Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture – 21
Tests for controllability – III

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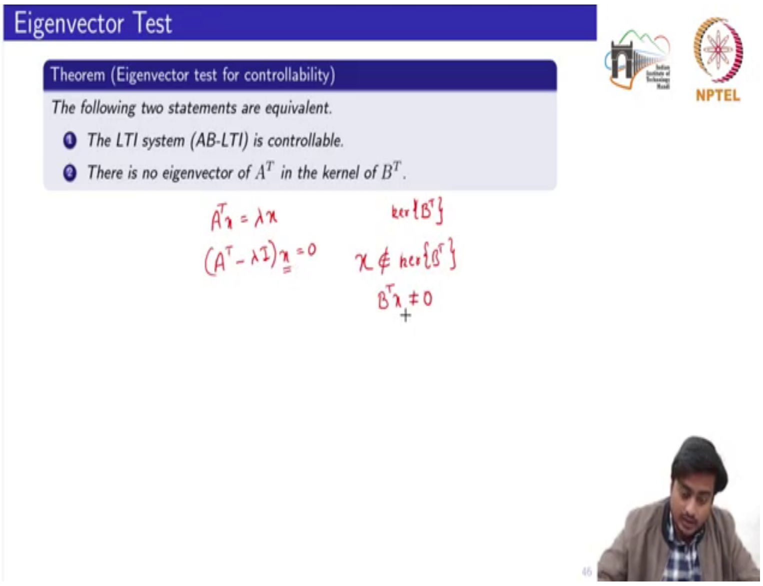

So, after introducing the two properties of the invariant subspace, we will use the eigen we will go through the eigenvector test for the controllability. So, this is the result that the LTI system that is the pair AB of the LTI system is controllable if and only if there is no eigenvector of A transpose in the kernel of B transpose. So, first of all try to understand that what is written in the statement and then, we will go through the proof of this statement.

So, it says there is no eigenvector of A transpose. Now if there is an eigen let say x be that eigenvector and saying x as an eigenvector, it would be written as A transpose x equal lambda of x which is associated with some eigenvalue lambda which you have been seen as the computation of the eigenvalues also A minus lambda I to x equal 0, right.

Now, this eigenvector which is given by x here it should not be in the kernel of B transpose. Now kernel of the B transpose is basically a linear subspace. So, let us write this kernel of B transpose, ok. This is the subspace and it says that this x should not belong to the kernel of this B transpose which is to say that the B transpose x should not be equal to 0 because if this eigenvector x belongs to the kernel of B transpose, then in that case we would have B transpose into x equal to 0, right.

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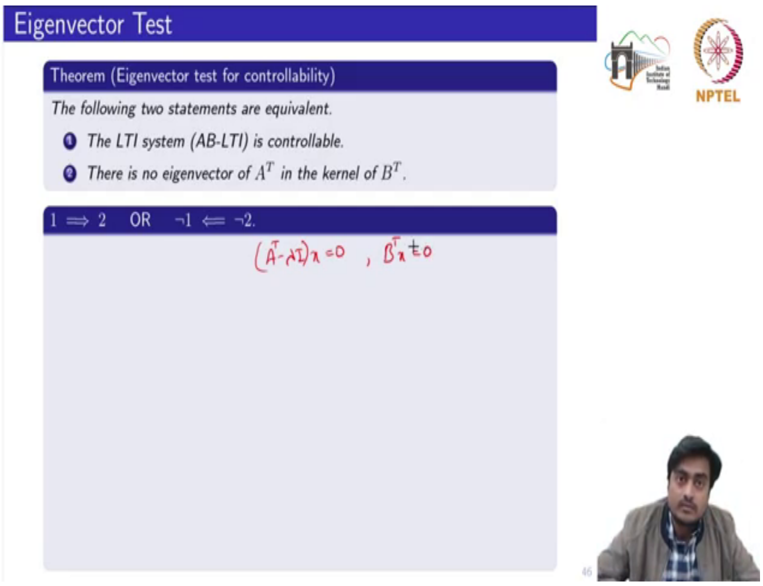

So, what we need to show here that there is no eigen or let us say there is an eigenvector x which satisfy this equation, but does not satisfy this equation, ok. So, this is what we want to show here. So, let us see the proof which says one implies to that the AB pair is controllable. If the AB pair is controllable, then it implies that there is no eigenvector of A transpose in the kernel of B transpose.

Now, we will prove this through negation that is the reverse implication that if there is an eigenvector of A transpose in the kernel of B transpose, then it implies that the system is not controllable ok. Now, saying that there is an eigenvector of A transpose in the kernel of B transpose implies that A transpose minus lambda I x equal 0. This is the first equation. In the second equation is this one. So, both the equations are satisfied.

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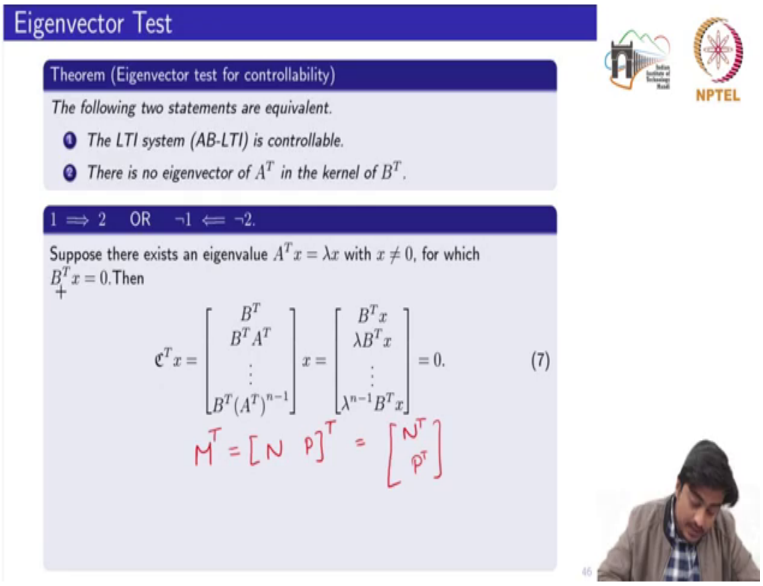

So, suppose there exists an eigenvalue lambda associated with the eigenvector x with non-zero eigenvector for which B transpose x equal to 0, ok. This is the negative statement of the second one. So, let us write the multiplication of the controllability. The transpose of the controllability matrix into x.

Now here I have used the property of the transpose of the matrix let us say we have some matrix M which is given by some N and P ok. Now if I take the transpose of this one, it would be equal to N transpose and P transpose and so, this the same property I have used here because here we are taking the transpose of the controllability matrix and it would become B transpose B transpose A transpose up to B transpose into A transpose to the power N minus 1 into x.

Now, I have taken this x inside this these block matrices and we would have the first term as B transpose into x. The second one would be B transpose into A transpose into x. Now A transpose x is already equal to the lambda x so, we lambda x scalar. So, we have replaced here by lambda into B transpose into x ok. Similarly up to lambda to the power N minus 1 B transpose x, ok.

Now, if you pay attention here that we already have this assumption that B transpose x is equal to 0. So, all these parts would be it would become a zero vector which is in 0. So, what does this mean? That x is an eigenvector which is in the kernel of the C transpose, the controllable, the transpose, the controllability matrix.

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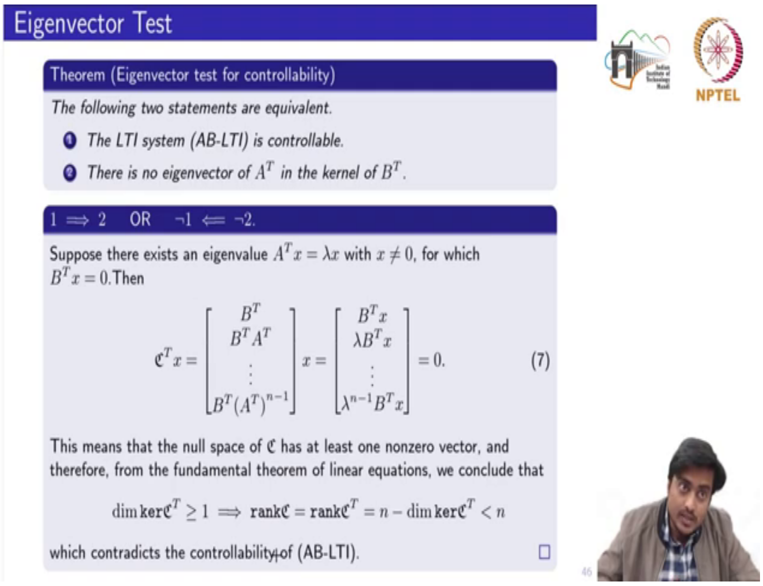

Now, it means that the dimension of that subspace would be at least one because we have already assumed that x is a non zero vector and for a non zero vector, it satisfies that the C transpose x is equal to 0 and it that is say that x is also in the kernel of C transpose and which is non-zero. So, the dimension or the nullity of that subspace is at least 1 or greater than equal to 1.

Now, if we use the fundamental theorem of linear equation I could write the if you remember that we have nullity plus rank is equal to the number of columns of that matrix. So, here we have the rank of C transpose is equal to the number of columns which is n minus the dimension of the kernel of C transpose which is the nullity.

Now, here we already know that the dimension is at least having one value. There is non-zero vector meaning to say that the rank of the C transpose would be definitely less than n, ok.

Now we know that if we take the transpose of the matrix and if we compute the rank, the rank one change meaning to say that the rank of the controllability matrix would definitely be equal to the rank of the C transpose and the rank of C is less than n mean which contradicts the controllability because for the controllability we should have the rank of the controllability matrix to be equal to n, right. So, this proves the first implication.

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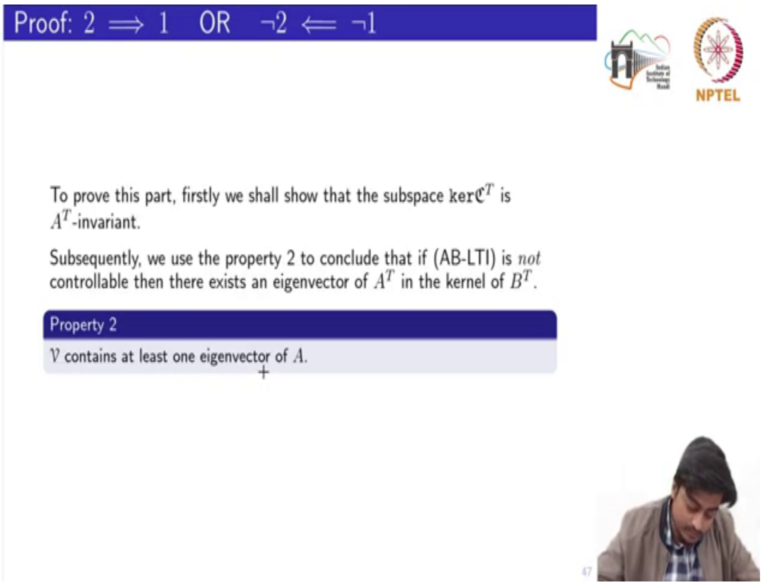

Now, the second one if there is no eigenvector or that is to say or the negative statement that if the system is not controllable, then it implies that there is an eigenvector of A transpose in the kernel of B transpose. So, we will prove this part by using two arguments. The first one that first of all we will show that the subspace, the kernel of C transpose is A transpose invariant. Now using the property 2 which is written over here that the subspace v which is already a variant contains at least one eigenvector of A of that matrix.

So, then we use the property to finally conclude that if the pair A B is not controllable, then there definitely exists an eigenvector of A transpose and the kernel of B transpose.

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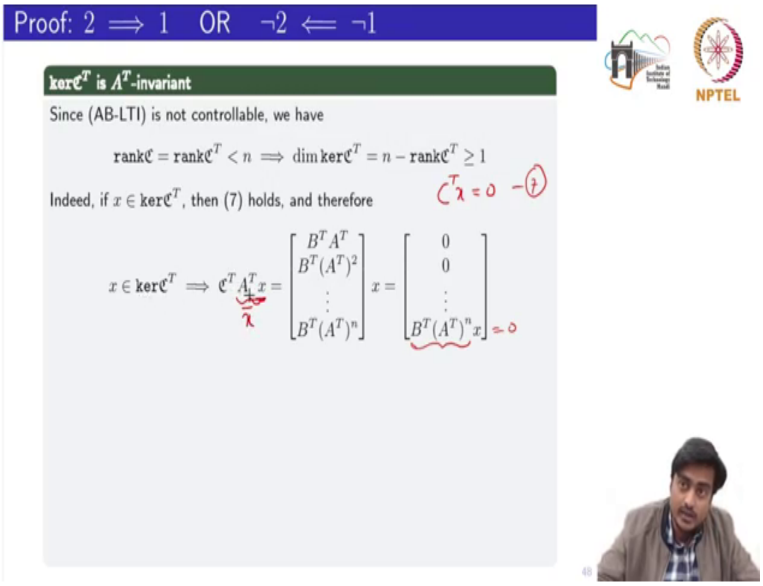

So, let us see. So, since we know already with the assumption that the pair AB is not controllable, we would have the rank of C which we can write also equal to the rank of the C transpose should definitely will less than n. If it is equal to n, then it would be it would have be become controllable implies that the dimension of the kernel of C transpose would be greater than or equal to 1.

Now, let us see if x belongs to the kernel of C transpose. If x belongs to the kernel of C transpose, then the equation 7 hold. This equation 7 is nothing, but the C transpose into x is equal to 0 which we had seen earlier. So, that question is number 7 and therefore, we could write that x belonging to the kernel of C transpose implies that let us denote this vector let us say denote this vector by x bar.

If you recall the property of the invariant subspacs, it means that if x belongs to the subspace kernel of C transpose and if this x bar which is given by A transpose x also belongs to the kernel of C transpose, then it implies that this subspace which is the kernel of C transpose is A transpose invariant, ok.

So, I can write this one is also the C transpose into A transpose into x where this A transpose x would be multiplied by every element of the block matrix. So, we would have B transpose into A transpose. So, the power of the A transpose would increase by 1 in all the block matrices, ok.

Now, earlier we have already seen that the all these elements except A to the power t sorry A transpose to the power n x is not necessarily equal to lambda to the power n into x. If it would have been, then it could also be 0.

So, we only need to show now that this part is equal to 0. If we are able to show that this part is equal to 0, then it means that C transpose into x bar is also equal to 0 right which by which we will finally conclude that this subspace is a transpose invariant, ok. So, let us see.

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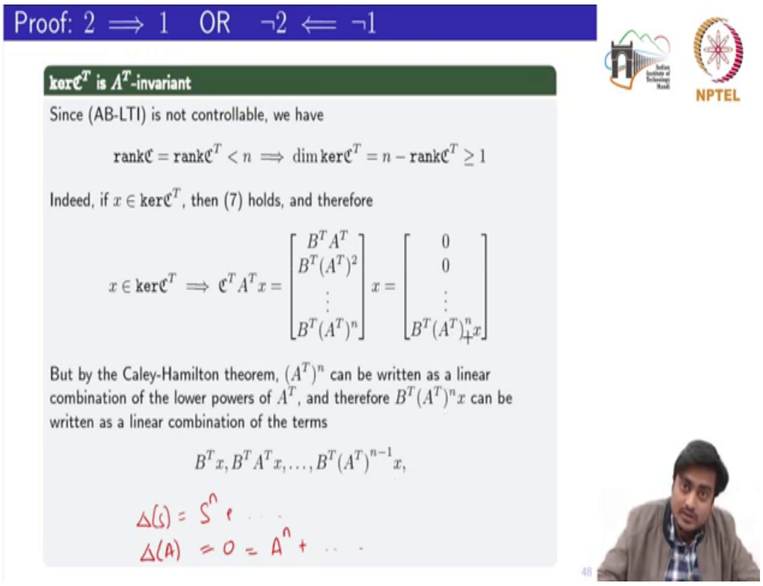

Now, by using the Caley Hamilton theorem if you recall the Caley Hamilton theorem for given matrix A if we compute the characteristic polynomial by delta S S to the power n plus these elements, then it implies that the polynomial by put by replacing S by A would be equal to 0.

So, now using this theorem we can write A transpose to the power n as a linear combination of the lower powers of A transpose, right. Why? Because it is equal to 0. So, it would contain sum A to the power n plus the lower a terms. Now if it is equal to 0, I can write A to the power n as the linear combination of the lower the terms. So, using this argument we can use this term and since we know that all these lower elements are already 0 which implies that this term would definitely equal to 0, right.

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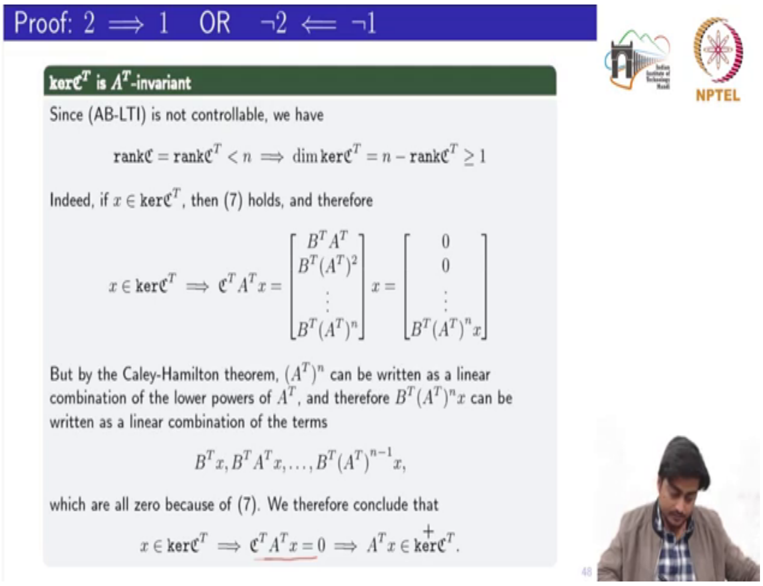

So, this is what it says. So, x belonging to the kernel or this subspace we have shown that this subspace is A transpose invariant because this equation has been satisfying, meaning to say that this vector A transpose into x also belong to that subspace, ok.

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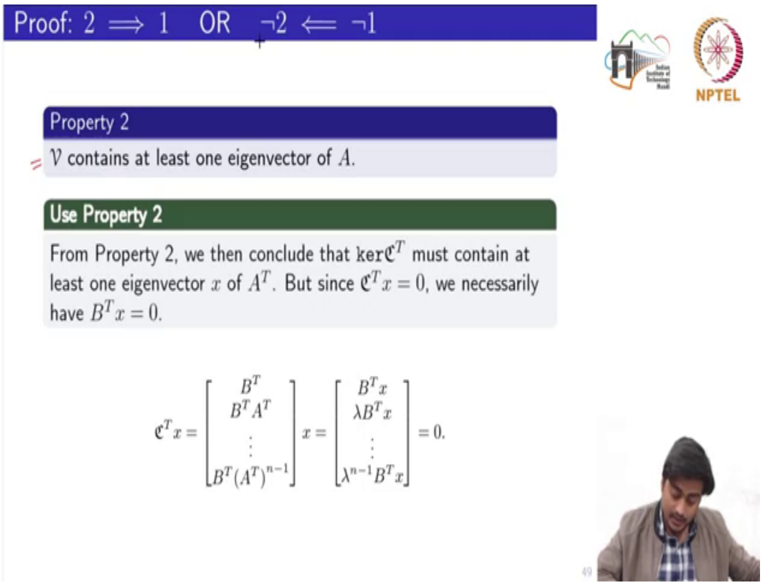

So, they shows that the subspace A transpose invariant. Now using the property 2 we then conclude that the subspace. So, let us visualize this v which is already A transpose invariant which is kernel of C transpose.

So, kernel of C transpose must contain at least one eigenvector of A transpose because this subspace is A transpose invariant if it contains one eigenvector of A transpose and since C transpose x equal to 0 implies this would only be possible if B transpose x is equal to also 0 which says that the second statement is also nullified or we have the opposite of the second statement.

So, this completes the overall proof of the controllability of the AB pair in terms of the eigenvector of the A transpose not belonging to the kernel space kernel of B transpose.

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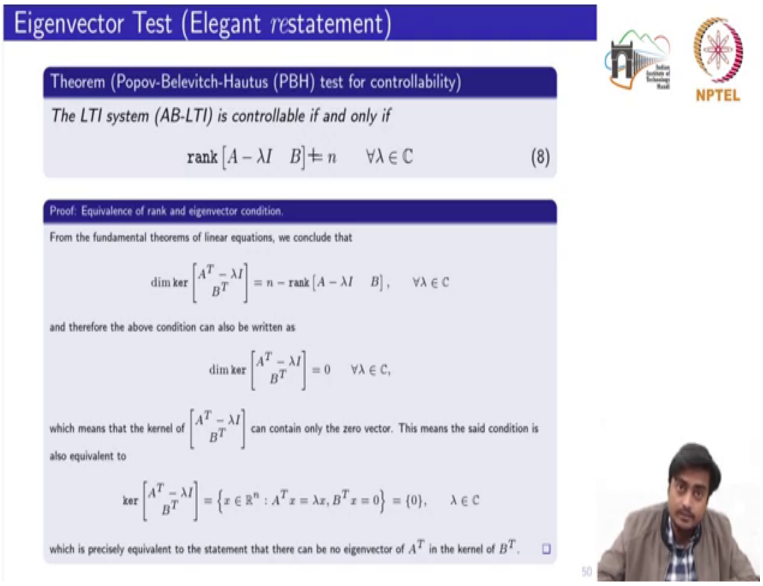

So, here there is another result which is I would say the elegant restatement of the result what we had just discussed. So, it says that the AB pair is controllable if and only if the rank of this matrix A minus lambda I B is equal to n for all lambda belonging to the set of complex numbers, ok. Now this test is also called the Popov Belevitch and Hautus test or in short the PBH test.

So, this is just the restatement of the statement what we had said that the eigenvector of the A transpose should not be in the kernel of the B transpose ok. So, we can see a quick proof or the equivalence between these two statements. So, here we will show the proof by establishing the equivalence between this rank condition and the eigenvector condition which we had seen in the previous result.

So, now again using the fundamental theorem of linear equation, we conclude that the dimension of the kernel of this matrix I can write as equal to n minus rank of A minus lambda I B here. In fact, I have used two properties; the one that the orthogonal complement property has also been used here.

So, now if we suppose that this rank condition has been satisfied, so we would have n minus n would be equal to 0 for all lambda belonging to the set of complex numbers. Now again using the same argument we can say that the kernel of this matrix can contain only the zero vector because the dimension is 0. If the dimension is 0, then it would contain only the zero vector which is equivalent to saying that the kernel of this matrix which is given by the set x belonging to the n dimensional set of real numbers.

Both these conditions have been satisfied that is A transpose x into is equal to lambda x and B transpose x is equal to 0 and it should be equal to 0 because it contains only the zero vector; only the zero vector which is precisely equivalent to the statement that there can be no eigenvector of A transpose in the kernel of B transpose ok. This finish is the proof.

So, for the eigenvector test we have two proofs two tests given the matrices A and B, either we ensure that there is no eigenvector of A transpose in the kernel of B transpose or by computing the eigenvalues of the matrix A and for all the eigenvalues of A if this rank condition has been satisfied and this it says that for all the eigenvalues belonging to the set of complex numbers.

Now these eigenvalues could be stable eigenvalues and could also be unstable eigenvalues. Now if this rank condition is satisfied for all the eigenvalues meaning to say that the system is controllable ok.