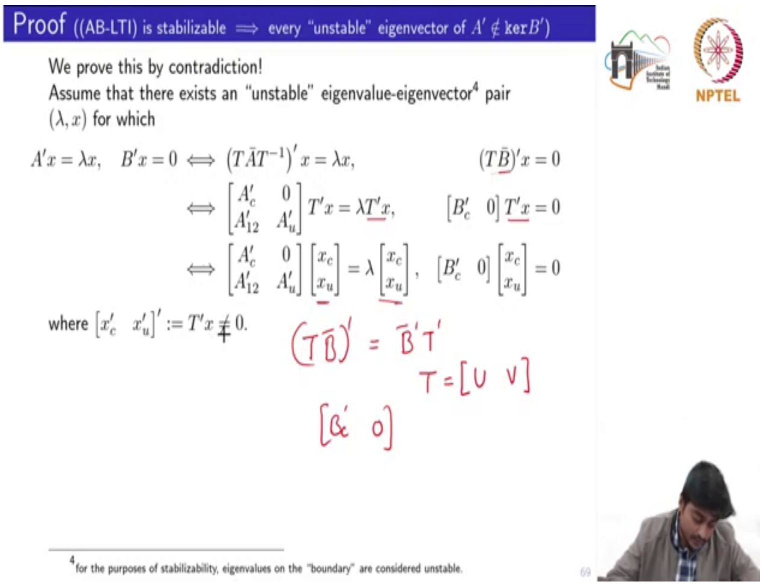

So, the proof this is a if and only if proof. So, first of all we will see this implication that if the pair AB is stabilizable then it implies that every unstable eigenvector of A transpose is not in the kernel of B transpose or does not belong to the kernel of B transpose, ok. So, we will prove this by contradiction that is to say that there exists an unstable eigenvector of A transpose which is in the kernel of B transpose implies that the pair AB is not stabilizable, ok.

So, assume that there exists an unstable eigenvalue eigenvector pair lambda x for which these two equations would satisfies this, A transpose x is equal to lambda x meaning to say that x is that eigenvector corresponding to an unstable eigenvalue that is why we are calling it an unstable eigenvalue eigenvector pair lambda x, ok. And since if this eigenvector is in the kernel of B transpose we would also have B transpose x is equal to 0, which is equivalent to saying that I can replace this A matrix and B matrix by its decomposed counterparts which is your T A bar T transpose oh sorry T inverse and here B would change to T into B bar, ok. Again, expanding this T A bar T inverse I would have this decomposed form with respective transposes T transpose x is equal to lambda times T transpose x, ok.

Now, note that here that I can write. So, first of all this first of all see this one we have this T B bar transpose which I can write as B bar transpose and T transpose, ok. Now, if you recall that we have represented T as U as this matrix as this matrix, where we know that B bar is your B C transpose and 0. So, I can write this part as equal to this part which is I mean equal to 0. So, and T transpose x is nothing, but your x bar which is given by here x bar says that the decomposed form of x c and x u, ok. Similarly, t transpose x would become x bar the rest matrices would remain the same similarly it would happen here, ok, where we know that this x bar is equal to t transpose x and is not equal to 0. Clear up to this point.

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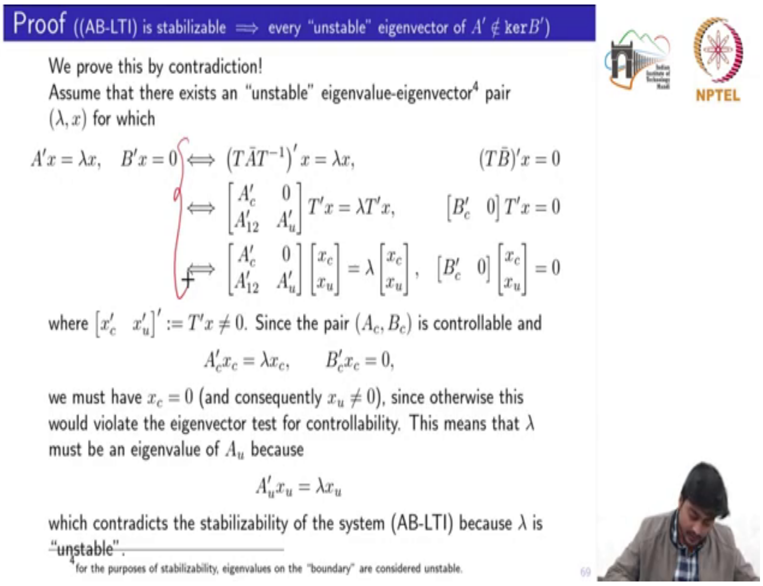

So, moving forward since we know that the pair A c B c is controllable already, meaning to say that there is an eigenvector x c which satisfy these two equations and since x c should be 0 only because it cannot satisfy both these equations, if it for this eigenvalue if we have this x c B c transpose x c should not be equal to 0, right. And if there is some x c satisfying this equation it must be 0, ok. So, x c we have 0.

Now, note that here that in this equation we have x c and x u, x c and x u is not equal to 0, right and we already know that x c is equal to 0. It means that x u should definitely be not equal to 0, right therefore, we have consequently x u is not equal to 0 because the complete vector itself is not equal to 0, ok, since otherwise this would violate the eigenvector test for the controllability, right.

So, this says, this means that the lambda must be an eigenvalue of A u because A transpose u x u is equal to lambda x u which contract contradicts the stabilizability of the system because lambda is unstable, ok. So, this completes the proof of this of this part. The second part would be the other way implication.

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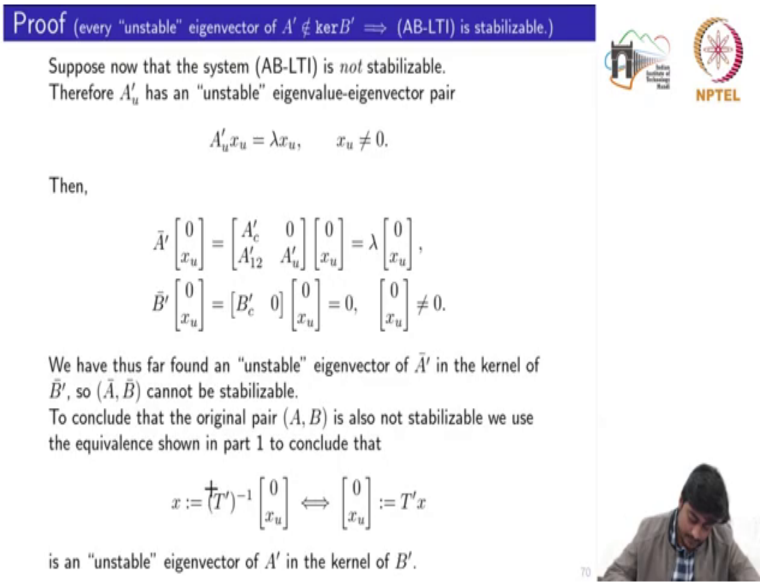

That is to say that every unstable eigenvector of A transpose if it is in the kernel of B transpose then it implies that the pair AB is stabilizable. Again, we will follow the negative negation approach that suppose now if the system is not stabilizable, so we want to prove that there is an unstabilization vector of A transpose which belongs to the kernel of B transpose, ok. So, suppose if the system AB is not stabilizable, so therefore, what we had seen in the last part of the result that A u transpose has an unstable eigenvalue eigenvector there which satisfy this one being x u not equal to 0, ok.

Now, here I have explicitly written that x c is already equal to 0 and x u is not equal to 0 that is why we are not writing x c and we are writing 0 in place of x c. This A bar transpose into x bar having x c is equal to 0 is equal to the transform matrix and the transformed state, which would be nothing, but equal to lambda times u x c. Similarly, if I expand this thing it would become B transpose, B bar transpose into this state vector in fact, it become equal to 0 because we have x c equal to 0, but we also know that x bar is not equal to 0. So, we have thus far found an unstable eigenvector of A bar transpose in the kernel of B bar transpose, so this pair A bar B bar cannot be stabilizable.

Now, since both these pairs A B and A bar B bar algebraically equivalent. So, when we say that A bar B bar cannot be stabilizable it implies that A bar B bar oh sorry A B, the pair A B is also not stabilizable, but we can see to compute that the original pair AB is also not stabilizable we use the equivalents shown in part one to compute that for that equivalence is basically this equivalence. And in fact, it is straightforward also by taking the definition of algebraically equivalent that proving that A bar B bar cannot be stabilizable, in fact, implies that the original pair A B is also not or cannot be stabilizable, ok.