Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture - 22
Tests for controllability – IV

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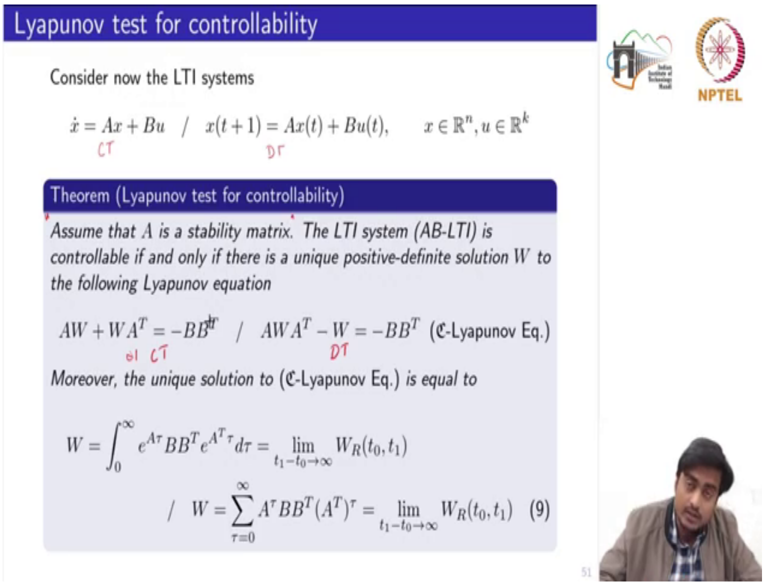

So, now we will start with another Test of the Controllability, what we say the Lyapunov test. So, consider now, the LTI systems there on the left hand inside, it is the continuous time and here its the descript time, where we have using the same variability. Whenever, we; whenever it is simplicity clear that the in which domain we are we talking about the test ok.

So, it says, that assume that A is a stability matrix. So, it is so, note this point because you will going to use this argument. Assume that the matrix A is a stable the LTI system that is the pair AB is controllable if and only if there is a unique positive definite solution W to the following Lyapunov equation.

So, this Lyapunov equation is in the continuous time domain; excuse me, continuous time domain and this one is in the discrete time ok. Here, we have use that the Lyapunov equation with respect when, we are talking of the Lyapunov equation in the context of testing the controllability, because if you recall that the Lyapunov equation, we have studied while stating the stability result that Lyapunov equation is slightly different from this equation ok. Though both of them are the Lyapunov equation because, they can be represented in either way.

Moreover, the unique solution to this; to this Lyapunov equation is given by this W and this is the reachability Gramian computed over the limit t 1 minus t naught tending to infinity again this is in the continuous time and this is in the discrete time.

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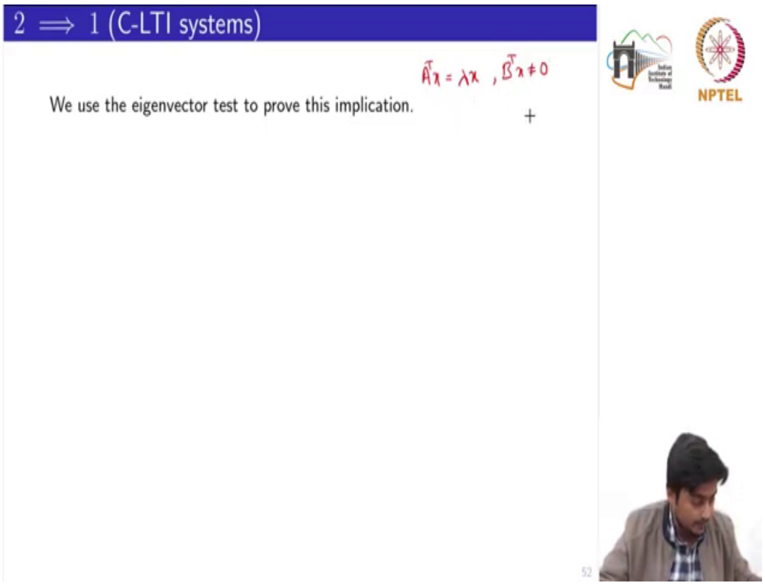

So, we shall see the prove. So, here the first statement is basically that the system is controllable and the second statement that W is a solution to this equation which is unique and positive definite ok. So, we will show the proof for the continuous time for the discrete time systems it can be prove in a similar way.

So, first of all, we will show that the second implies one that f W is a solution to that Lyapunov equation implies, that the system is controllable ok and we use the eigenvector test to prove this implication. So, just to recall for the eigenvector test that associated to lambda, there is an eigenvector of A transpose x, which satisfy this equation and does not satisfy or does not satisfy that B transpose x is equal to 0 ok this is was the eigenvector test.

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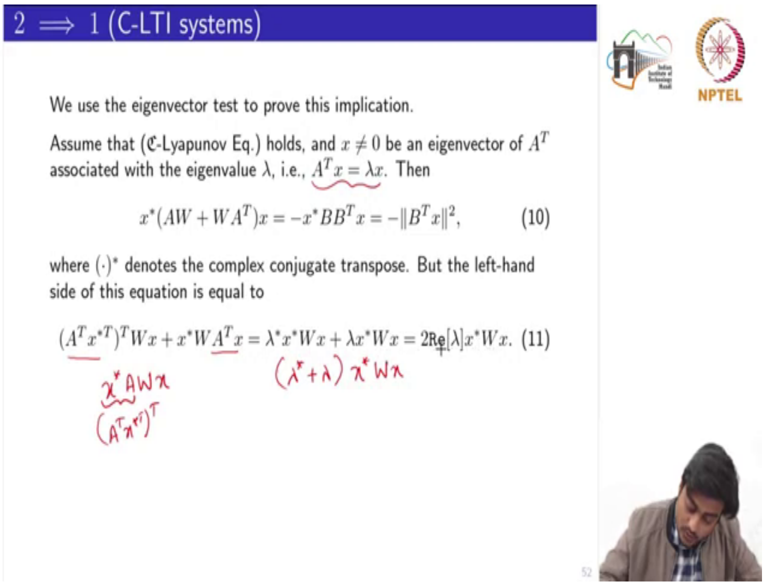

So, we start with the assumption that the Lyapunov equation holds, which is also this implication. And x is a non 0 vector or eigenvector of A transpose associated with the eigenvalue of lambda that is, it satisfies a transpose x is equal to lambda times x ok.

Then, we can write if you see the left hand side of the Lyapunov equation, we can use the, which is also a matrix. And it is an assumption already in this implication that W satisfy this equation that equation ok. So, I can write that matrix in the quadratic forms, which is x transpose this weight matrix into x, which is nothing, but equal to minus x complex conjugate transpose, BB transpose into x ok. Again using the property of the norm, I can replace this whole quantity by the squared know of B transpose x.

So, we will play with this quadratic form. So, see from at the left hand side that the first term we would have is x star A W x ok. Let us write this is for the clarity that we would have x star

A W x ok. Now, this term I can write as A transpose, x star transpose and its transpose only this term. Because it would become x star into A which is nothing but this one ok.

So, by just taking the transpose I replace this part the rest of the part W x remains as it is plus we would have x star W A transpose into x is equal to. Now, using this equation we have A transpose x is equal to lambda times x. So, I have replaced this part and this part by lambda times x using this equation.

So, here we would have lambda star x star, because lambda is an eigenvalue and taking the and it would be a complex number also ok. Lambda star x star into W x plus lambda into x star W x. Since x is lambda is a scalar, I could write this lambda star plus lambda into this quadratic form. And this is nothing, but twice the real part of that eigenvalue ok. The eigenvalue is already real then it stay as it is otherwise the imaginary part would go away ok. So, finally, I would have twice into real part of that eigenvalue into that quadratic form.

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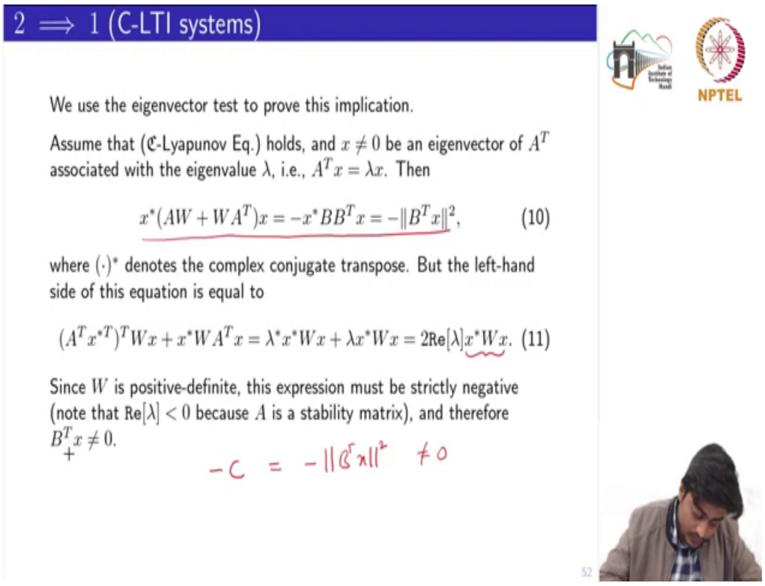
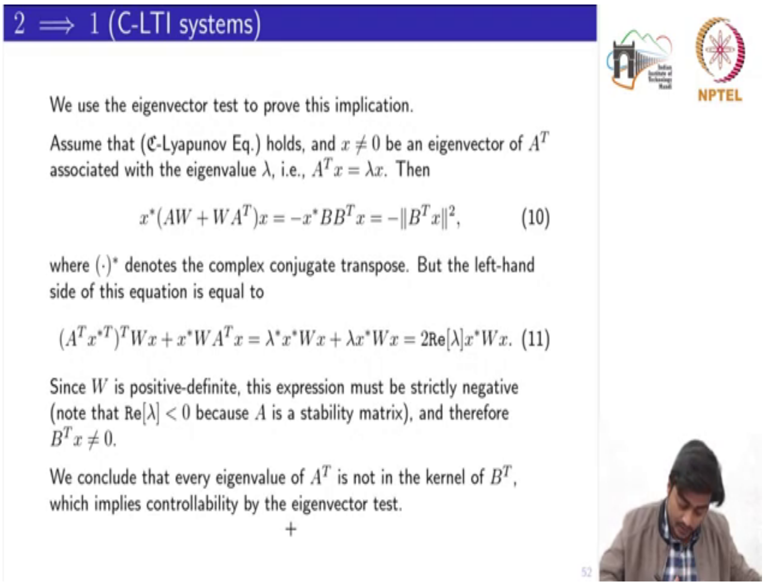


So, since W is positive definite this expression, which is given in the equation number 11 must be strictly negative, why? Here, we have x star into x W is already positive definite. So, this part would definitely be positive. Now, here we are taking only the real part of the eigenvalue of lambda and since we had there is an assumption that A is a stability matrix. So, the eigenvalue must definitely be on the left hand side. So, this term would definitely be negative ok. If it would be negative then it will never be equal to 0, because on the right hand side we have minus of norm BB transpose into x.

So, if let us write this just for the clarity that on if I see this equation on the left hand side, we have just shown that it must be strictly negative. So, it must be some negative C, where C is a positive number ok. This I can write as B transpose x square ok, meaning to say that B transpose x would never be equal to 0. It would be equal to 0 if an only if this is equal to 0 which is not right. (Refer Slide Time: 08:27)



So, we have just shown that there is an eigenvector of x, sorry there is an eigenvector x of A transpose for which this B transpose x not equal to 0 is satisfy ok. Which implies that the system is controllable because with that eigenvector test we have the if an only if condition for the controllability.

Now, for the other way implication that the system is controllable implies that the that W is a solution of the Lyapunov equation being the unique and positive definite. So, we have assume that the A B is controllable.

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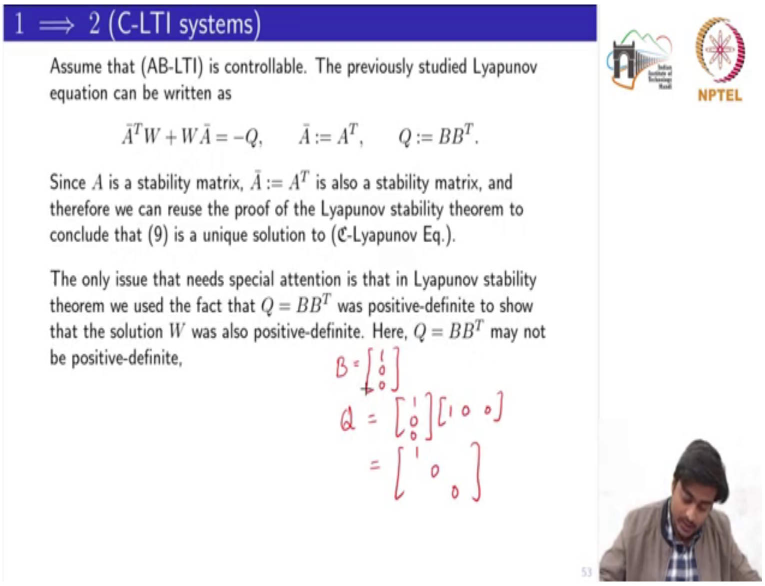

So, this previously studied Lyapunov equation is this one. The lyapunov so, this previously studied is the Lyapunov equation we have studied while discussing the stability results in terms of Lyapunov. So, that equations basically given by this one.

So, this equation if I see that equivalence between this equation and the Lyapunov equation, we have introduce in the in this result, yeah this is just a parametric form in terms of A and B matrices. So, if I replace this A bar here by A transpose and this Q matrix by BB transpose, I would obtain the Lyapunov equation, which we are talking about in this result ok.

So, here since A is a stability matrix, A bar would also be stable because taking the transpose would not change its eigenvalues ok. The location of the eigenvalues would remain the same and if a is a stability matrix then a bar would definitely B. And therefore, we can use or use the proof of the Lyapunov stability theorem, which says that given a matrix a positive definite

Q matrix. If there exist a solution of solution matrix W being unique and positive definite satisfying this equation. Then the eigenvalues of this A bar would strictly be on to the left hand side ok.

So, we can use that result to prove that W would be a unique solution to this Lyapunov equation what we are speaking in the context of controllability. But now, there is a slight difference here, that here Q is basically given by B into B transpose ok. So, now, B could be any matrix let suppose, if my B is a vector.

Just for an example, which says that we have a single input and if I compute this Q which is 1 0 0 and B transpose 1 0 0 it would be 1 0 0 right. And the off diagonal all the of diagonal elements are also 0. So, here it is not necessary that Q is also positive definite; Q here could be positive semi definite also right. So, we just need to take care of that thing; that here because Q is now having a specific form of B into B transpose, it is not necessarily to be positive definite.

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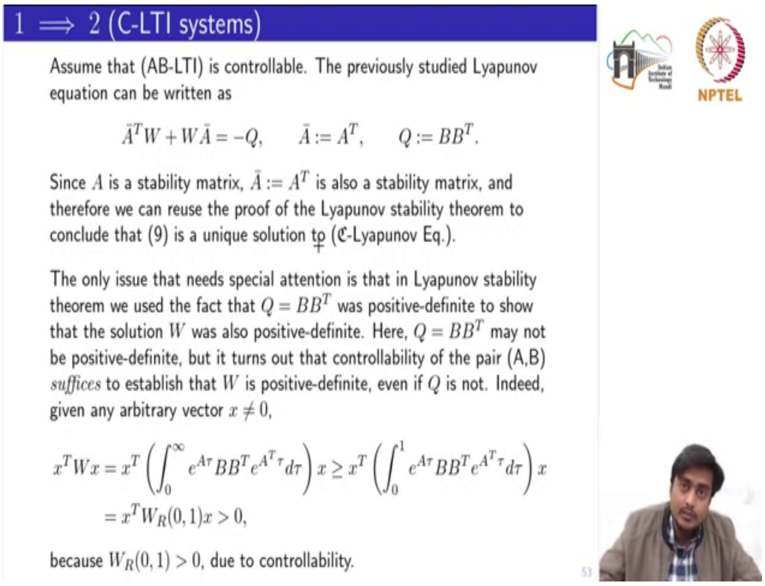

But it turns out that the controllability of the pair A right, which is already an assumption. That the controllability of the pair A B surfaces to establish that W is positive definite even if Q is not, it is important here. So, let us see how.

So, indeed given any arbitrary vector x, which is not equal to 0 and I write this quadratic form x transpose W x ok. Now, W I have use the same W which has been given in the result. So, we need to prove that W is unique and positive definite and satisfy this Lyapunov equation. In spite of knowing that Q is not necessarily a positive definite matrix ok, so we have this one. Now notice here, that here we are computing the integral from 0 to infinity ok.

Now, if I and that is nothing, but a non square; it is nothing, but a non square with inside with the integrate which meaning to say that it is always be positive. So, if I am computing the integral over.0 to infinity and if I compute that integral from 0 to 1 I would defiantly have this value would be greater than; if I compute that integral over 0, 0 to 1. Because I know that the integral is a positive quantity ether it would be equal or would be greater, whenever I compute this integral over 0 to infinity. And this is nothing, but your reachability Gramian x transpose W R from 0 to 1 into x. And being the controllability its says, that the reachability Gramian must be positive definite.

So, I can have that this greater than equal to 0 because this result is already known to us that the controllability Gramian should definitely be positive definite. And it is already an assumption that the A B is controllable, which implied that this condition would definitely hole. If this hole, then it means x transpose W x is a positive definite ok. So, we are just shown that W which, we have used as this one. Which is nothing, but the solution of the Lyapunov equation is positive definite.

Now, the second one is the uniqueness, for the uniqueness we could use the previously stability result. Where the idea behind showing the uniqueness is, we could use two matrices. For the same Q we can define two matrices, let us say W and W bar. Now, the rest is to show that W and W bar are not different, but they both are equal, which shows or which stabilizes the uniqueness. So, that part we would not be discussing again, but you can refer to the Lyapunov stability theorem part.

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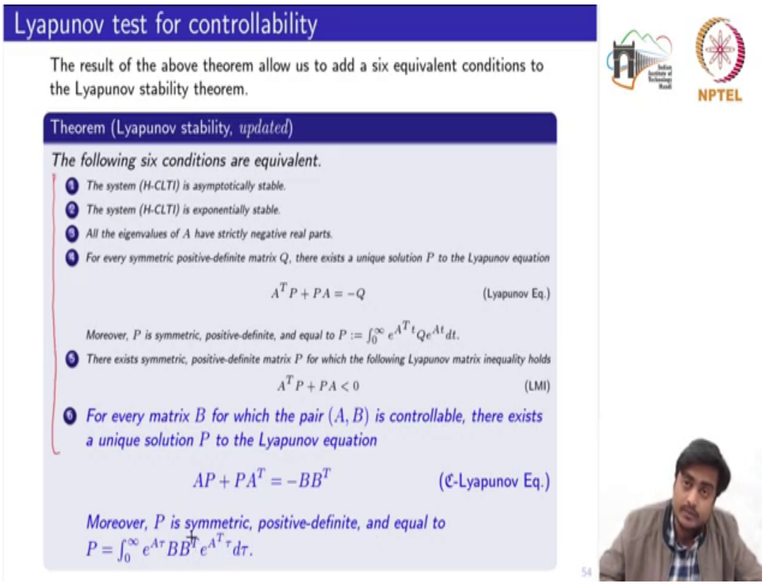

So, the result of this theorem, in fact allow us to add a 6th equivalent statement to the Lyapunov stability theorem. So, now we have; we have just updated that Lyapunov stability this result that the system this homogeneous CLTI is asymptotically stable, thesystem is exponentially stable.

So, the first five statements in fact, we have already shown the equivalence in the stability week, that all the eigenvalues have strictly negative real parts for a given Q, P is a positive unique positive definite unique solution to this Lyapunov equation. So, this Lyapunov equation, we meant by the previously studied Lyapunov equation. And the fifth statement that this matrix is negative definite ok.

Now, we have added this sixth statement which also includes the controllability. So, all these sixth statements in fact, stabilizes both the results stability plus controllability. That for every matrix B for which the pair AB is controllable, there exists a unique solution P to this Lyapunov equation in the context of the controllability. So, that is why we have name these two Lyapunov equations are differently.

The first one which is the stander one in the context of Lyapunov stability, the second one in the context of the controllability ok, other it just a representation otherwise both of them remains same. More over P is symmetric positive definite and equals to this one um. Here we are just replace Q by BB transpose.

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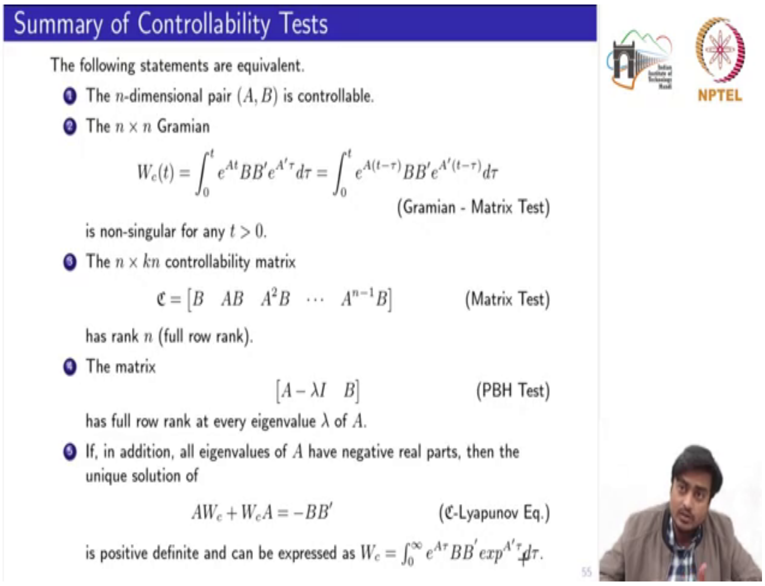

So, now the controllability test is finished, this is just summary of all the controllability tests we have studied that the following statement are equivalent the n dimensional pair A B is controllable, which is equivalent to saying that the square matrix W c which we have define as the Gramian, which can take either this from or this from. If we noted that the here we have use the t and t minus tau. This part you can prove it quite straightforwardly.

So, this is we have define as the Gramian matrix test. The Gramian matrix test we have also studied in the contest of linear time varying systems. The third statement says that the n cross k times n controllability matrix, where k is the dimension of the control input the. This matrix has rank n or say that it is the full row rank this is again a matrices. So, the all these five results are in fact, the LTI case. Matrix test we have the non singularity of the Gramian in for the linear time varying systems.

But the matrix test where we have explicitly computed the matrices was weaker result for the linear time varying case, but for the L T I system it is a strong result. That again we have the if and only if condition for the non singularity or the full row rank of this controllability matrix. Four you can also say the eigenvector test or the matrices because both of them are related that the matrix this A minus lambda I for all the eigen values of the matrix A. B has full row rank at every eigenvalue of A which we have said that this is the PBH test; Popov Belevitch and Hautus test equivalent to that we had additional condition that there is no eigenvector of A transpose in the kernel of B transpose.

The 5th if in addition all eigenvalues of A have negative real part, then the unique solution of this equation, which is the Lyapunov equation in terms of the controllability test is positive definite and can be expressed at this one. So, this finishes the controllability all test related to the controllability.

Now, we have same additional results, which stabilizes the invariance and the decompositions of the controllability, because so for what we had seen that, if this rank condition is stratified that the system is controllable. Now, we also need to answer that if the rank condition is not satisfy, then under what circumstances we can still say that our system is controllable.