Linear Dynamical Systems
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Tutorial on Controllability: Part I
Lecture – 20
Tutorial - 3

So, now we will be starting with the Tutorial of Controllability week. So, the controllability week is discussed during the theoretical lectures. It is finely divided into two parts: part I, part II, which would be covered over a period of two weeks; that is week 3 and week 4. And today we will carry out the tutorial for the part I and second part would deal with the tutorial controllability part II covering the rest of the lecture slides.

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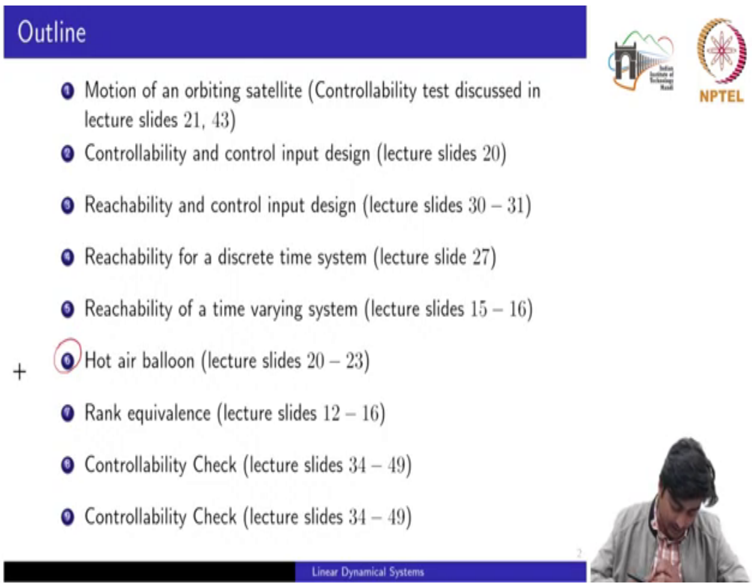

So, this is the outline for the tutorial problems. So, the problems which we would see today would consists of the controllability test from the lectures slides 21 and 43. We would also have the in control input design, the reachability for the discrete time, and the time varying systems. One practical example is considered in problem number 6 which is the hot air balloon. Then, the rest we do with the controllability check under different conditions and also the rank equivalence condition.

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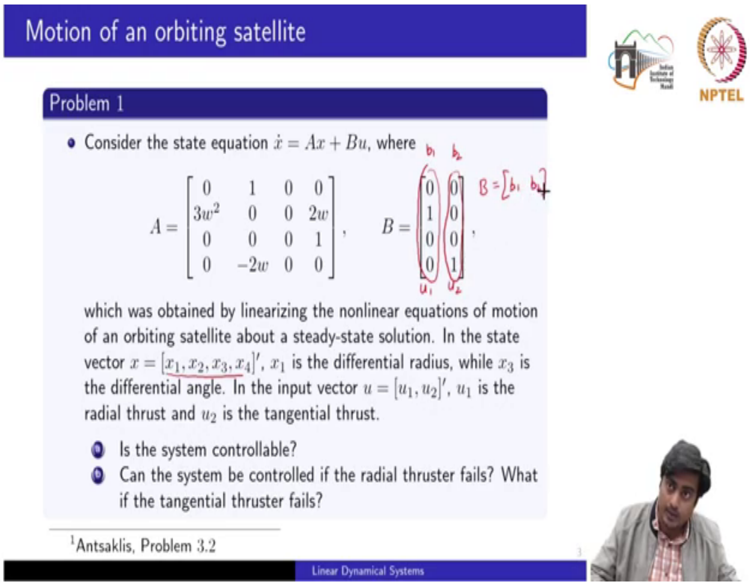

So, this problem 1 deals with given the matrices A and B in the linear time in variant system, which obviously was obtained by linearizing some non-linear equations of motions of an orbiting satellite around some steady state solution. So, four states have been identified: x 1 to x 4 with their physical significance the non-linear equations. We have two inputs u, which is divided into u 1 and u 2 where u 1 is the radial thrust and u 2 is tangential thrust. So, the first part of the problem is we want to test whether the system is controllable or not.

Second part is quite interesting, in the sense that we want to ensure the controllability of the system under one of the inputs. Say for example, if u 1 which is the radial thrust, if the radial thrust we have lost complete control over the radial thrust. So, can we say that the system is still controllable by using only the control signal u 2 and vice versa. So, this problem was taken from the book Antsaklis the problem 3.2. So, this book was also important to the one of the references books, so if you want to go through other problems you can take a look at it. So, let us see.

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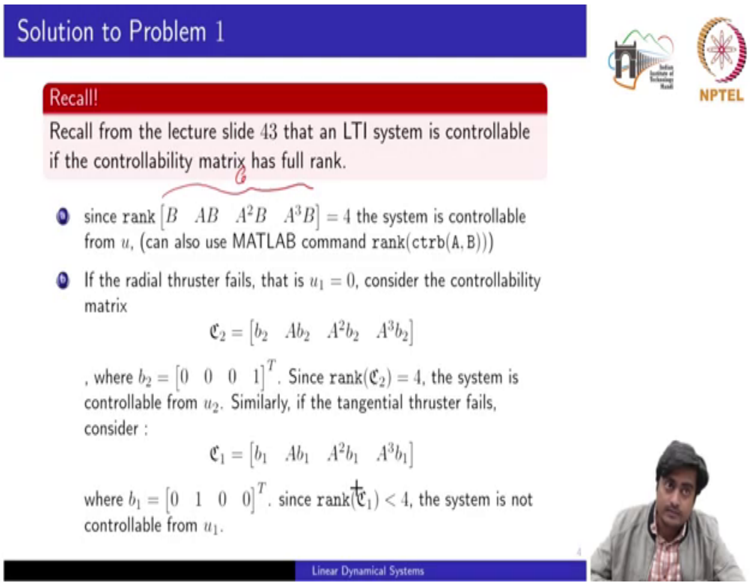

So, from slide number 43, from the lecture slide we know that this LTI system is controllable if the controllability matrix is a full rank right. So, this is the pretty much attended test, because we are already provide with that A and B matrices. So, we can compute the, first of all the controllability matrix and then we try.

So, since the dimension of the A matrices 4 cross 4, this would be your controllability matrix and it will compute the rank of this matrix you would see that is equal to 4 which is equal to the dimension of the state by system. We can also use the MATLAB command, by using this which is the rank of the controllability matrix of A and B. So ctrb is the function, is the inbuilt function in MATLAB which you can use to compute the controllability matrix even the matrix is A and B.

So, since we have identified if we are using both the controls the system is completely controllable. Now we want to test if the first input fails; mean it to say even if u 1 is completely equal to 0. Then if you pay attention to the B matrix, so this B vector is corresponding to u 1 and this one is corresponding to u 2. So, if u 1 fails then we would help the or we can label this else b 1 and b 2.

So, our B matrix is basically two column vectors b 1 and b 2 ok. Now, if u 1 fails then we do not have excess to b 1, but we have excess to b 2. Similarly, if u 2 fails then we have the excess to b 1 instead of b 2 ok. So once, u 1 is equal to 0. So, in that controllability matrix we would be using only b 2 vector instead of using the complete B matrix. So similarly, we compute the controllability matrix and then we can compute the rank then you can also compute by yourself that the rank of this material is equal to 4. Meaning to say that, even if u 1 is fails that you still you still had the power to control the system by using only the u 2 signal.

Now if the other input fails. So, you compute the controllability of matrix and we see that the rank is less than 4. Meaning to say if u 2 fails then you cannot completely control the system by using only the input u 1, ok.

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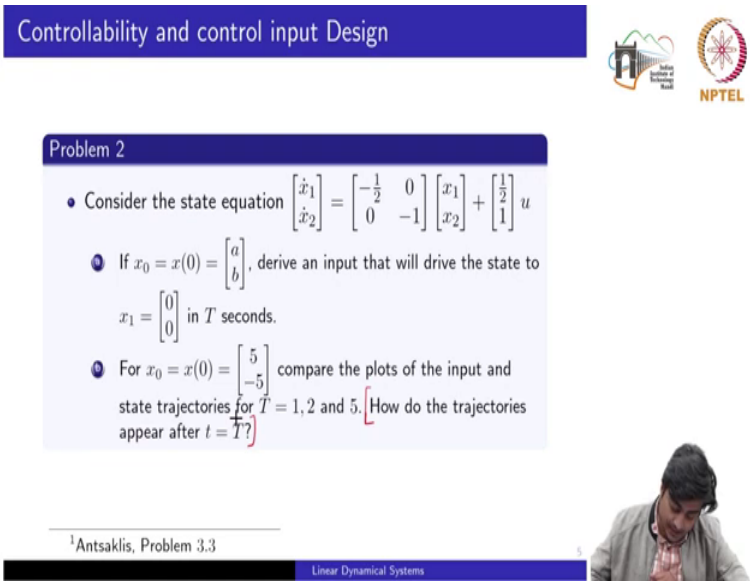

The second problem is the follow up problem is 3.2 of the same book, where we are again provided by the LTI system A and B matrices. So, given the initial conditions at T is equal to 0 a b; b we want to derive an input that will try the state to the origin in T seconds ok.

The follow up part of this; so, the first part only computes the parametrized solution in terms of a and b. The second part this a and b is taken as 5 5 and minus 5. And we do a further analysis by computing that how the input and state trajectory would behave if we specify three different values of capital T; that is 1, 2 and 5. And how do the trajectory appear after.

So, this is pretty much important part the last part, because by solving this problem we would also come to one conclusion which we have also discussed during the lecture slides. That controllability is not in fact equal to the to ensure that the system would be stable, because we here the idea if we want to find a control input that can take a non-zero initial condition to zero initial condition, but it would stay there for the rest of the time it is not mandatory.

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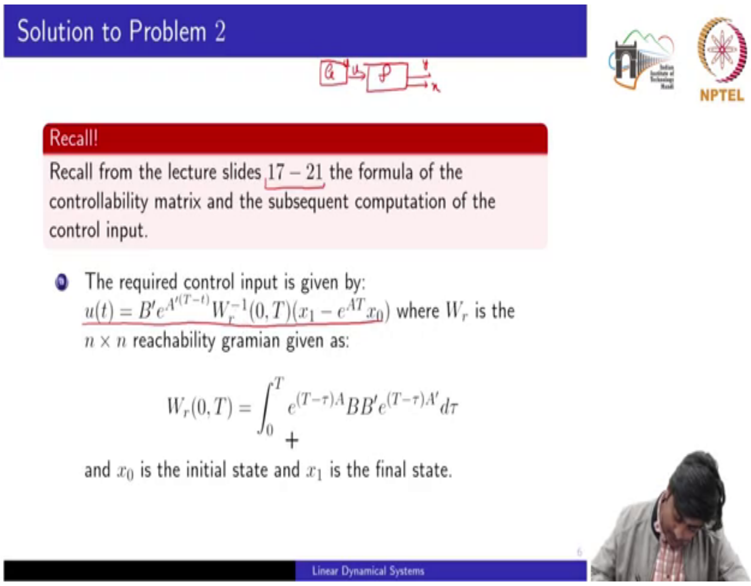

So, by looking at the lecture slides from 17 to 21 you can compute the control input by using this formula. So, in fact we have explicitly computed this formula to compute the control signal in terms of the gramian matrix. Now here one important thing to notice is that this input it is basically an open loop input. Open loop input in the sense say let say we have a plant which interacts with the input output signal is u and y and one internal state variable is x, ok.

Now, this input which has been designed here it is taking some, let say we call it the controller which is generating the u signal. Now u signal we call it open loop because this control or this controller which is another sub system is not taking any information either from y or from x, ok. So, we can compute the control input using this formula which requires to compute the gramian matrix whose formula is given here.

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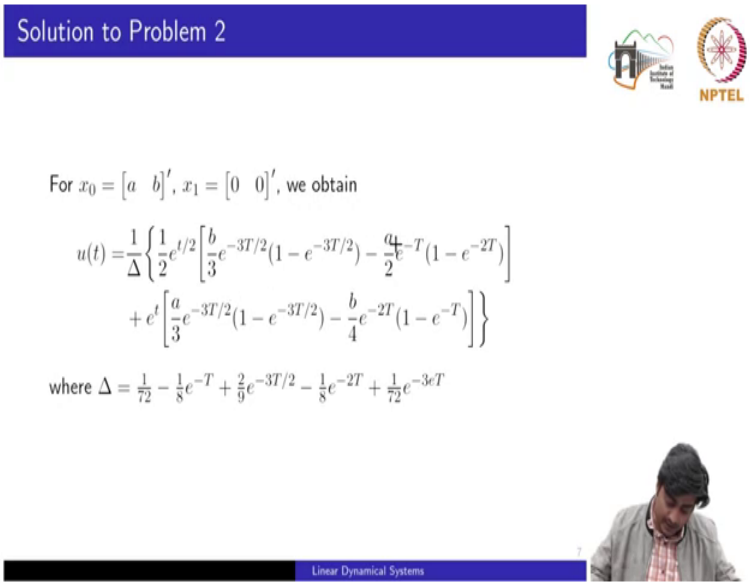

So, once all the computation is done, we put the initial conditions as a and b and the final condition at time T is 0 0. So, after putting all these equations and the parameter values you would find this complicated view, where there is delta is given by this. So, this is just simplification of solving this integration involved in the gramian matrix, but once we go to the part b which involves further putting the values of T and the initial conditions that are a and b.

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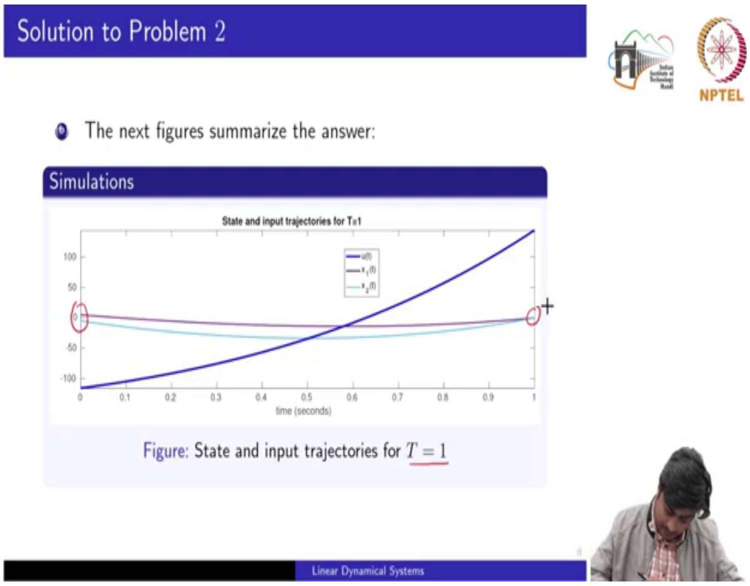

So, the first part is; so we directly plotted the figures because once you have the complete formula for the u you can also compute an analytical expression. Now once we have the analytical expression and we put it into the we give it to the plan, we also conclude that trajectories x 1 and x 2 to show that whether they are reaching to the origin in the finite amount of time.

So, here the first case is; we have taken when T is equal to 1 and the initial conditions are 5 and minus 5 ok. So, computing that control input and applying to the plan we see that in one second the state trajectory have reached to the origin ok. Because, we have already computed their system is completely controllable.

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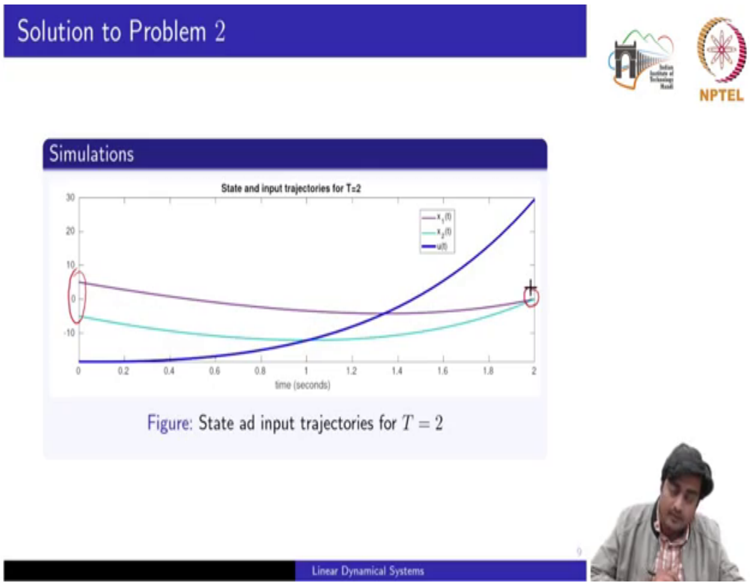

Now, if we see at T is equal to 2, this is the input signal which we are seeing in the dark blue. And again starting from the initial conditions we reach to the origin in two time seconds. So, one thing here is interesting to note that once we have larger the time from T is equal to 1 to T equal to 2; one straight thing is to notice that the control signal the energy required in the control signal is quite less than what it has been there in T equal to 1. We will also do some computation or to quantify the amount of energy used in the control signal.

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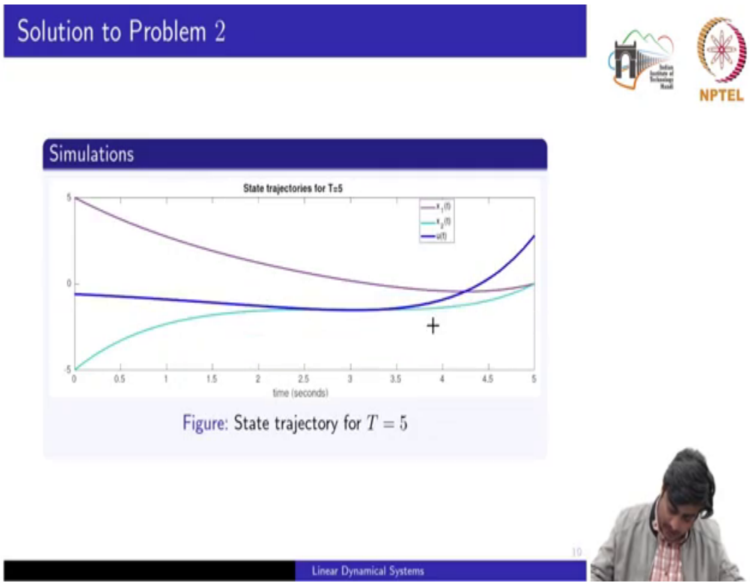

Similarly, we compute for T is equal to 5. Again starting from minus 5 and 5 we reach to the origin in 5 time seconds where the control signal is the given by this. So, since the system was controllable this these trajectories only shows that how the control and the straight trajectories behave, whenever we want to see that the initial conditions are reaching to the origin in certain amount of time.

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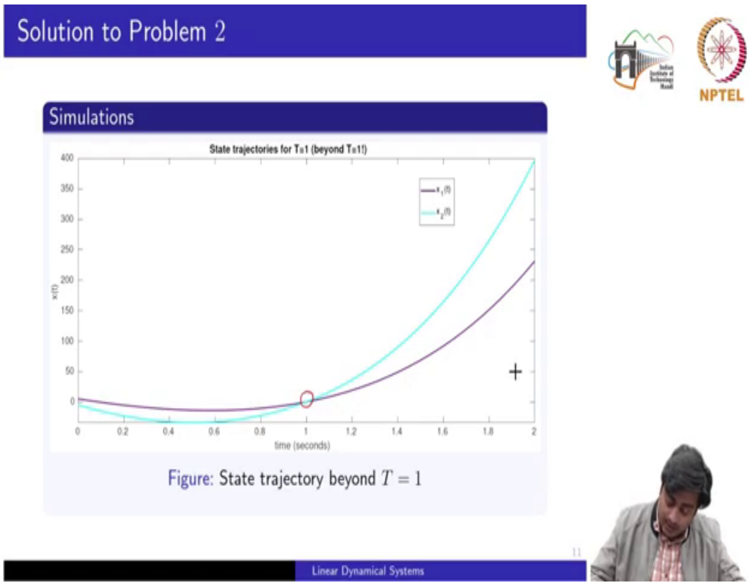

But, the interesting point here to see that; what would happen if we want to simulate the same system beyond T is equal to 1. So, up to T is equal to 1 we had seen that it has reached to 0, but if we keep applying the same input what we had computed, we would see the system basically this is to infinity it is not necessary that it will stay at 0. Now, if the state trajectories are staying at 0 until sorry, T times to infinity then it mean the system is stable system.

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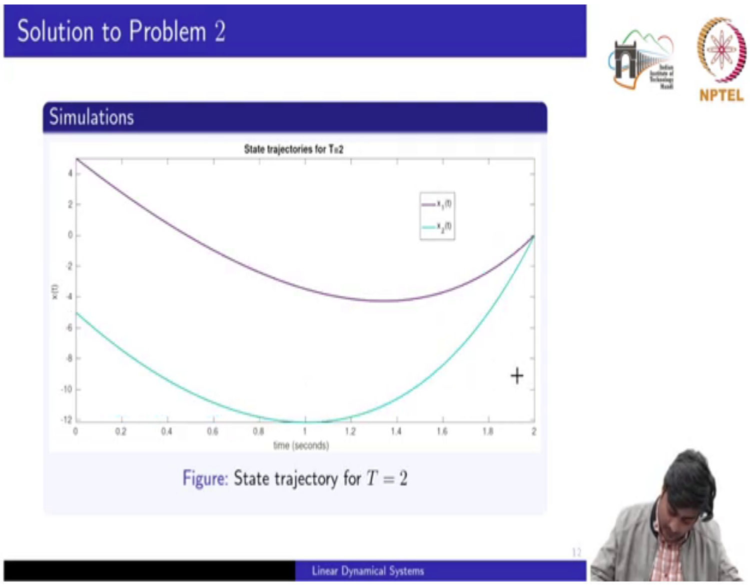

So, this trajectory is for T is equal to 2.

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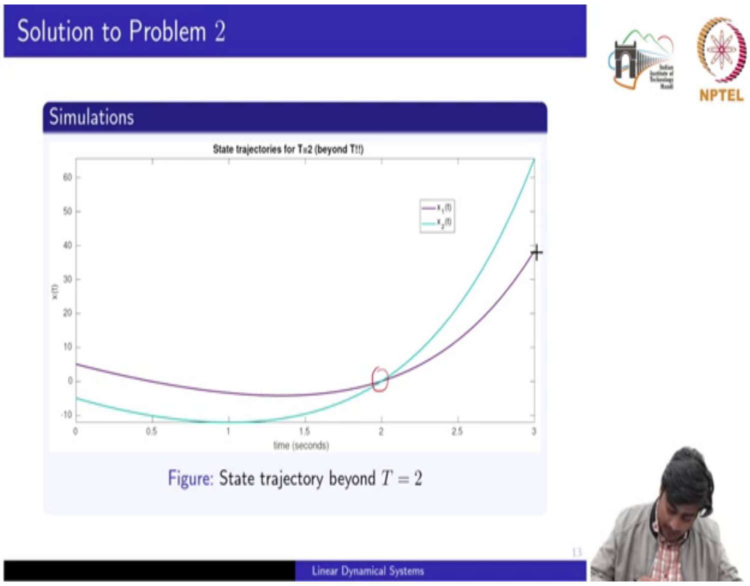

Now, it we see beyond 2, again we notice that the system is exploring towards to infinity.

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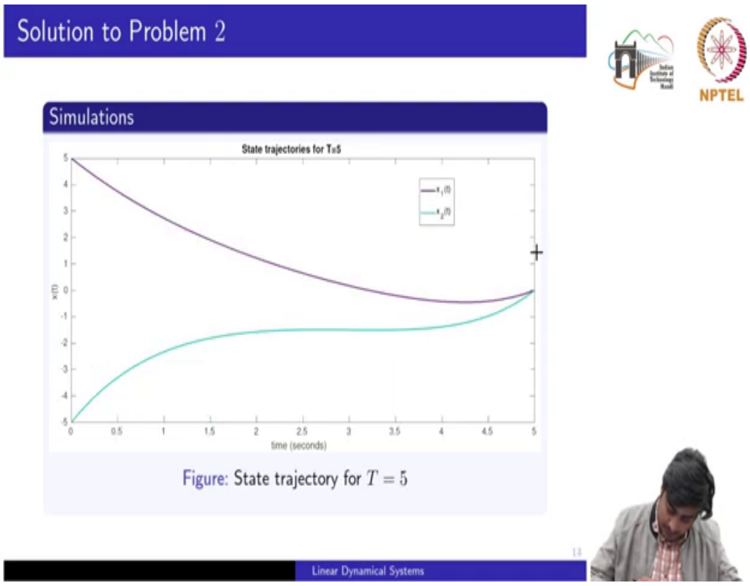

Similarly, these trajectories are T is equal to 5.

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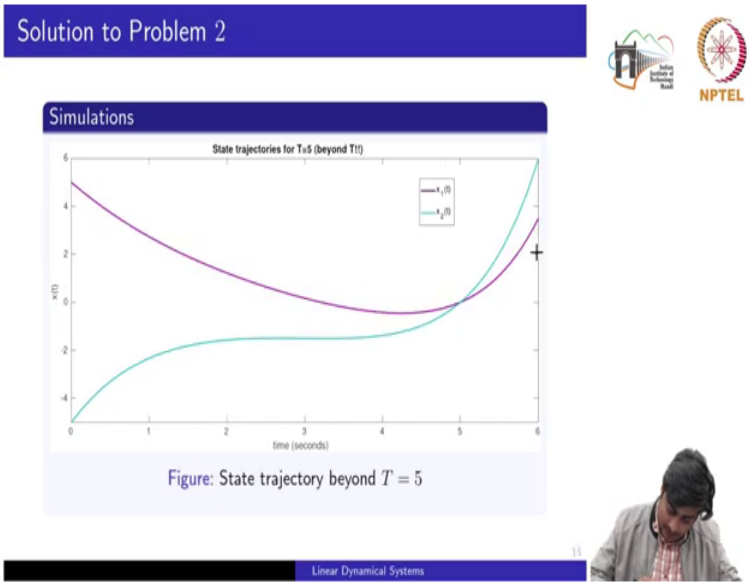

And after 5 reaching to 0 they can take any path. Although, the system was unstable that is why for the same control input what we had computing the trajectories are reaching towards to infinity.

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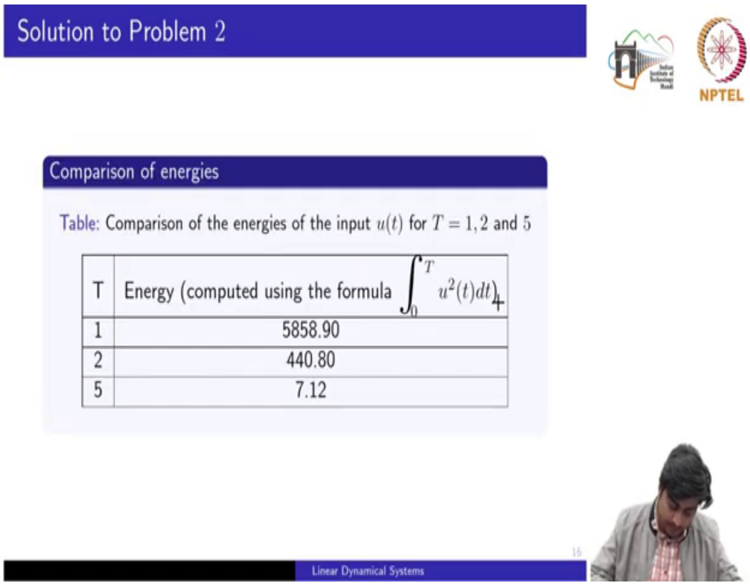

T	Energy (computed using the formula $\int_0^T u^2(t)dt$)
1	5858.90
2	440.80
5	7.12

So, you can also compute the energies of the input signal. This you can also visualize by the amount of efforts the actuator has to put into to steer the trajectories from non-zero initial condition to zero initial condition. So, if the time we have taken is less; meaning to say that the control input has to be at quite fast in comparison to if you want the system to be a bit slower.

And, because we here we need that in T is equal to 5 seconds the system can reach from non-zero initial condition to zero initial conditions. Which means that it would take more time and the control actions would be slow. This is also the reason that the energy for T is equal to 5 is may less than what energy is required at is equal to 1 ok.

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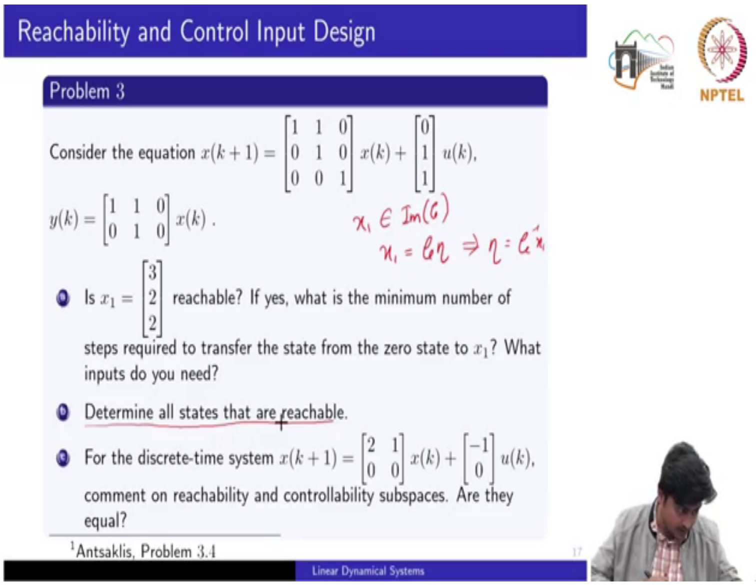

The problem 3 is again the subsequent problem of the same book. Where, now we are considering discrete time system with a b c matrices are given by this. So, the first part of the problem is we want to know whether x 1 which is given by this is reachable. If you recall the definition on the reachability that from the zero initial condition we want to reach to some non-zero initial condition and here we have specified that condition.

So, starting from the origin we want to determine that whether there exists any control input such that we can reach to this to this x 1 at t 1. So if yes, what is the minimum number of steps required to transfer the state from the zero state to x 1. Since this is a discrete time system we can compute the number of steps also, and what inputs do you need right. And, the second part is determine all the states that are reachable. Third is a separate problem which we will come on to later.

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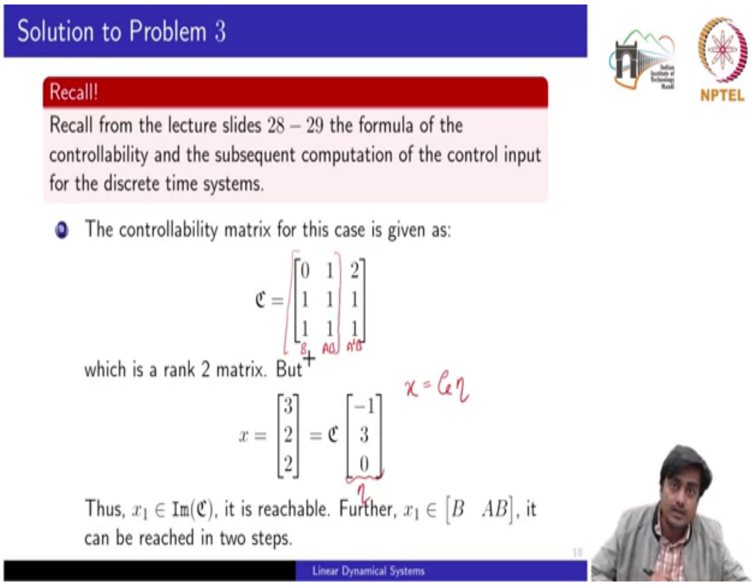

So, first we will discuss the part a in part b of this problem. So, if you recall from the lecture slides number 28 to 29 you can directly use the controllability test as a rank test to determine whether the system is controllable or not. But, at the same time we you need to recall that for the continuous time system the concept of reachability and controllability coincides, but for the discrete time system it may coincides or it may not.

So, first of all we compute the controllability matrix which is given by this one; by using the standard formula which we had seen in the problem one. And computing the rank of this controllability matrix we find that it is a rank deficient matrix, because it is not of rank 3. Now, if we want to test whether this state at T is equal to t 1 is reachable. Meaning to say that this x 1 that this x 1 should belong to the image of this controllability matrix, right. And to say this that there should be some eta vector such that this equation is satisfied.

So, if you are able to find, if we put this x 1 which is given here we have the controllability matrix with us. Now we are able to find that eta vector then this state is reachable. Now, one straight way to compute this eta is would be possible if we compute directly this one, but we cannot take the inverse of the controllability matrix because it is a rank deficient matrix it is non-singular or it is singular matrix.

So, by an by computing ad-hocly this or searching for this eta we find that for this specific eta which is minus 1 3 0 we see that this equation is satisfied. Meaning to say x is C eta, where it is now given by this meaning to say that this x 1 belongs to the image of the controllability matrix ok. This state is reachable. Now since the rank of 2, we could also say that x 1 would also again belong to B AB hm. We get this part is B this is AB and this is A square B. Now if the rank is 2 we can only take this part the rank of this sub matrix let say would also be 2.

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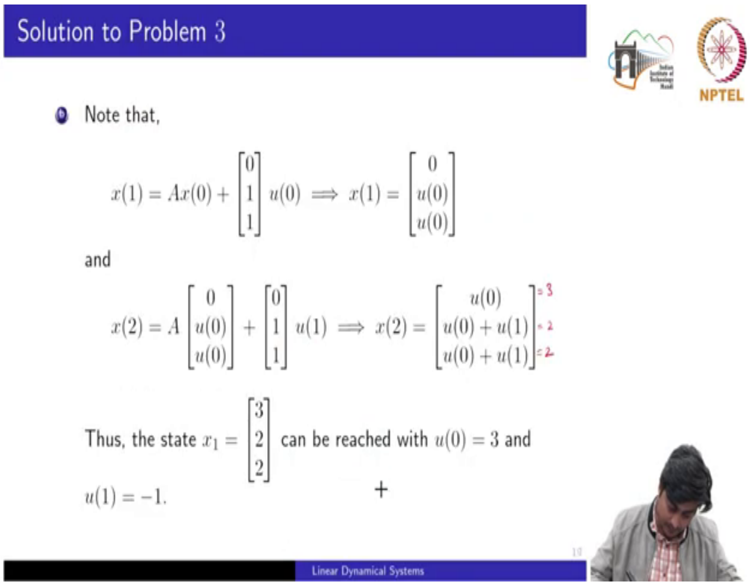

So, for doing the computation of that how many number of steps are required and what are the values of those step of the control signal, we would parameterize the solution of the discrete time system. Since it is a discrete time system we can write it as an algebraic equations. So, starting from k is equal to 0, we write x 1 is equal to A x 0 plus B u 0 where x 1 is now given by this. Similarly, x 2 would be A x 1 where x 1 has been computed over here plus B u 1. So, we can again parameterize x 2 in terms of u 0 and u 1.

Now if we see this step the first element is 0, but we need the first element to be 3. So, only in one step we cannot reach to that state value. But if use pay attention here selecting u 0 as 3 and u 1 as minus 1 you would see that u 0 plus u 1 would be 2 and similarly this would be 2. So, only in two steps we can reach to the value which is specified initially 3 to 2. And the values of the control signal would be 3 and minus 1 ok.

So, going back to part 3, here another discrete time system has been chosen with different AB matrices and also the dimension the system. We want to comment on the reachability and controllability subspaces. Now, at the same time we also need to comment upon that whether both the concept are equal for this state system or not.

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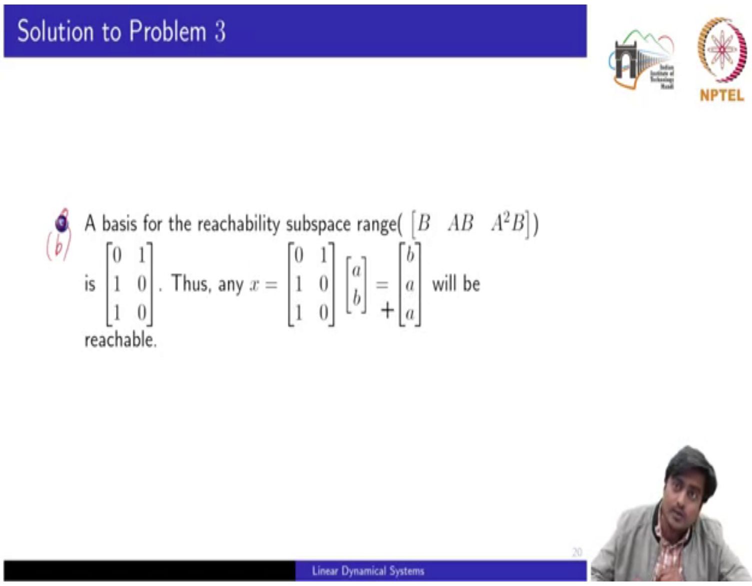

So, before that I think we have skip this part which is determined, all states that are reachable so here it is written b and so it should be b. So, you can compute the basis for the reachability subspaces which is given by the range space of this controllability matrix and this basis you can select, so that they these two vectors are linearly independent vectors. Now there could be any two linear independent vectors which you can choose as a basis and then writing x for those a b you could find all the combinations of x.

Now choosing this as a basis which are basically the first two the first one is the 0 1 1 element and another we have chosen such that it becomes a linearly independent with the first vector. So, substituting this here we see that all those x’s which are having this arrangement b a will be reachable. Now also there all the elementary operations you could perform over these elements would also be reachable, because they will form another basis of the same space of the controllable range space of the controllability matrix yeah.

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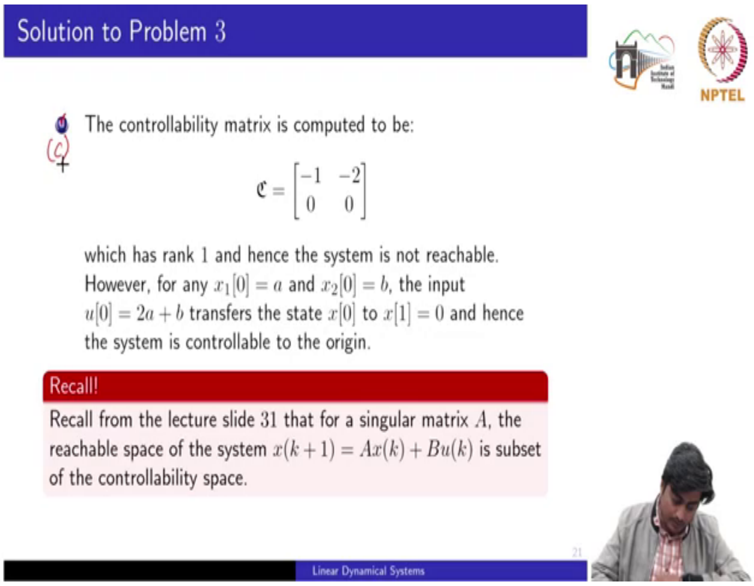

So, this is the C part. So, if you compute the controllability matrix for the matrices given here by this a and b you want to comment on the reachableity and the controllability subspaces, which you can do by computing the rank of the controllability matrix. So, this has been that here. Now it is clearly visible that the rank of this matrix is 1, meaning to say with the system is not controllable.

But at the same time you would notice that the matrix a is a singular matrix. And we if you recall from the lecture slides we have said that the that the reachable subspace is basically a subset of the controllable subspace. So, it means you can find some input which can still transfer the state from x 0 given from a and b to the origin, right. And the system is controllable to the origin, but it is not reachable ok.

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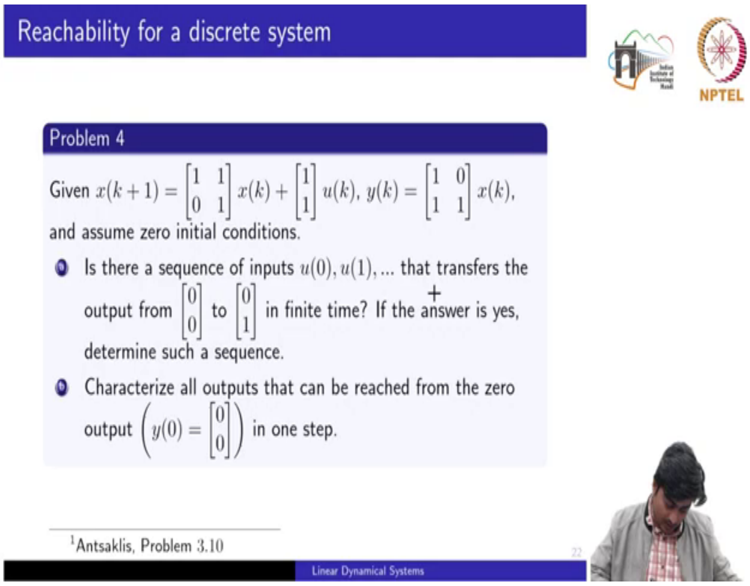

The problem 4 discusses the reachability for a discrete time system. Again, we have taken the AB matrices and the C matrix with 0 initial conditions the discrete time. So, here the problem is to find that whether they are there exist a sequence of inputs that transfers the output from origin to some non-zero value in finite time. If the answer is yes determine such a sequence. Second part is with the characterization of all outputs that can be reached from the 0 of put in one step.

So, here one important difference you will notice that so far we have talked about the controllability and reachability of the state vector. Now here we talk about the output vector. So, the first question arises here that whether the formula what we had studied of the rank condition is also applicable to determine the or let say the reachability of output hm. That a can compute the controllability matrix and then determining the rank would ensure that this system is output reachable and similarly, the characterization is possible.

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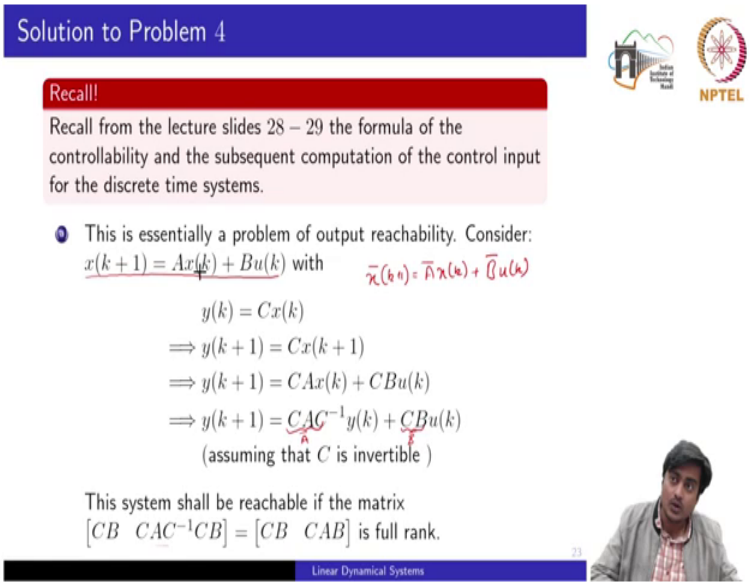

So, the answer would be no first of all, because we have computed the AB matrices for this state space system; where, the state vector is x and AB matrices are being a part of this algebraic equation in this way. So, we need to parameterized this state space or this state equation in terms of the output equation to finally use the rank test. This you can do by using; so if you recall the output expression we have y k is equal to C into x of k.

Now if I advance this k-th time to k plus 1-th time both sides we would have y k plus 1 equals x k plus 1. Now since this is the state equation I can use this equation here to finally substitute at A x plus B u ok. Now, from here if my C is a nonsingular matrix I can write x k as C inverse y k and CB u k would stay as it is. So, this part and this part would be my A bar and this part would be my B bar. And now I can replace my y by y which is an output variable by let say some x bar.

So, this would become x bar k plus 1 A bar x of k plus B bar u of k. Now this equation become almost similar to the state equation what you what we have been seen from the beginning. So, using these matrices A bar and B bar I can compute the rank of the controllability matrix, which would be given by CB if I substitute this B bar and A bar; A bar is this part C AC inverse and B bar is CB.

So, the controllability matrix to test whether my output is reachable or not this matrix would become CB and CAB. So, I need to check the rank and computing the rank we notice that it is the full rank matrix.

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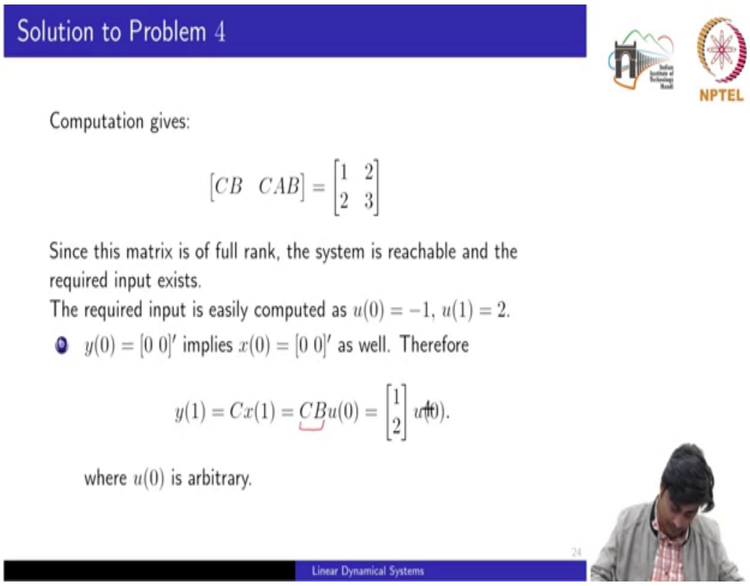

Meaning to say that the system is reachable and the required input exists. Now in a similar way what we had done for the previous problem, you can compute that how many number of steps are required and what are the control signals; which can take the origin to some non-zero state at some time T. So, the required input we are computed as u 0 is equal to minus 1 and u 1 is equal to 2 would take you to the to the desired state, right

So, if you recall the second part it says that characterization of all out puts that can be reached from the 0 output in one step. So, this origin for y 0 would be 0 implies that x 0 would also be 0. Therefore, if we want to reach in one time steps I would write y 1 is equal to C of x 1 and C and x 1 is given by B u 0, because the if you see this equation I just substitute directly k is equal to 0. Now, k is equal to 0 we have x 0 is equal to 0, and u 0 would come as it is because u 0 is a would become a parameter.

So, simplifying the CB matrix we would have one two and some parameter u 0. Now for any u 0 this is the y which we can reach from the origin in one time steps. So, if you chose u 0 is equal to 1 then y 1 would become 1 and 2 respectively, ok.

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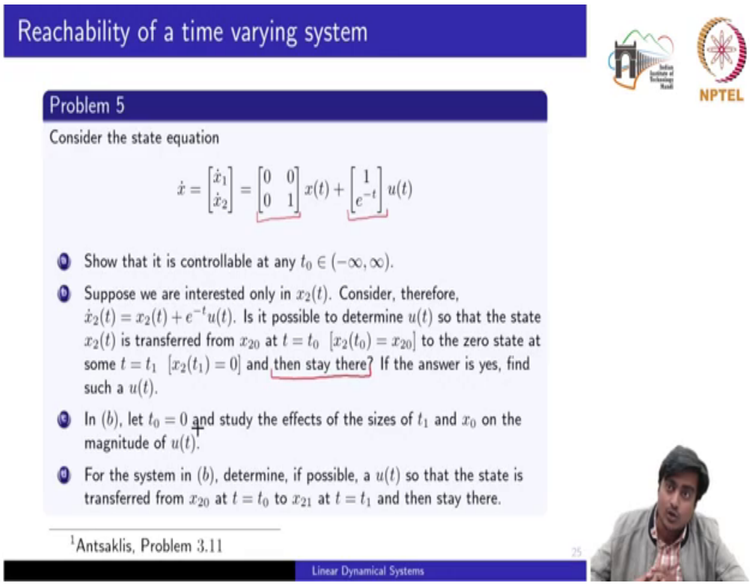

Problem 5 deals with the reachability of a time varying system. So, given in LTB system with a matrix which is a constant matrix, but b matrix there is a time varying matrix because of the inclusion of the exponential term. So, the first part that we need to show, that it is controllable at any time t naught. So, the b c d part we will discuss one-by-one.

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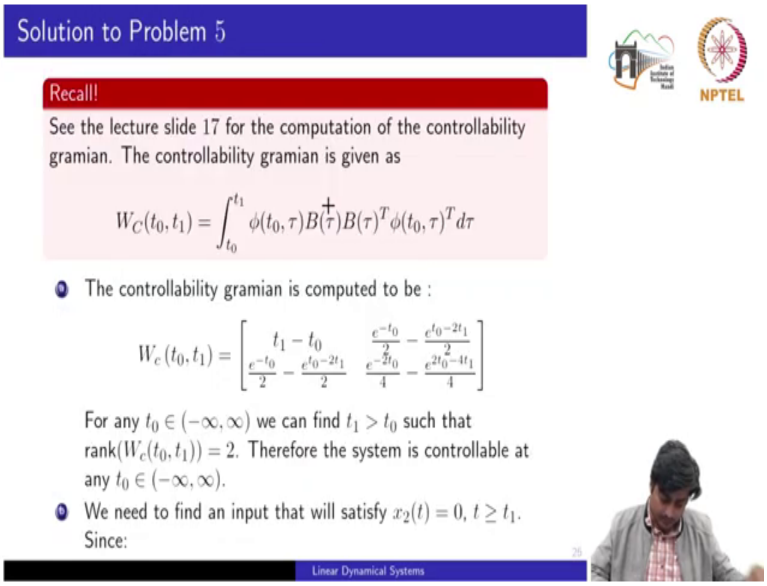

So, first we check the controllability. Now, for the controllability if you recall one of the key result that we need to first we need to compute the gramian. And if that gramian is having a full rank for all t naught and t 1; meaning to say the system is completely state controllable. The controllability gramian can be computed by using this formula where this phi is the state transition matrix.

Now, since we have this a matrix which is pretty much simple you can directly compute the state transition matrix without any difficulty. And by putting that matrix and multiplying with this B matrix we finally obtain this controllability gramian. Now for any t 1 and t naught you can check the rank of the matrix you would find the ranker is always 2; meaning to say that the system is completely controllable.

The part b of this problem is just suppose we are interested only in x 2. So, we discard the first equation and we considered that x 2 dot is equal to x 2 plus e to the power minus t into u t. So, the problem is that is it possible to determine u t. So, that the state x 2 is transferred from x 2 0 to the zero state at some time t is equal to t 1 where x 2 of t 1 is 0 and then stay there, right.

So, if you recall the previous problem when we computed the control input we have ensured that during that time the states would reach towards to 0, but it was not possible to guarantee that the states will stay at 0. But here we need to design a control input and ensure that once it reaches to the origin the state it would stay at 0 afterwards ok. So, if the answer is yes; find such a u of t. So, let see. So, we need to find an input that will satisfy x 2 is equal to 0, for call t greater than equal to t 1 because x 2 at t 1 should reach to 0.

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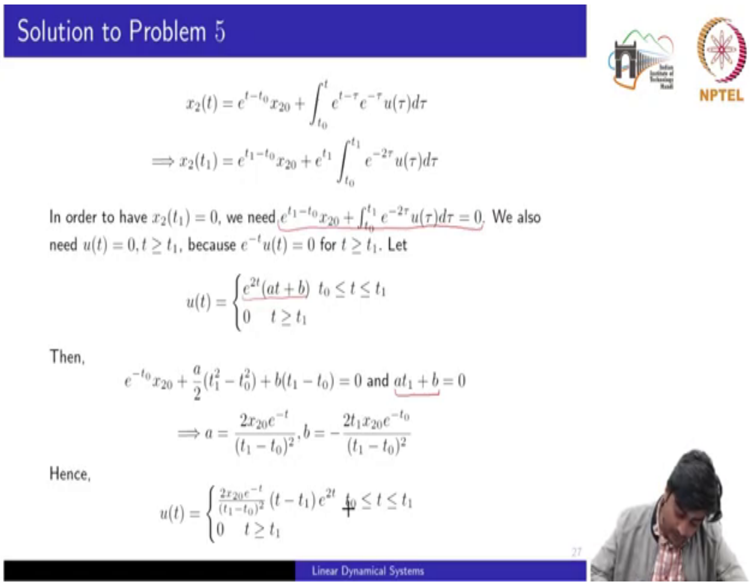

Now if you see the solution of only this is state equation we can express the solution in this way, where the state transition matrix would be an exponential term because of the a non dependence of the a matrix 1 time t. Now putting t is equal to t 1 in this equation and equating this part equal to 0, because we need to ensure that x 2 at t 1 should be equal to 0. So, for that we require that the right-hand side should also be equal to 0.

And once it reaches to 0, if we are not supplying any input to the plan after that time t 1 we could ensure that it would always stay at 0 ok. So, let us see. So, first of all we define some input which is given by e to the power 2t into a t plus B for all time between t naught and t 1 with this input first of all we need to show that it reaches to the that x 2 at t 1 is equal to 0 at time t 1. And afterwards x 2 should be equal to 0 ok.

So, this required two equations to be solved. So, the first one we have obtained from here, we just need to substitute u T into this expression which we have specified here, and since this computation is between t naught to t 1 we would use this u t. Now the second part that u t should be equal to 0 for t greater than equal to t 1 which would happen if a t 1 plus b is also equal to 0.

Now, we have two unknowns in two equations solving and these are the linear equations. So, solving these two equations we obtained a as this and b is this. Substituting this a and b into this u we finally obtain this control input which ensures that the state of the system would reach to 0 and would stay there afterwards.

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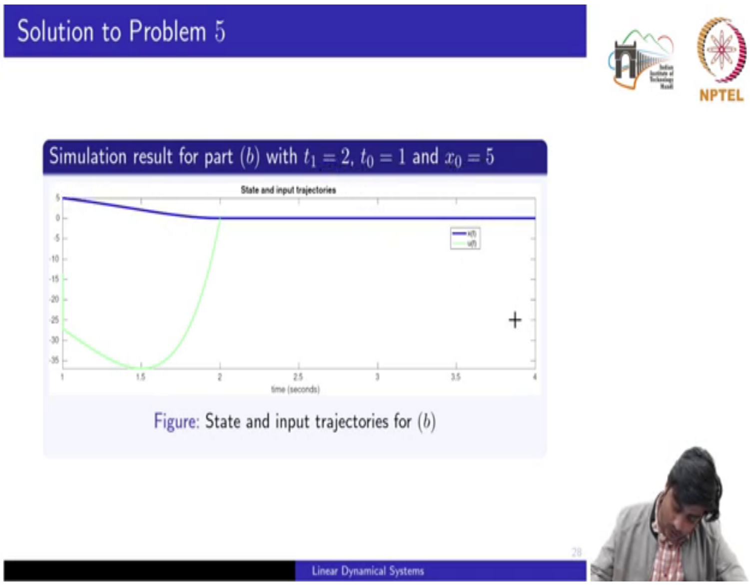

This you can also visualize in this simulations; that we have specified t naught as equal to 1 and x naught which is x at t naught is equal to 5 and we have chosen at time into t 1 is equal to 2 ok. So, we need to ensure that starting from t naught the state should reach to 0 by time t 1 and afterwards it should stay at 0 ok. So, supplying the same control input what we had computed. You would notice that that is add objective have been achieved.

Now, the third part of the problems is that in part b let t naught is equal to 0, and study the effects of the sizes of the t 1 and x naught on the magnitude of u t. So, here you should recall one thing that in the previous in one of the previous problems we had computed the control energy. So, we had notice that if the time is longer then the control energy is less. So, the same behavior we are expecting it here. Why? Because, now there is more time in which we need to ensure that the system reaches to the origin. So, the control actions could be slower instead of being faster. So, let us see.

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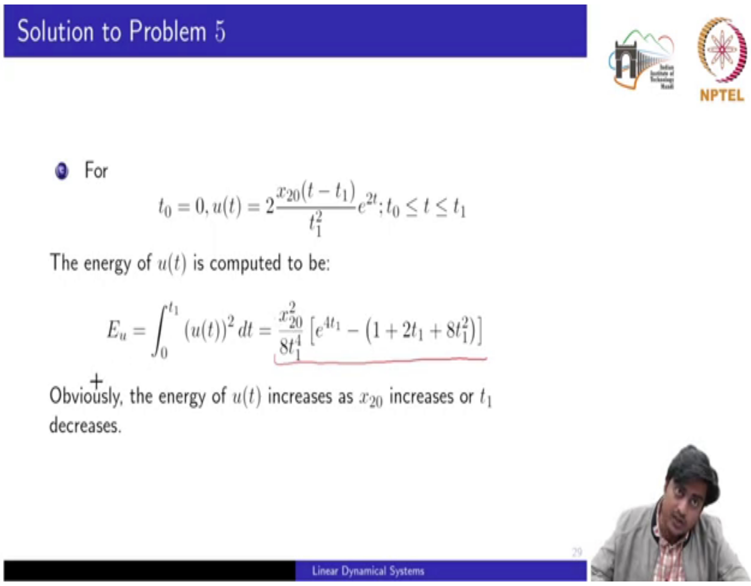

So, for t naught is equal to 0, we already have the control input which we have computed. And if I compute the energy from 0 to t 1 because afterwards it is 0; after t 1 it would T 0 and seeing and computing this in finite integral we computed this formula in terms of t 1 and x 2 naught. So, we noticed that the energy of u t increases as x t naught increases, if we keep that t 1 constant then as x t naught increases the energy would increase. Now keeping x t naught as constant as t 1 decreases the energy would decreases; almost the similar kind of the behavior what we had seen in one of the previous problems.

Part d, that for the system in b that is considering only the second state equation determine if possible u of t, so that the state is transferred from x t naught to x 21 and then stay there. So now, this is a more generic formula which we are looking for, because in party b we need to ensure that x 2 at t 1 should reach to the origin, but in the part d x 2 of t 1 could be anything right it could be 0 also. So, whatever the results we are going to obtain in part d would also be applicable to part b directly.

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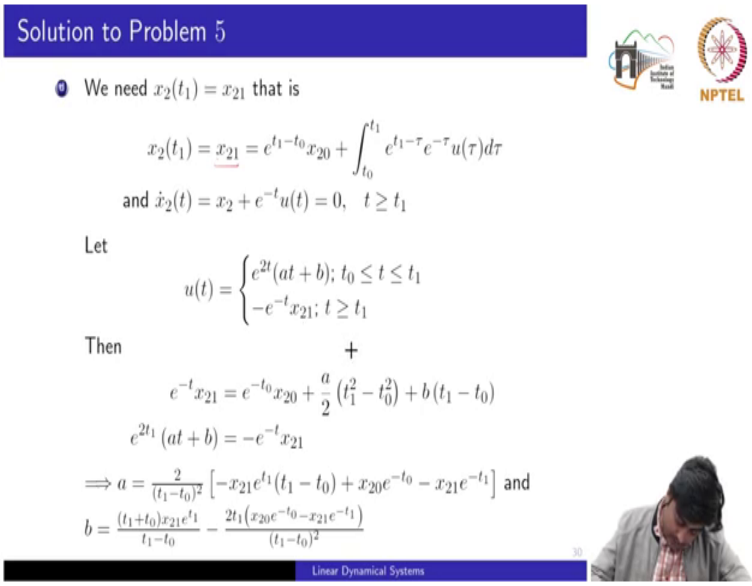

So, again it requires the computation of the control input where we need to ensure that x 2 at t 1 should be 0. In fact, here it should not be 0, but x 2 of t 1 should be some value let say x 21 and this x 21 could be 0. And since we want the x 2 to stay at x 21 after t greater than equal to t 1; meaning to say that that derivative of that equation should be equal to 0, so that it becomes a constant value starting from that initial condition at t is equal to t 1.

Because, if I see this equation this equation would have its initial condition at T is equal to t 1 and at t is equal to t 1 we have already ensured that it has reached to x 21. So, if I put the derivative of this equation equal to 0 meaning to say that this trajectory would stay at x 21 only. So from here, I specify u T as same as before that at the part 2t a t plus b between t naught to t 1 because we have already seen that this u t applying within this time would take me to that x 21 ok. And if I put this equal to 0 I would obtained u of t as minus e to the power minus t x 21 ok.

Again simplifying these expressions; so finally, we obtained this a and b. That if I put this a and b here this control signal which is again an open loop control signal, there is no feedback path which would be the topic of the week 5. That is open loop control signal as given as that is add objective; that the state trajectory has reached to x 21 at time t 1 and its states their proverbs, ok.

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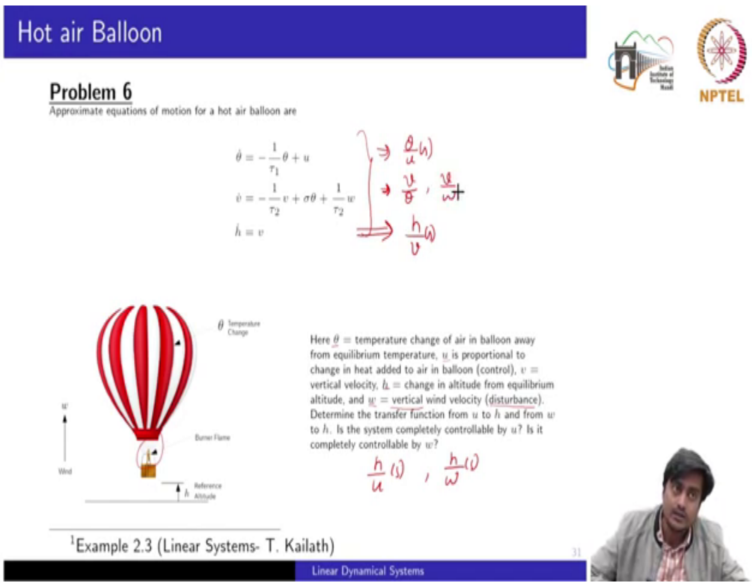

So, the problem 6 is a hot air balloon, where the approximate equations of motions are given. This problem is taken from the book on Linear Systems by Keller which is an example 2.3. So, these equations are the linearized equations around some operating conditions. So, theta u h and w have some physical significance; that theta is the change in the temperature of the air balloon, u is the change in heat which is being added here; so that this balloon can fly, w is some vertical wind velocity.

So, here the point which we pay attention that we have taken only the vertical wind velocity because the wind three components, but here considering only the vertical component and this is acting as disturbance ok. And h is the change in altitude from the equilibrium altitude. So, first of all we need to determine two transfer functions from both the inputs where u is the in fact a control signal and w is a disturbance. So, we need to compute u by; excuse me h by u of s, one transfer function another transfer function is h by w. And second part of the problem is that we need to ensure or we need to determine the controllability from both the individual inputs u and w, right.

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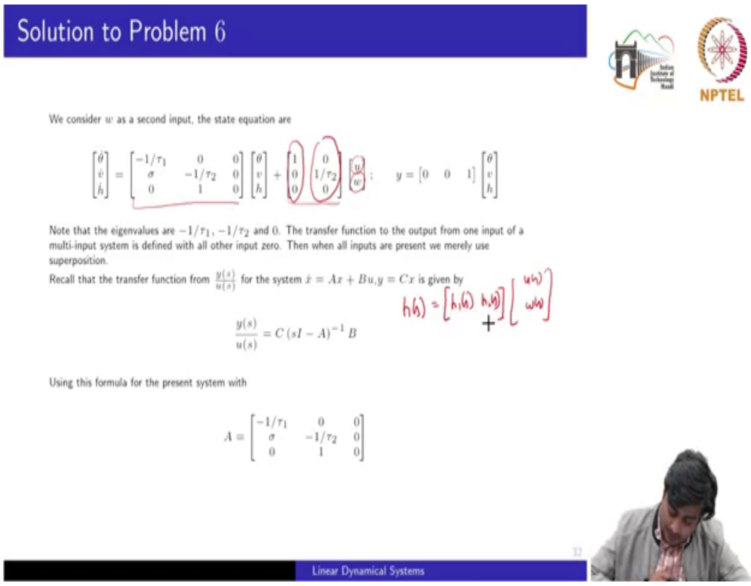

So, all that those three differential equations we can expressed them in terms of the state space representation, where the state variables would become theta v and h and this is the a matrix and the b matrix with respect to two inputs u and w. Now, when it is go by the physical significance of these two control inputs we know at the outset that by using u you can control the system. But also it would be interesting that whether we can control by w also right, because w is basically a disturbance which is the wind velocity. So, if you are lucky then the wind can take it to the desired height, right

So, if we compute the eigenvalues of the system. These eigenvalues are given at minus 1 by tau 1 minus 1 by tau 2 and there is one eigenvalues which is located at the origin. So, the transfer function either you can compute by using the subtended formula which we had seen in many times that is C into s I minus A inverse into B. So, by putting this B as this one time and another there is you would obtain two different transfer functions with respect to u and w; so, using this formula for the present system with.

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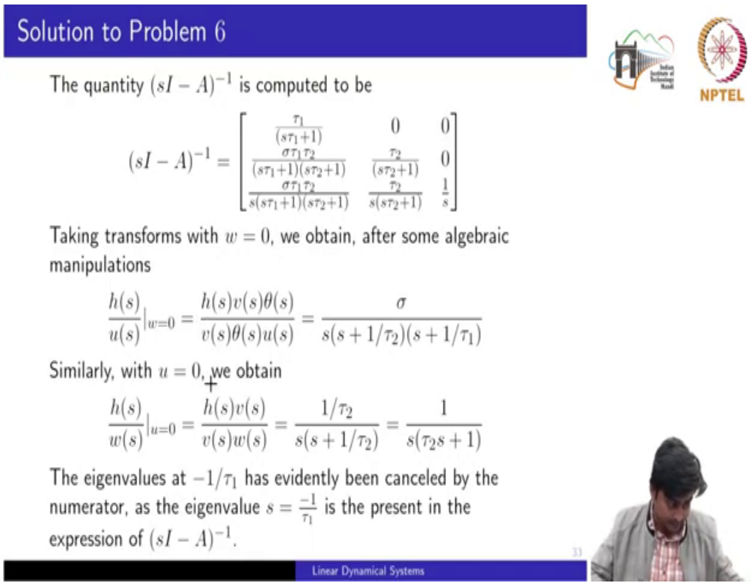

First, we can see the computation of the s I minus A inverse. So, you can compute that matrix which is given by here. Now, since here we have two different inputs. So, the first test which you can do: once you have the AB matrices knowing this AB matrices I can compute the controllability matrix and then compute the rank of that matrix. So, this could be a straight test, but here try to visualize the controllability of the system without performing any tests first of all.

So, first we will compute the transfer functions in the sense that let us say we would have h of s. If we are considering as two inputs it could be a vector of u of s and w of s. Let us call it h h 1 of s and this is h 2 of s. So, we need to compute these two transfer functions. So, say applying the superposition principle we can compute h 1 by putting w is equal to 0 and compute h 2 by putting u equal to 0. And then compute two different transfer function when putting it into the vector it would give me the complete representation of the system in terms of the transfer functions ok.

So, the similar way we have done here that computing h by u by putting w is equal to 0. So, since three equations were involved we have computed h by v. See here, so from here I can compute h by v which could be basically an integrator, from here I can compute theta by u the transfer function again by considering the initial conditions equal to 0, and from here I can compute basically v by theta and v by w individually, ok.

Now putting all this expression h by v v by theta and theta by u we obtained this transfer function. Now, putting u is equal to 0 and computing h by w, so we need to substitute h by v and v by w which is this transfer function. So, here one thing to note here that all the eigenvalues of this matrix s I minus A inverse which is an important matrix, because this is nothing but your state transition matrix. If I take the inverse of this matrix it would give me the state transition matrix. And we know that almost all the characteristics are embedded into the state transition matrix.

So, if I compute the poles of this matrix we have the poles at 0 minus 1 by tau 1 and minus 1 by tau 2. Now, if all the eigenvalues are visible in the transfer functions; meaning to say that the transfer function that I can control that system represented by the transfer function ok. So, if I see the transfer function h by u, I see all the eigenvalues are present here.

Now, if I see the transfer function of the one over h by w we see that one of the eigenvalues 1 by tau 1 or tau 1 s plus 1 is not available; meaning to say that it has been cancelled out by some part in the numerator. So, if I see the input output signals that is h and w that mode or that eigenvalue is basically hidden from the input output signals. So, you can also visualize that how that how that eigenvalues been cancelled out. So, for example, let us write this equations.

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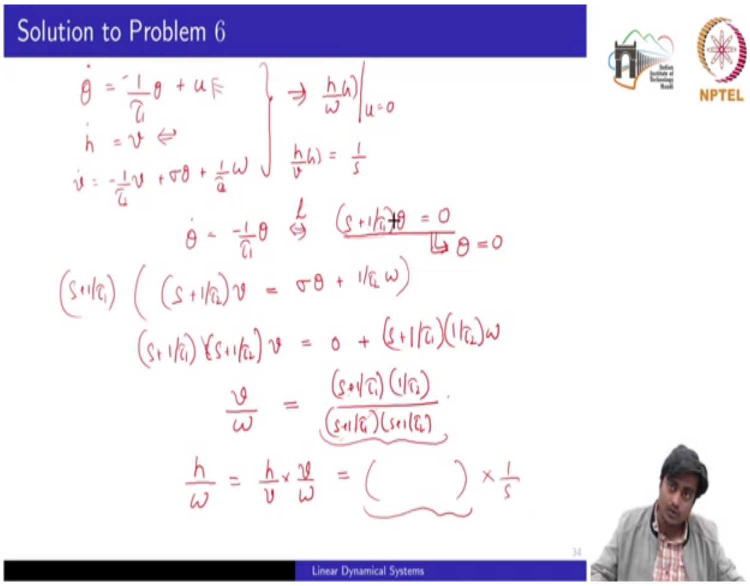

So, the first equation is; let us write this here theta 1 dot sorry, theta 1 plus u another one we have h dot is equal to v tau and w: I think we have written correctly. So, 1 by minus 1 by tau 2 v sigma theta plus 1 by tau 2; so, that should 1 plus tau 2, ok. So, now from here we need to compute the transfer function h by w of s by putting u is equal to 0 ok. So, if I see this equation.

So, this one is pretty much standard, if you take the Laplace transform both side it is nothing but an integrator. So, we would have h by v of s is 1 by s. Now, if I substitute when I substitute u is equal to 0 in this equation I would have theta dot is equal to minus 1 by tau 1 theta. And taking the Laplace transform I would get s plus 1 by tau 1 theta equal 0, because u is taken as equal to 0.

Now, coming back to the third equation and taking the Laplace transform I would help s plus 1 by tau 2 v equal sigma theta plus 1 by tau 2 w. So, all these variables are the function of s. So, either we can use the hats symbol which we have been using in all the our demonstrations. So now, we notice that s plus 1 by tau 1 into theta is equal to 0.

So, there would be two outcomes from here. So now, if this tau 1 is positive if this tau 1 is positive then I can say that theta would definitely be equal to 0. But if I do not substitute or if I do not assume anything on the positivity or negativity of this tau 1, then because if tau 1 is on the or this eigenvalue is on the right-hand side then it is no longer possible that theta would approach towards to 0. But if the eigenvalue or the root of this part is on the left-hand side then theta would definitely be equal to 0.

So, without making any assumptions and knowing this fact; so, we multiply the whole equation by s plus 1 by tau 1 ok. So, here I would have s plus 1 by tau 1 sigma, where sigma is a scalar so s plus 1 by tau 1 into theta would become equal to 0 and here I would have s plus 1 over tau 1 tau 2 tau ok. Now computing the transfer function from here v by w as we would have s plus sorry; 1 by tau 1 and in the denominator again we would have 1 by tau 1 and s plus 1 by tau 2 is a function of s.

Now this v by w once multiply by this h by v I would have h by w as h by v into v by w or in the s w. So, you would notice in this part we would have 1 by s, right. So, the eigenvalue 1 by tau 1 which was hidden somewhere is now explicitly visible to us, which is being cancelled out because this numerator and denominator part on numerator denominator part would cancel out each other. So, if I see only the signal h and w I will not be able to see this mode from the signals, ok.

So, now it is pretty much clear that the system is controllable from u, but it is not controllable from w, ok. But at the same time we should notice that if s plus 1 by tau 1 polynomial is having the roots on to the left hand side then this theta would always be 0; even if it is not controllable then we have another concept which is condition then controllability, ok.

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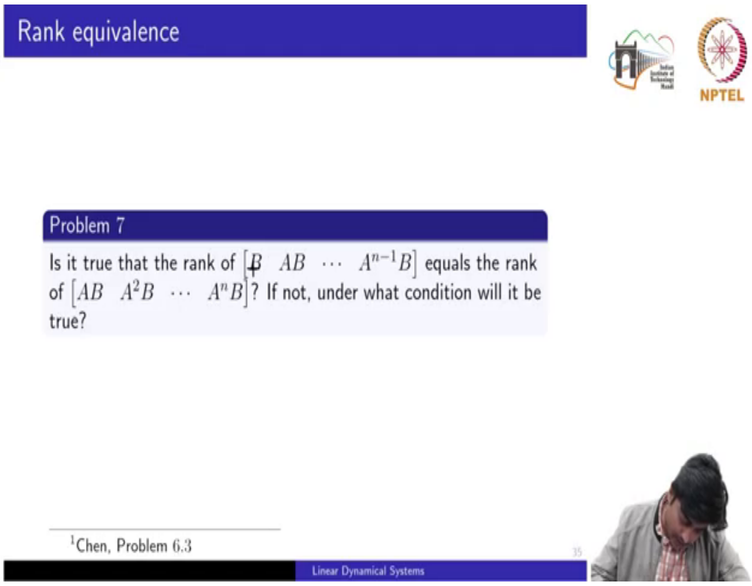

The problem seven is that is it true that the rank of this matrix equals the rank of this matrix, and if not under what condition will it be true. So, this problem is taken from the book by C T Chen and problems 6.3. So, if you pay the close attention to this matrix you would noted that there is nothing but controllability matrix. Now, if I multiply by a matrix the whole controllability matrix I would obtain this.

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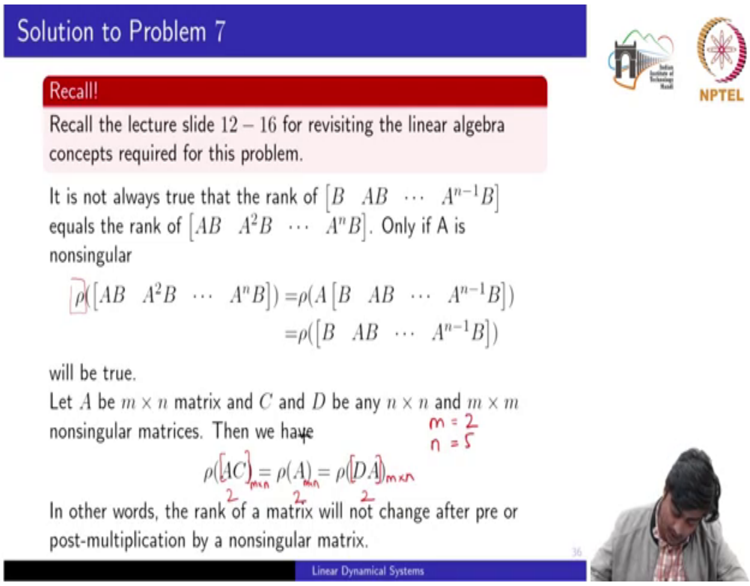

So, in one of the in one of the results we have noticed that I can express. So, this rho we have used as the rank. So, rho means that I am interested in computing the rank of this matrix. So, the rank of this matrix I can express as the rank of A and the rank of the rest of the matrix. Now, if the rank of the rest of the matrix or in fact this is the matrix of which I am interested in computing the equivalence with the previous matrix.

So, I can express this matrix as the multiplication of these two 1. Now if A is a nonsingular matrix if A is a nonsingular matrix then I can express the rank of only this matrix which is nothing but the matrix the controllability matrix ok. So, under the condition that A is nonsingular I could say that the equivalent holds otherwise if A is a singular matrix then the equivalence will not hold. So, you can do a quick example or a quick check also.

Say let suppose A which is not relevant which is not related to here that A is some m cross n matrix and there are other two matrices let us say C and D which are the square matrices of dimension n and m and both these C and D matrices are nonsingular matrix. So, then we would have the rank of the matrix AC would be equal to the rank of A and it would be equal to the rank of the matrix D and C; D into A.

So, to do a quick check say suppose m, we have we are taking m is equal to 2 and n is equal to let say 5. Now, this the dimension of this AC matrix would be your m cross n, because A is m cross n and C is n cross n, your DA matrix would be dimension of m cross n also. Yes, it could be having the same dimension as this AC and rho A is m cross n. Now since m is lesser number than n, so the rank of A would be equal to 2 and the rank of DA again would be equal to 2 and similarly the rank of AC would also be equal to 2.

So, this equality would always hold which is equivalent to saying that pre-multiplying and post-multiplying a matrix by a nonsingular matrix would not be able to change the existing rank of that matrix, ok.

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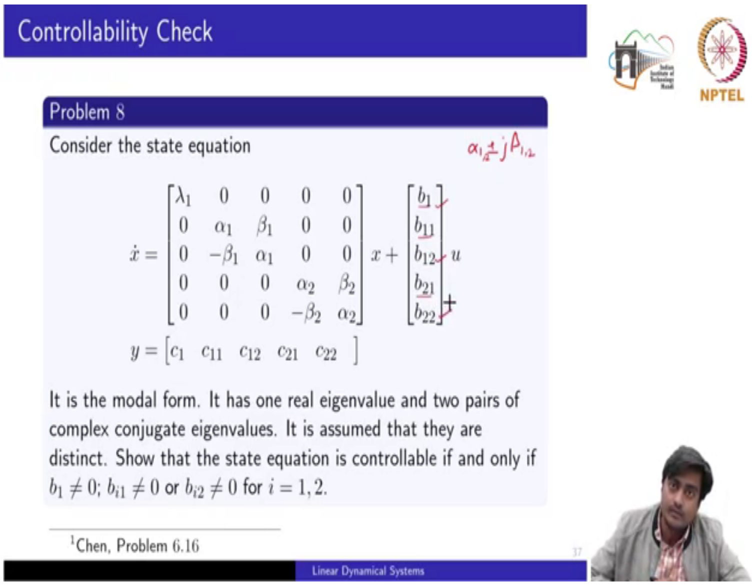

So, problem 8 deals with the controllability check. So, this is an important problem. In fact, this is also one of the important results which is speaks about specifying the controllability condition in terms of computing the Jordan form of the matrix. So, so far if we speak about the controllability checks of an LTI system, we had studied the rank test the previous test and some by characterizing their species also, by computing their gramian.

Now this test is in terms of the Jordan form. Let us say for a given a matrix it will be compute its Jordan form Jordan canonical form then by seeing only the distribution of the b matrix we can directly comment on the controllability of that state space system ok. So, pay attention here. So, this has been already a specified in the model form. So, it has one real eigenvalues which is lambda 1 and two pairs of complex conjugate eigenvalues. So, this has been expressed as in terms of the model form, because if because eigenvalues occur in the complex conjugate pair. So, the eigenvalues are alpha plus minus j beta; so, j beta 1 and similarly for 2 ok.

So, all 5 eigenvalues are distinct. So, here we need to show that the state equation is controllable if and only if b 1 b 11 and b 21 are non-zero or b 1 b 12 and b 22 are non-zero ok. So, directly by checking the condition of the B matrix we can comment on the controllability of an LTI system. Once the a matrix is given into the model form ok. So, we will see the proof of this, but this is also one good result.

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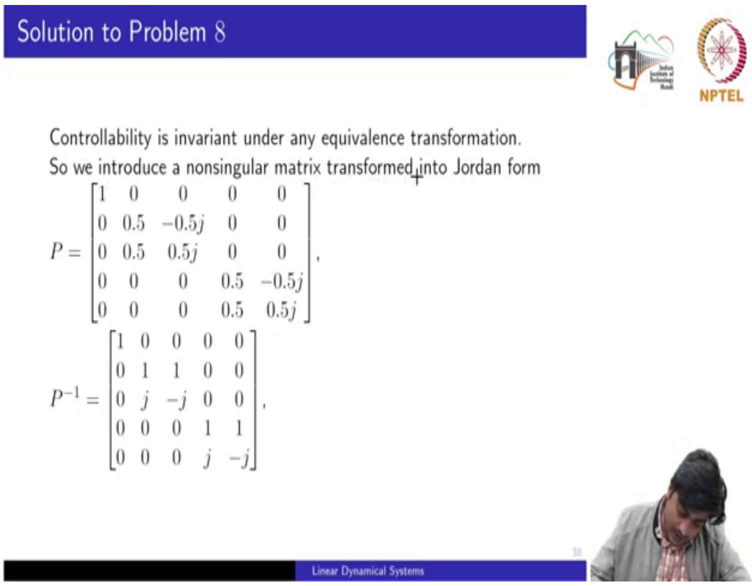

So, we know that the controllability is invariant under any equivalence transformation. So, we can introduce and nonsingular matrix transformed into Jordan form. So, this matrix we have been using P, computing this P inverse we get this.

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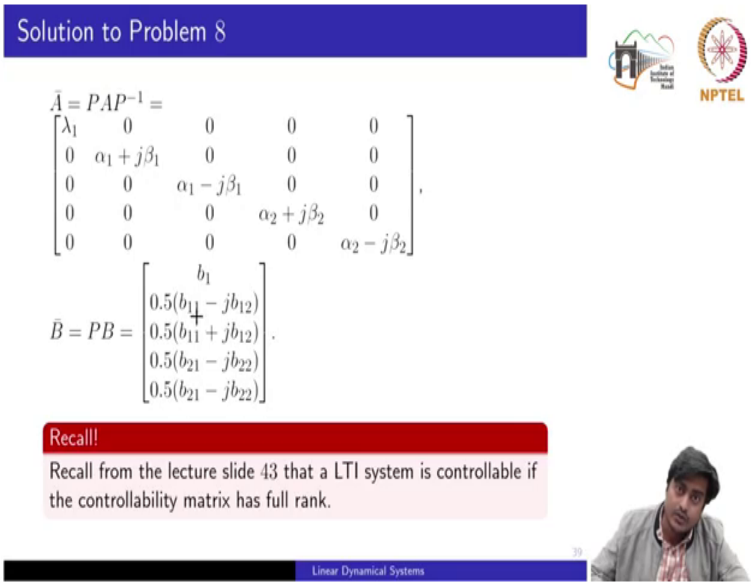

So, computing this A bar, we see we have this P AP inverse which we have been seen from the beginning. So now, this matrix becomes a complex matrix. And from the beginning we know or we knew that there are five distinct eigenvalues, where two eigenvalues are the complex conjugate pairs and the 5th one is the real eigenvalue.

So, I can express all these five distinct eigenvalues in the form of a diagonal in the A matrix, but if I do not want to express my A matrix is a complex matrix I can transform into real matrix by using that model form which was already given in to the question. So, it the A matrix does not contain any complex terms.

So, similarly applying the transformation on the B matrix we would obtain B bar. Now recall from the lectures slide number 43 that the LTI system is controllable if and only if the controllability matrices full rank. So now, on this A bar and B bar we can apply the controllability test that is the rank test and from there only we would obtain the conditions on this B elements of the B matrix.

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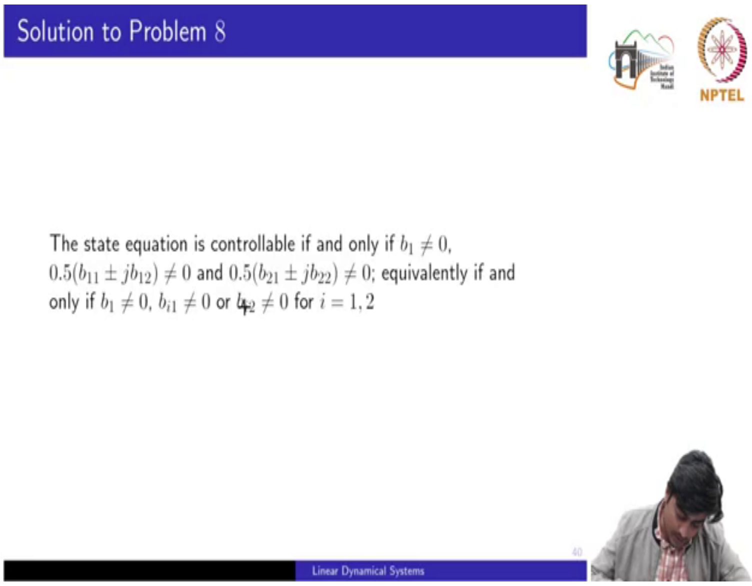

So, you would notice if the state equation is controllable if and only if b 1 is not equal to 0 and these two part have also not equal to 0. Which is again equivalent to same that b 1 is not equal to 0 or either of the combination of the b part associated with the eigenvalues are also not equal to 0, ok.

So, this is another good test to do a check on the controllability, that if it is possible to compute the model form; then, by looking only at the B matrix we can comment directly on the controllability or you compute the rank of the controllability matrix.

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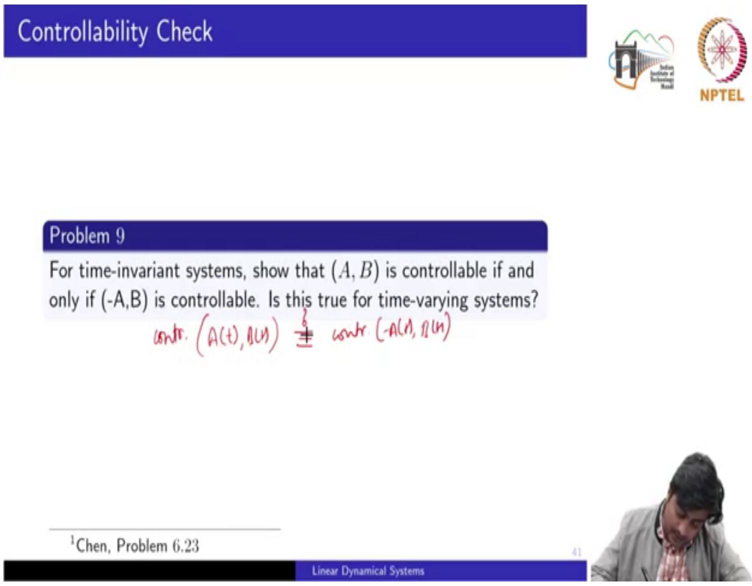

So, this is the last problem which again is of the controllability check. That for time in variant systems we want to determine that the pair A B is controllable if and only if minus A B is controllable. Meaning to say, that if I take a negative A matrix then that new pair I could also say is controllable or not ok.

Now, we also need to show that is this true for time varying system. Meaning to say, that A t B t pair or the controllability of A t B t pair is equivalent to the controllability of minus infinity; whether this hold or not. So, we need to show in the time invariant case and we also need to show for the time varying case, ok. So, let see.

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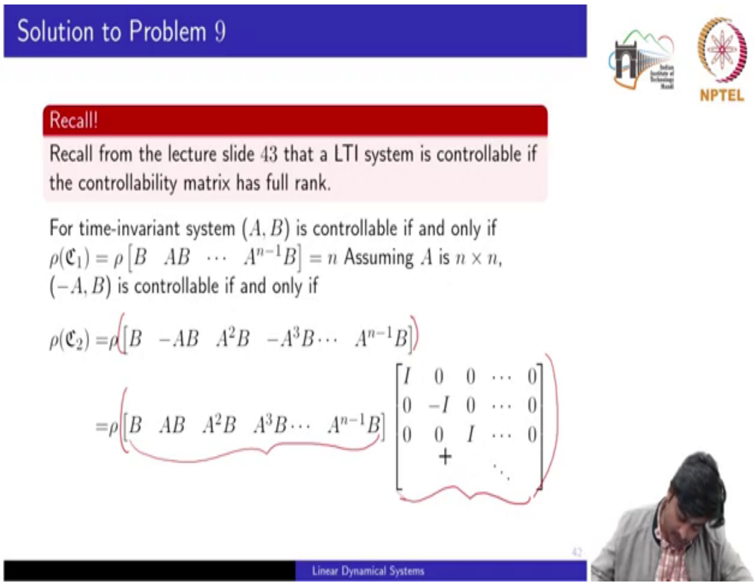

So, for time in varying system we know that the pair A B is controllable if and only if the rank of the controllability matrix is equal to n. Now assuming that A is n cross n matrix minus A comma B pair is controllable if and only if, if I form a controllability matrix of that pair which is given by B minus A B here we have A square so it come positive and similarly in this way.

Now again I can express this whole part as this part, where I have extracted the negative terms in terms of another matrix which is composed of the identity matrix of appropriate dimensions and the rest this is. Now this is rho is basically to compute the rank. Now, if you recall the previous result this matrix is a nonsingular matrix and this matrix is what we already know that it is controllable. So, the rank of these matrix is already equal to n. And since it being the pre multiplication and post multiplication by a nonsingular matrix does not change the rank.

So, the rank of this matrix would definitely be equal to the rank of the original A B pair or the controllability matrix formed by the original A B pair right. So, by changing the sign of the A matrix does not change or does not affect its controllability properties in the LTI case, ok.

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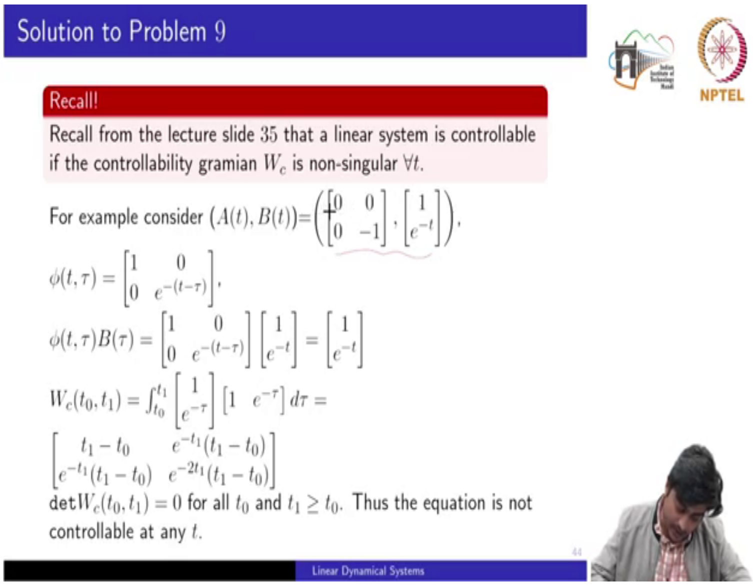

Now, seeing for the LTV case for some particular example let select a of T and B of T as these matrices. For this a matrix we can compute the state transition matrix which is pretty much easy to compute. And then computing the controllability gramian and checking the rank for every t 1 t naught we found that this pair A of t and B of t specifically given by these combination it is not controllable at any time t.

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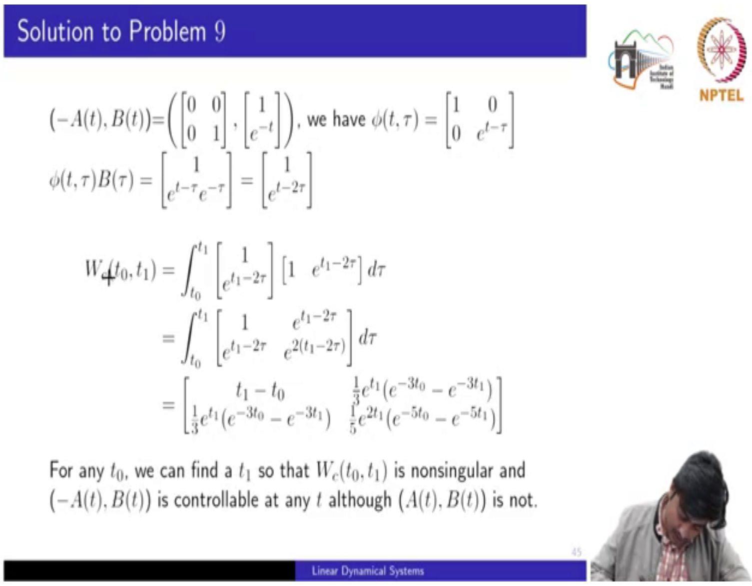

So, now we will check the controllability of the pair with the negative A sign which is given by this, so this will become positive the rest of the element would stay as it is 0. So, for this new pair we have the state transition matrix given by this and the controllability gramian is given by this.

So now, you would notice that for any t naught we can find it t 1, so that w c the controllability gramian is nonsingular and this where is controllable at any t although the pair original pair A B was not. So, that equivalence holds for the LTI case, but that same equivalence does not hold and we are talking in the time varying case, ok.