Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture - 23
Controllable Decomposition - I

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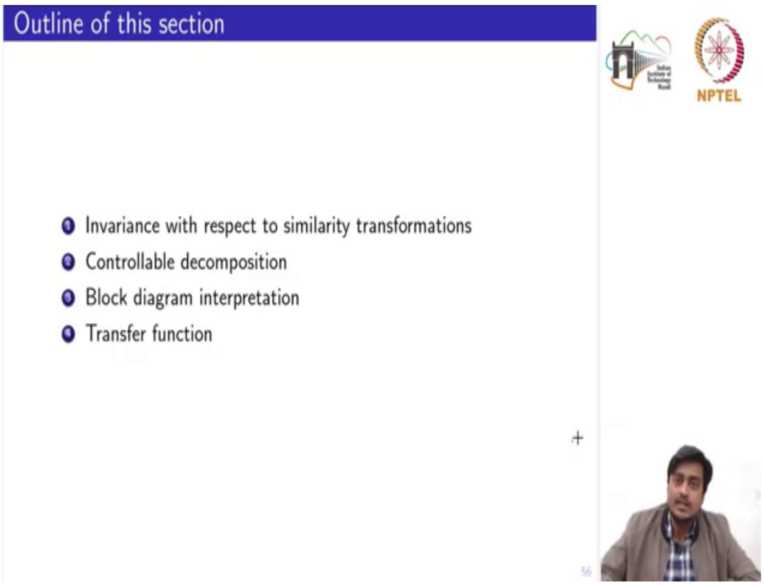

So, now we will start the next sub section of the overall controllability topic, here in this the next topic is the Controllable Decomposition which is drawn on the top of the slide and the outline of the section is. So, we will cover four major aspect in this controllable decomposition first of all the invariance of controllability with respect to similarity transformations. So, here the question we want to answer that if you remember that in the first week we used different representations of the system by used by computing different a b c d matrices.

Now, if the question here is if we have two different AB pairs and they both are related with some similarity transformation, does it also implies that if the first pair is controllable, the second pair would also be controllable? So, here you want to show the invariants. The second one is the controllable decomposition. So, this is the most important point of the section where we would be going to show that if the system or if the rank of the controllability matrix is not equal to n, then it means that all the states let us say we would have n number of states, right that all the n number of states are not controllable um.

Now, the question arises here that are some of the state controllable. So, here we would be going to answer that thing by using some decomposition. The third one we will see we will visualize this decomposition through some block diagram representation. And finally, the equivalence of this controllable decomposition in terms of the transfer functions because if you recall from the 1st week given two different representations of the same system, we have proved that the transfer function would remain the same right.

Now, here the question arises if I have obtained a decompose a controllable decompose form whether the transfer function of those two different presentations are equal or not ok. So, this is we were going to answer in the fourth one.

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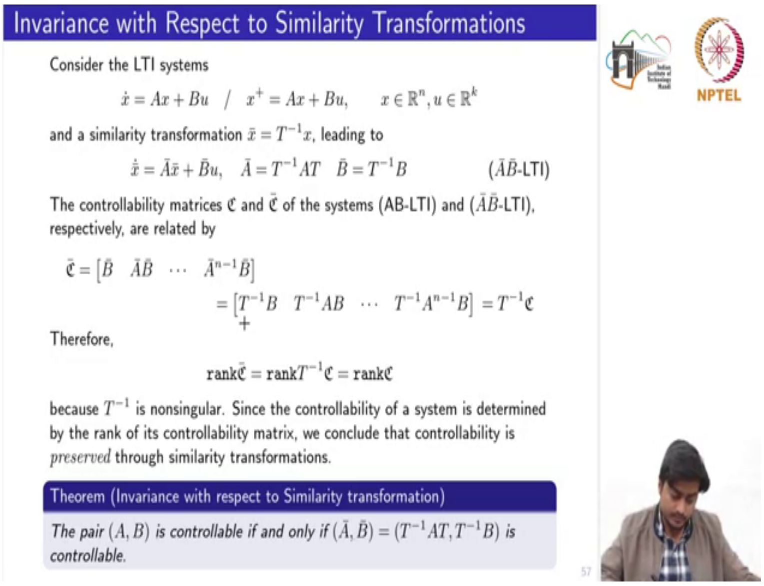

So, let us start. So, we have this considered the LTI systems. Again the continuous time and the discrete time are separated by a slash and we are using the similarity transformation that these. So, we will talk about mostly of the continuous time because for the discrete time the results would directly whole. If there are some differences in the continuous time in the discrete time we would take them separately similar to what we had taken earlier while computing the covariance between the reachability gramian and the controllability gramian, ok.

So, here we have two different representation AB and A bar B bar and x and x bar. So, now, we have a similarity transformation related by this T matrix and the property of this T matrices that it is a non-singular matrix. So, x bar is equal to T inverse x. So, if I put or I put x is equal to T into x bar into this equation I would obtain this A bar B bar and there is also relation between this A bar A by T inverse A T and B bar is equal to T inverse B, ok. So, we denote this one is pair A bar B bar of the LTI system and this one is the pair AB of the LTI.

Now, we will compute the controllability matrix for both the systems. So, this is the controllability matrix of the pair A bar B bar where we have used all the matrices with the bar over n. Now, in this part we will replace B bar and A bar by these two equations into this controllability matrix, then you would obtain T inverse B T inverse AB and similarly up to the n minus 1, ok.

Now, since T inverse is common in all these matrices, so I can take it T inverse outside and the remaining part if you see carefully it is nothing, but the controllability matrix of this AB pair, ok. So, I can write it as T inverse of the controllability matrix of this pair AB. Now, the controllability matrices of the pair A bar B bar and that of pair AB are related again by this similarity transformation matrix T and we know that T is a non-singular matrix.

And, if the rank of this matrix which is composed by matrices A and B is full rank that is that is if it is equal to n, then the rank of the c bar would also be equal to n because the multiplication of a matrix with the non-singular matrix does not change the rank of that matrix, ok. So, we would have the rank of c bar is equal to rank of T inverse c is equal to rank of c, ok. The rank condition will not because T inverse is non-singular. So, since the controllability of a system is determined by the rank of its controllability matrix, we conclude that controllability is preserved through similarity transformations.

So, even if I use any similarity transformation matrix and transform the representation from one form to another form, the controllability is preserved. And, this is the key result that either you say that the controllability is invariant under any similarity transformation or you say the pair AB is controllable if and only if this pair A bar B bar is also controllable where A bar B bar are specifically given by or in terms of AB and the similarity transformation matrix.

So, the proof of this theorem is in fact given above. So, we started with the so this is what we call the proof by construction and then finally give the result ok.

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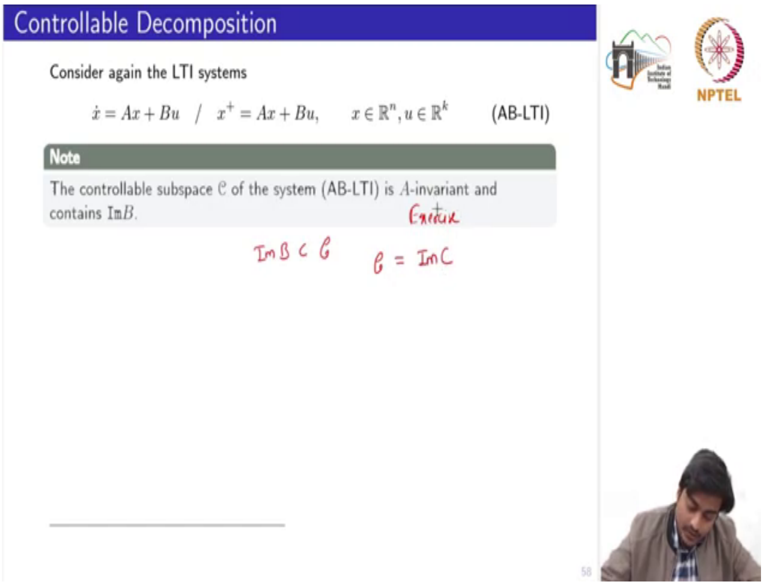

Now, coming onto the controllable decomposition part; so, consider again the LTI systems note that here that the controllable subspace c this is matcal c of the system AB, LTI is invariant A invariant and contains image of B ok. What does it mean? Try to visualize this thing that the image of B is in fact subspace of this controllable subspace. So, any vector or any element belonging to the subspace would definitely belong to the subspace and this subspace is invariant.

So, you can take this you can prove it by yourself. So, you can take this as an exercise and show it by yourself that the controllable subspace c which is and c if you recall from one of the result that c is nothing, but equal to the image of the controllability matrix, and this image of this controllability matrix contains the subspace image of B. So in fact the proof is pretty much straightforward, but you can show this the first part A invariant we have already shown that the this controllability matrix is A invariant right. So, you can take this is an exercise and see and solve by yourself or to prove this property.

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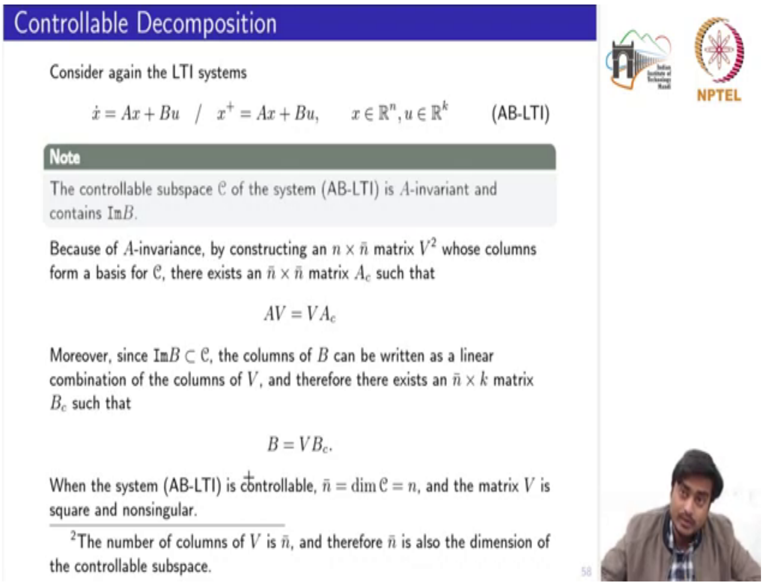

Now, because of the invariance of the or a invariance of the subspace we can construct an n cross n bar matrix V where the n bar is nothing, but the number of columns of V matrix whose columns forms a basis for this controllable subspace um. This is the in fact the one the first property of any linear subspace being A invariant or any matrix invariant here it is A. So, there exists an n bar cross n bar matrix A, c such that this equation has been is satisfy ok. AV is equal to V into A c.

Moreover, since the image of B is contained in the controlled controllable subspace we know that the columns of B can be written as a linear combination of the columns of V. Now if it can be written as a linear combination of the matrix V, I can write it as B is equal to sum V into B c ok. So, meaning to say that therefore there exists an n bar cross k matrix B c such that this equation satisfied and this is nothing, but saying that B is a linear combination of all the columns of V.

So, now this is an important point that when the system Ab-LTI is controllable you would have n bar is equal to n which is nothing, but the dimension of the controllable subspace and the matrix V. In that case would be A square matrix because the dimension of matrix V is n cross n bar and we would be A square if the system is controllable. Now, here the question arises what with what we have started that if the system is not controllable?

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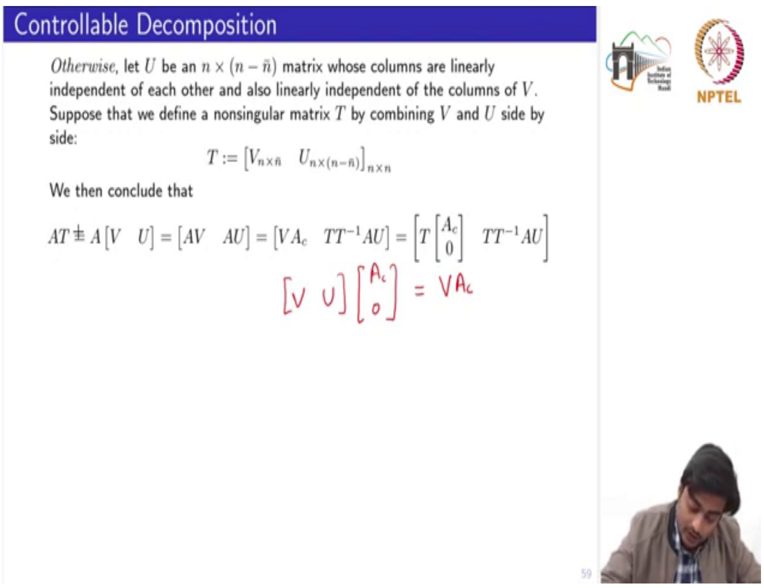

If the system is not controllable, then let us say we define A matrix U. If the system is not controllable it means that the rank is not equal to n, but it is less than n ok. So, we define another matrix U which is an n times n minus n bar matrix whose columns are linearly independent of each other and also linearly independent of the columns of V. So, this U matrix has these two properties that all the columns are linearly independent to each other and also to the matrix V.

So, using these two matrices U and V which we have defined earlier I can write another matrix T which is a non-singular matrix are given by cascading this V or concatenating these two matrices V and U V is the n cross n bar and U is n cross n minus n bar. So, the overall matrix would be a square matrix or dimension n.

Now, see here then the multiplication of the matrix A into T i can write as equal to A into V U taken from over and since both of them are matrices I can individually multiply this matrix with these matrices. So, I would have AV and AU AV by using that property I can write it as V into A c and since T is non-singular T into T inverse would be an identity matrix. So, I have replaced identity matrix by T into T inverse into the second part, ok.

Now, V into A c I can write it as T in this matrix. Let us see if you have faced any difficulty. That T is your V U and this matrix is A c 0. So, this matrix is nothing, but V into A c because U into 0 would be 0, right. So, I can write V into A c is T into this matrix A c 0 and the rest part would remains as it is, ok; so, the part from A into T. So this is clear now.

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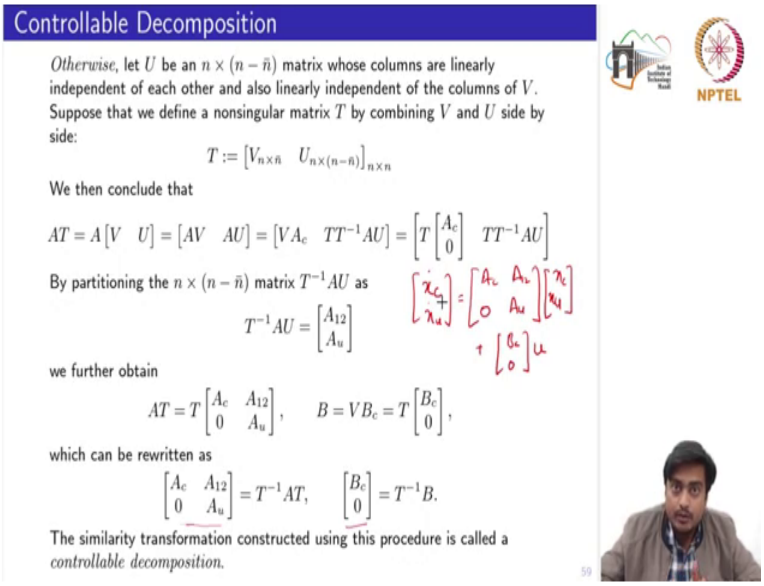

So, moving forward that this part T inverse AU, I can partition this matrix as a 1 2 and AU arrange in the column. So, finally I can write my matrix A times T is equal to T taken common from this part A c 0 and taken from this one which is nothing, but T into A 1 2 and AU. So, the remaining matrix would be A c 0 and A 12 U and V. We already have B is equal to B into V into B c and similarly as we have seen here I can write V B c is equal to T into B c 0.

So, now comparing we could write this matrix A c A 1 2 0 AU as equal to T inverse AT and B c 0 as T inverse BT, where T is nothing, but a non-singular matrix. So, it happens to become a similarity transformation with respect to this T matrix. So, we just have constructed A T matrix given the matrices A and V ok. So, this part is called the Controllable Decomposition. We will explore further property that with the introducing this specific T matrix which is given by here we have decomposed the AB matrices into controllable decomposition.

So, note that here that it should be clear from the visualization also say suppose if we have this A bar matrix and we say this is the B bar matrix and say suppose we denote x c and x u here we would have A c A 1 2 0 A U sorry x c and x u plus B c 0 u ok. So, here this is also called a canonical form by the way. Why? Because you see that this x u state is not at all influenced by u and it does not have any dependence on x c on which u has an influence ok. So, basically we are trying to go to also the part where I can extract the controllable part from the given AB pair, right.

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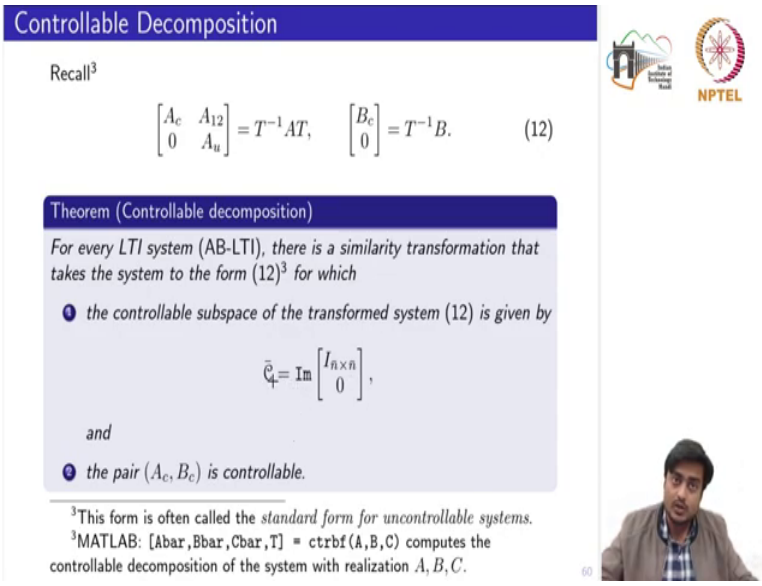

So, recall these two matrices we have just constructed. So, for every LTI system AB-LTI there is a similarity transformation that takes the system to the form 12 which is the decompose form for which the controllable subspace of the transformed system is given by the image of this identity in 0 and this identity is having a dimension of n bar and the pair A c B c is controllable ok.

Now, just visualizing this result the question what you ask from yourself that if I want to compute the rank of this A c B c pair, the rank would be equal to nothing, but n bar and n bar is less than n, ok. So, this form is often called the standard form for uncontrollable systems because we already know that n bar which is a lesson number, then n meaning to say that the system is not controllable. So, it is the standard form for uncontrollable system by introducing a specific similarity transformation matrix t, ok.

So, in the MATLAB you can also obtain this decomposition by using this CTB c t r b f command which computes the controllable decomposition of the system with realization a b c. So, A bar B bar C bar would be off this form and it would also give you the T matrix which we have computed by combination of V and U matrices ok. So, next we will see the proof of this one that how it becomes the rank of this control or let us say the dimension of the subspace is equal to n bar.

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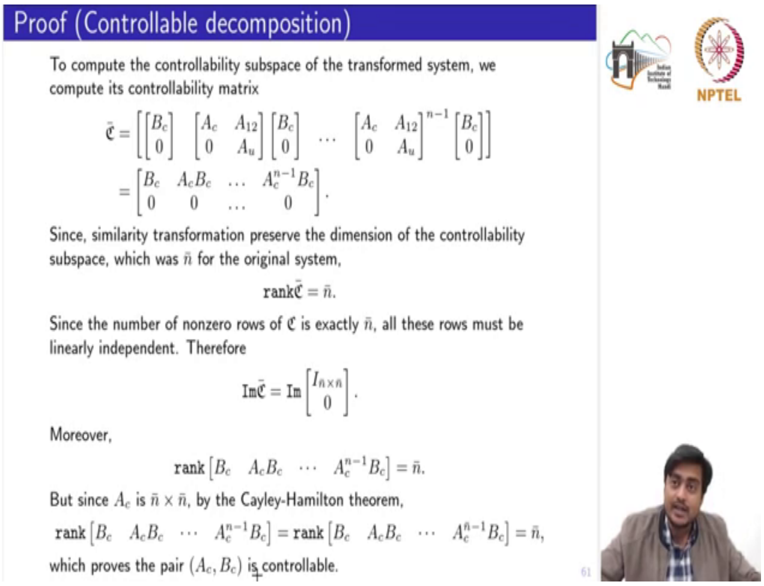

So, to compute the controllability subspace of the transform system, we compute its controllability matrix which is the c bar. Now here I have written all the by opening all these A bar B bar matrices. So, earlier when we studied this controllable decomposition we had over results in terms of bar matrices over it.

Now, here we know that bar matrices are nothing, but these parts ok. So, this one would be this one. So, here if you if I simplify this matrix I would have A c B c plus A 12 into 0 0 into B c plus AU into 0. So, the rest term would be A c B c and 0. Similarly if I go forward I would have A c to the power n minus 1.

Why? Because this is an upper triangular matrix and the power of the upper triangular matrix would be the power of the diagonal elements and the of diagonal elements could be could be would be computed. So, that is why we have written A c to the power n minus 1 directly. If there is some element some non-zero component here, then I could not write that A c to A c to the power n minus 1. So, so this point should denoted here. So, A c to the power n minus 1 B c ok. Now this row is completely 0. So, this one contribute any in the rank.

So, since similarity transformation preserve the dimension of the controllability subspace which was n bar for the original system. So, we would have rank of c bar is equal to n bar once we have come because we already know that there is a the rank of this controllability matrices n bar which is less than n. So, the dimension of the subspace is n bar already. Now we have extracted the subspace in terms of this controllable decomposition let us say that the rank of the c bar is nothing, but equal to n bar.

So, since the number of non-zeros rows of c is exactly n bar, all these rows must be linearly independent. Therefore, we would have image of the c bar is equal to the image of the identity matrix or dimension n bar and 0 the rest of the for the rest of the dimension. Moreover this one is also already equal to n bar which is not equal to n bar n where the system would have been controllable and since A c is A n is A square matrix of dimension n bar.

Then again by using the Cayley Hamilton theorem we would have the rank of this one equal to the rank of this one because this will not contribute into the into computing of the rank um. So, the rank of this rest of the matrix n bar which proves that the pair A c B c is controllable because we have extracted or we have discarded the matrices which were contributing towards the failing of the rank condition, ok.

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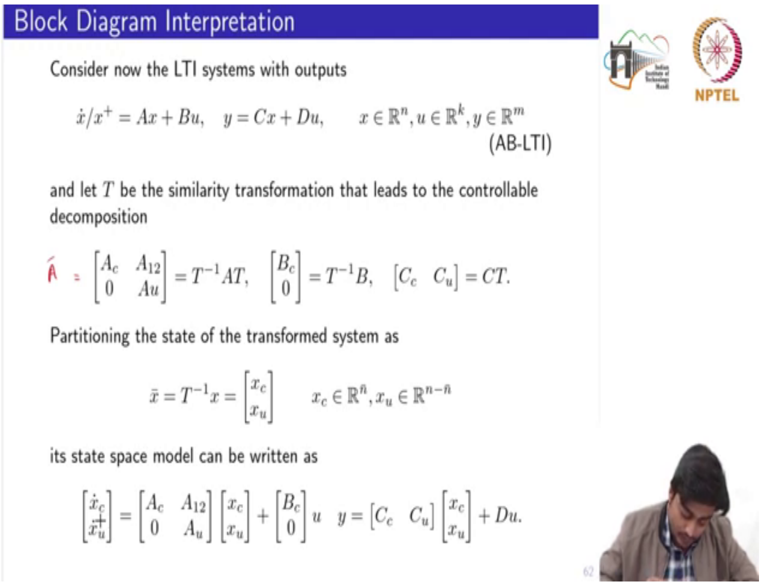

After seeing this controllable and uncontrollable decomposition of the pair AB we can go back to the same representation what we have discussed previously. So, let us see here we have this the LTI system continuous in this part here T is the similarity transformation matrix which decomposed the pair AB into this one and this is nothing, but your A bar pair A bar B bar and C bar and partitioning the state of the transform system as x is x c and x u.

So, the idea of an introducing this c and u in the subscript of the matrices in the vector from the beginning is to come to this point where we can explicitly show that which states are controllable and which states are uncontrollable and with respect to those controllable and uncontrollable states what are the associated matrices. So, A C is associated with the matrix with the vector x c and a u is associated with the vector which is uncontrollable state vector which is uncontrollable, ok.

So, we have x dot c x dot u this we have seen already. So, again I would like you to pay attention here that x dot c is being influenced by both the controllable and the uncontrollable sets because of these two say for example non zero elements and it has a direct influence of the control signal u because of this non-zero BC matrix ok. Now x u is only triggered by its initial condition not by x c and it does not have any influence by u. So, basically it is nothing, but a homogeneous equation x dot u is equal to a u x u, ok.

Now, so suppose if we have some non-zero element here if we have some non-zero element sorry say for example, if we have B matrix as it is being the zero element here. Now if this element is also non-zero if this element is non-zero, it does not imply that x u would be uncontrollable. Why? Because in that case x u is being influenced by x c and x u where x c itself is being influenced by u. So, so this could be an or loosely speaking could be an indirect control of the x u, right.

So, here we can see straightforwardly that you does not have u and x c does not have any influence of x u. So, this part is been completely removed from the controllable representation, ok. The D matrix will not change as we had seen in the similarity transformations.

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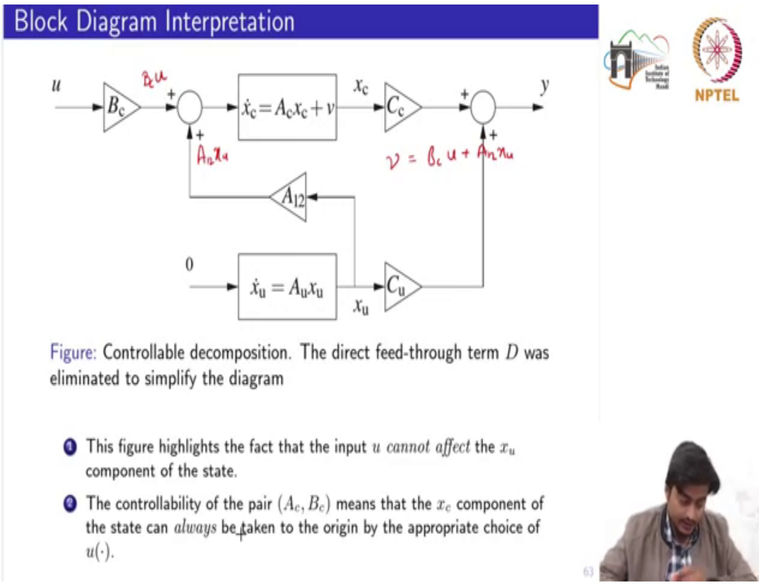

So, if I draw the block diagram here, so this part we would have B c into u and this part is been added which is A 12 x u. So, the addition of this B c u and A 12 x u we have replaced by n u ok. So, this n u is nothing, but your B c u plus A 12 x c and the rest is A c A c x c multiplied by C c, we would have the y matrix. So, you would see that x c is only being influenced by u and x u there is no you can say it is by 0. So, this is the uncontrollable part because you cannot control since the input would always be 0. You cannot manipulate the input, ok.

So, its there are two point which should be known, then that this figure highlights the fact that the input u can not affect the x u because if you noticed starting from u, there is no direction which is going towards x u here through any part right. The second point if the controllability of the pair A c B c means that x c component of the state can always be taken to the origin by the appropriate choice of u which is basically the definition of controllability right.

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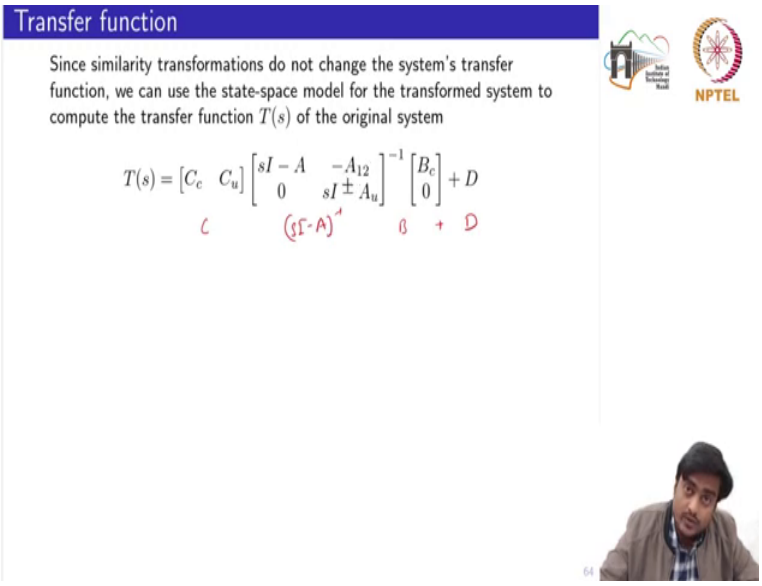

Let us see the transfer functions. So, since the similarity transformations do not change the system transfer function, this is we had seen in the first week when we were discussing about the equivalence transformations of the two different pairs. So, if we compute this T s which is your c s I minus A inverse B plus D. So, this is the overall transfer functions we had compared. So, again since it is an upper triangular matrix. So, the inverse of this matrix would be the inverses of these diagonal elements at least.

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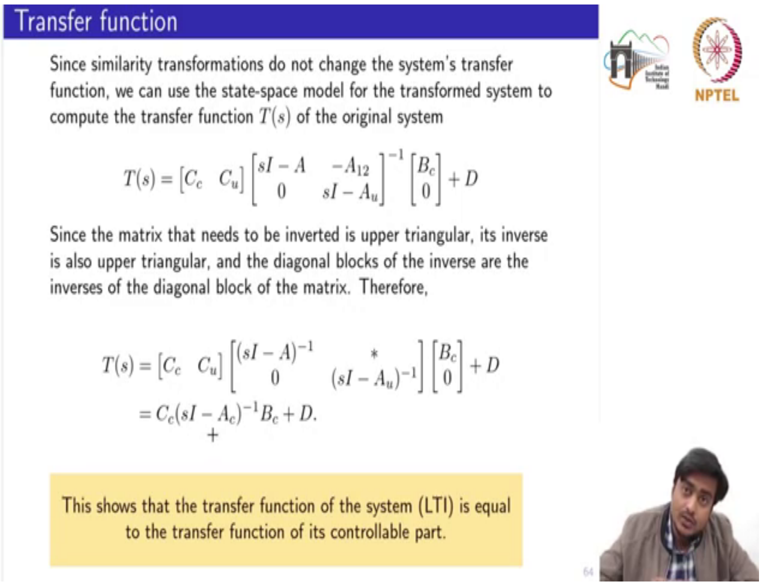

And this of diagonal element we can write we it should be compute, but it is of no use to us because the multiplication of this with this would be 0, ok. So, finally we would have this one C c s I minus A c inverse B c plus D right. So, since you can also use this property that under the any similarity transformation, the transformation does not change.

So, once we have the explicit representation of the controllable and uncontrollable representation and we know the transfer function is basically the representation only of the controllable part of the system what it also implies that whenever a transfer function is given to you that those transfer functions are always controllable, ok. So, if a transfer function is given to you those plants are always controllable in this would nothing be equal to the transfer functions of the original system.

So, it says this shows that the transfer function of the system is equal to the transfer function of its controllable part. So, this is the important conclusion. Now, if you can see that if a transfer function is given to you or a matrix of transfer function is given to you which we would possibly discuss in the last week when we discussed of the MIMO systems explicitly, then those systems are already controllable right because the system has been has been represented through transfer function ok.