Linear Dynamical Systems
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Week - 01
State-space solutions and realizations
Lecture – 02
Solution of LTV systems

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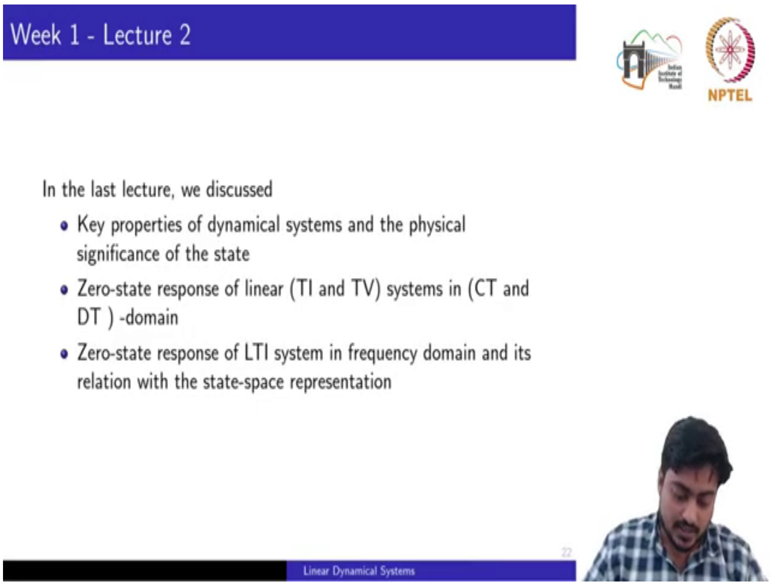

So, in the last lecture we discuss about the key properties of the dynamical systems and physical significance of the state. We also discussed about the Zero-state response of the linear system both in the Time Invariant and Time Varying case. Again, both in the Continuous Time domain and the Discreet Time domain.

Third, we discussed the particularly for the LTI system. We discussed a solution in the frequency domain because we saw that if we want to compute the response in the time domain that, then you need to compute the integral and the convolution. So, we use Laplace transform measure tools to compute the response in the frequency domain first and then, by taking the Laplace inverse, we completely take solution in the time domain.

We also saw the relation of the zero-state response with the state space representation of the system. The major issue where we stop in the last lecture is the solution of the time invariant time varying system.

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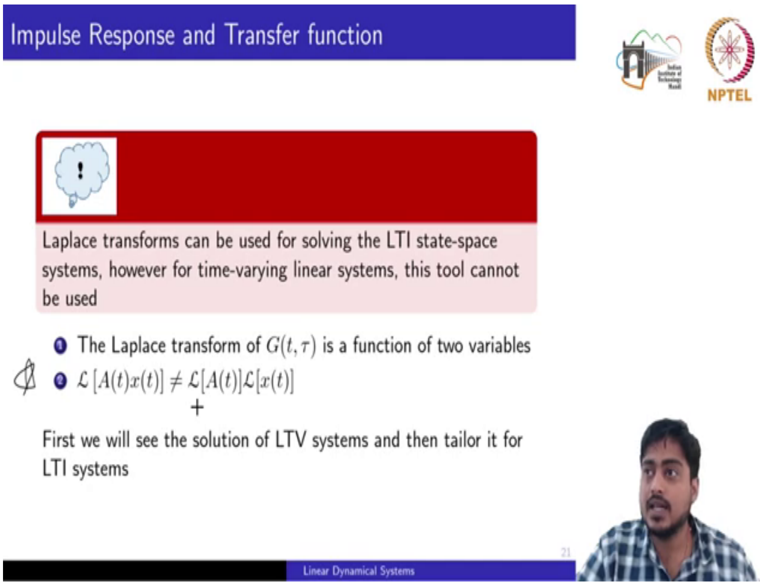

So, there were two issues to compute the solution of the time varying system that the Laplace transform of the function G t comma tau would now be a function of two variables and also the major issue is this point the second point, where the Laplace transform of the multiplication of the time varying matrix A of t into the vector x t is not equal to the their individual Laplace transforms.

So, today, we will see the solution of the linear time varying systems in the time domain and then, we would tailor it or tailor the solution for the time invariant case.

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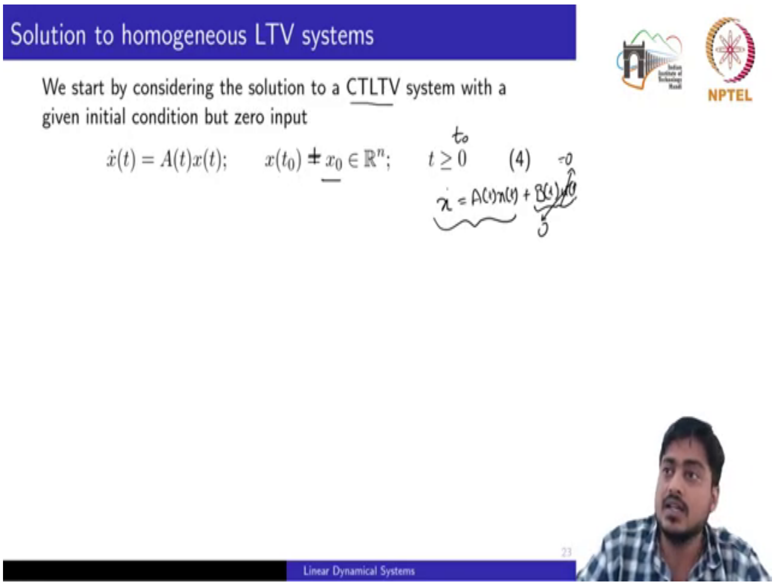

So, first we will start with the homogenous LTV systems. So, we start by considering the solution to a continuous time. So, this abbreviation is used for continuous time at linear time varying system with a given initial condition, but 0 input. So, we are not supplying any input to the system. If we recall the initial representation of the state space it was x dot is equal to A of t x t plus B of t u of t. So, now, we are taking that u of T is equal to 0. So, similarly this whole part would go to 0 and we are dealing only with this homogeneous part.

And the initial condition at time t is equal to t naught is denoted by x naught which is also again a n dimensional vector for all t and we are computing the solution for all t greater than or equal to 0 or in fact, it should be equal to t naught. You should treat this as t naught because if t naught is equal to 0, then we have the initial condition at t is equal to 0 other than that time t naught.

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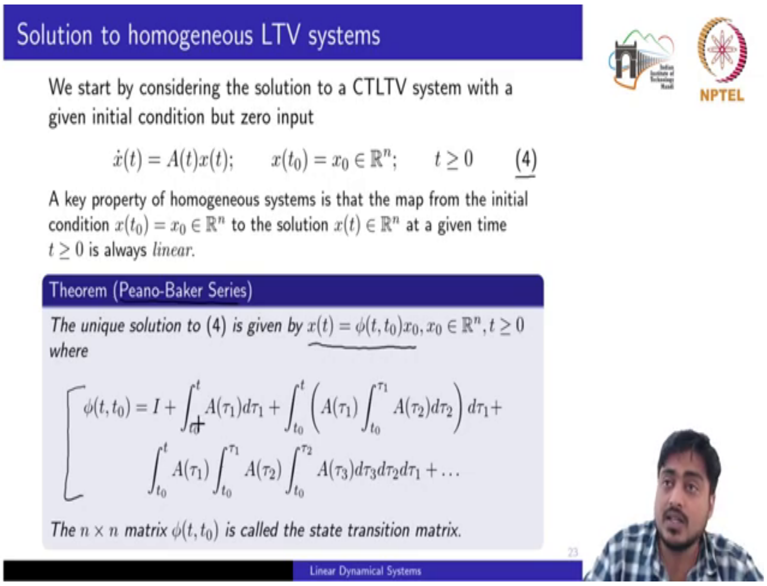

So, there is a very nice property associated with the homogenous system that when we compute the solution. The map from the initial condition and x t naught is equal to x naught to the solution x t at a given time is always linear; meaning to say that we can individually study the response of the state space system by first considering the homogenous equation and then, by considering the non homogenous equation because of the linear ethic property.

Now, we here introduce one theorem which also gives us the solution of the equation 4 or the homogenous LTV state space equation. The unique solution to 4 is given by x of t is equal to phi t comma t naught into x naught; where, x naught is the initial condition and this response is valid for t greater than and equal to 0, where phi t comma t naught is basically computed by this non ending series which we call the Peano-Baker Series. This n cross n matrix phi t comma t naught is called the state transition matrix.

Now, here you would see since A is the time varying matrix, we are having a we are having the integrals of this state matrix which the which is the A matrix as the non ending series.

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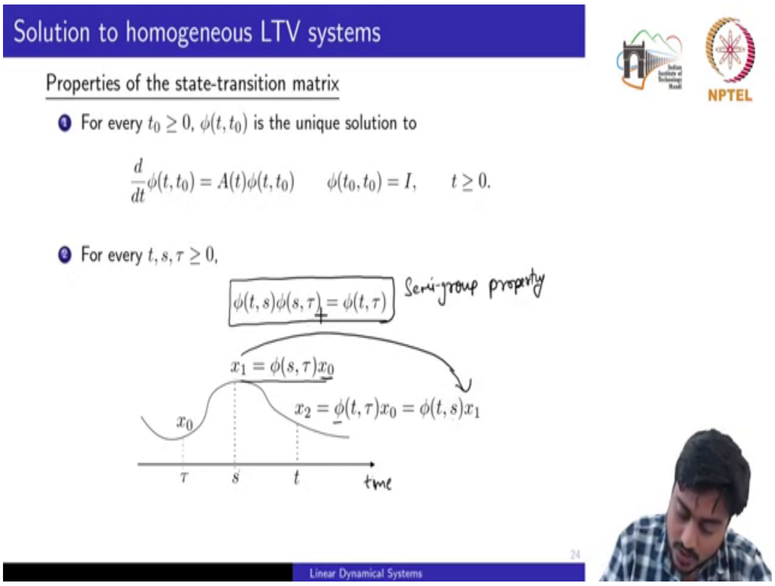

Now, let see further the properties of the state transition matrix because if you pay close attention to this series, it is quite complicated to compute all these integrals and we never know that how much integrals we need to compute. So, first we will see the properties that what are the properties or key properties of this state transition matrix and then, we will try to compute the close-it form solution of this state transition matrix.

So, the first property speaks that for every t naught greater than equal to 0, phi t comma t naught is the unique solution to this equation. This is a very nice property which we would explore later and phi t comma t naught also satisfy is equal to I. I, here is an identity matrix just to add in the last line that here I denotes the identity matrix of the suitable dimension.

Now, if the state transition matrix a is a matrix of dimension n cross n, then this I is an identity matrix of containing 1 in its diagonal of dimension n cross n. So, similarly here I is a n dimensional identity matrix for t greater than equal to 0.

Second, for every t, s and tau greater than equal to 0; the state transition matrix satisfy this property what we also called the semi-group property. Now, see let say this is the some state registry and on the x axis, we have which is the time axis. Now, given this initial condition x naught at t is equal to tau and I want to compute the solution at t is equal to s 1 which we denote by x 1.

So, this solution going to the previous theorem is given by the multiplication of the state transition matrix, which says that I want to compute the solution at t is equal to s given the solution at t is equal to tau and the initial condition at t is equal to tau is x naught ok.

Now, if I want to compute the solution let say at time t is at time t, then by x 2 I can represent in two ways. If I know x naught, then I can represent by the multiplication of the state transition matrix phi, at time t given the initial condition at tau multiplied by x naught or if my initial condition is x 1 because I know that this solution would be a unique solution.

So, in that case the state transition matrix should be computed by the response at t given the initial condition at s multiplied by that initial condition at t is equal to s. So, I could write x 2 is equal to phi t comma tau x naught or phi t comma x is equal to x 1. Now, if I put x 1 from here to here, it would give me phi t comma s into phi s comma tau is equal to phi t comma tau.

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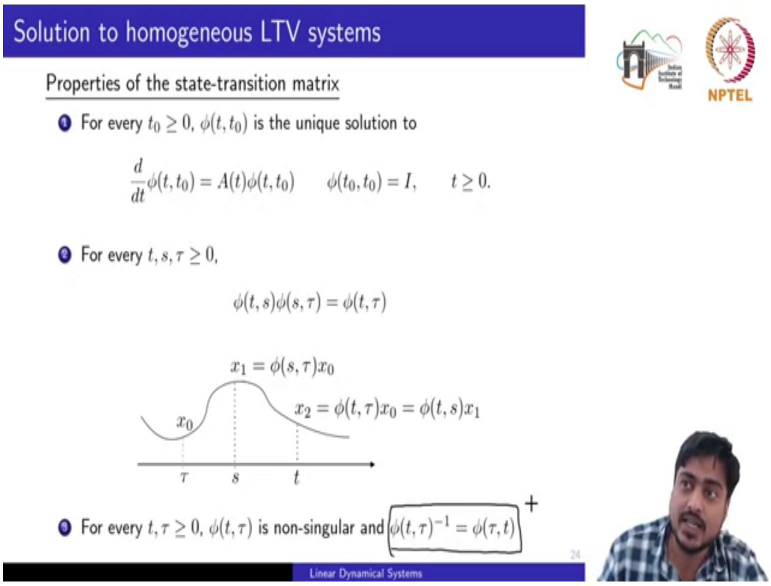

Now, the third property speaks that for every t and tau greater than equal to 0, we have phi t comma tau is non singular that is the state transition matrix is non singular and this relationship is always satisfied. Let phi t if I if we take the inverse of the state transition matrix, then it the order of the argument would change.

So, if we see if we pay attention to this phi it says that a we are seeing the response at time t given the initial condition at tau. Now, if I take the inverse of this state transition matrix, it gives me phi tau comma t; meaning to say that I am now taking the response at t is equal to capital tau, given the initial condition at time t ok. You can change the order. We would see the relevance of this property, when we will discuss about the discrete time solution of the time variance.

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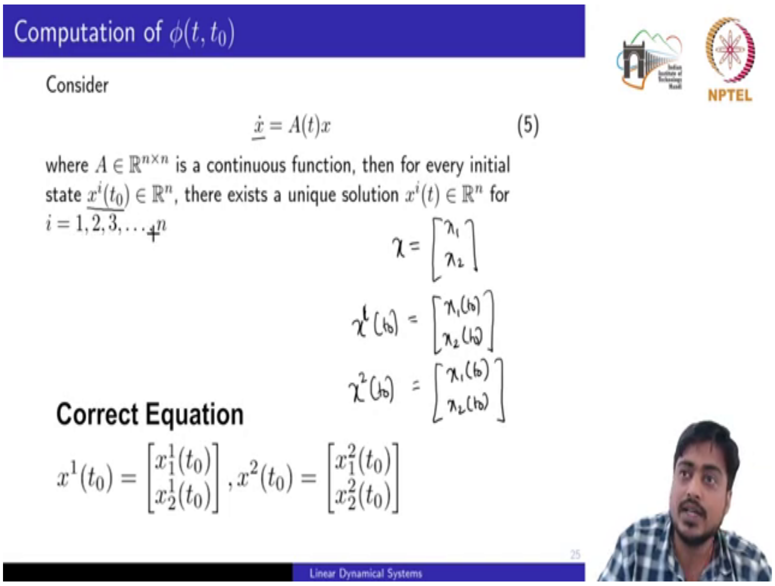

So, the important part here is how to compute the state transition matrix. So, first of all let us consider the again the homogenous system, where by x dot is equal to A of t into x. So, we know then for every initial state x i of t naught which is an n dimensional vector. Now, here I am taking different initial conditions ok. So, for example, if I say let suppose forget about this let say we have x, this original x is equal to x 1 and x 2 ok.

Now, this x i of t naught says that I have an n dimensional vector which I call let say x 1 of t naught which is again an n dimensional vector at time t naught ok. If I have x 2 of t naught meaning to say again it would be an n dimensional vector, but for another values right.

So, then for every initial conditions we could have different initial conditions which would be an n dimensional vector, then there exists a unique solution x i of t which is also again an n dimensional vector for all i's starting from 1 to the dimension of the state; from 1 to n ok.

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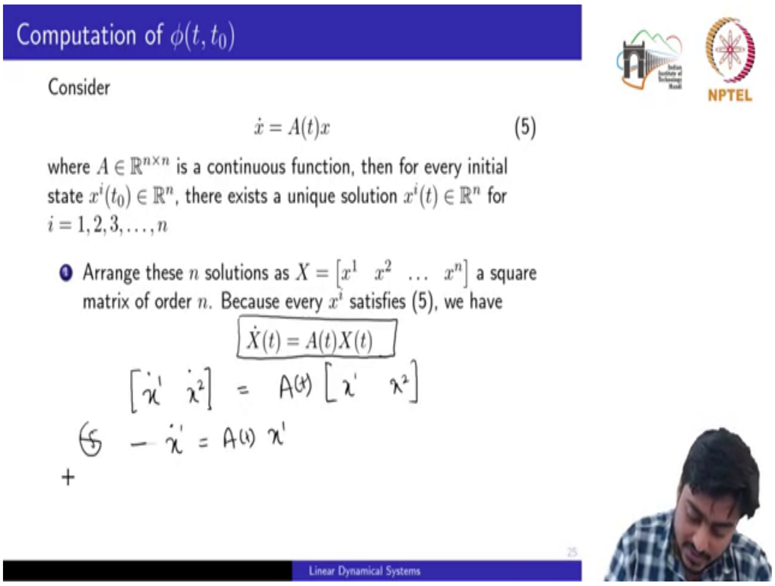

Now, as a first step arrange all these n solutions as capital X, which is equal to all these n dimensional vector arrange in columns. So, this would yield a square matrix of dimension n cross n because every vector X x1, x2 up to x n they are all n dimensional ok because every x i satisfies phi.

This is what we have computed. We can write X, capital X dot is equal to A into capital X right. If here X, we can see for example, let say just consider the case of only two initial conditions ok. So, this I put write A of t, this would be a derivative in an x 2.

So, in the preliminary step, we have computed that x dot 1 is equal to A of t x 1 and this 1 in the superscript denotes the different initial condition, but the state would remain the same. Basically, this is equation number 5. So, if I club all these equations 5 for different initial conditions having a different unique solution, then it would give me this equation right.

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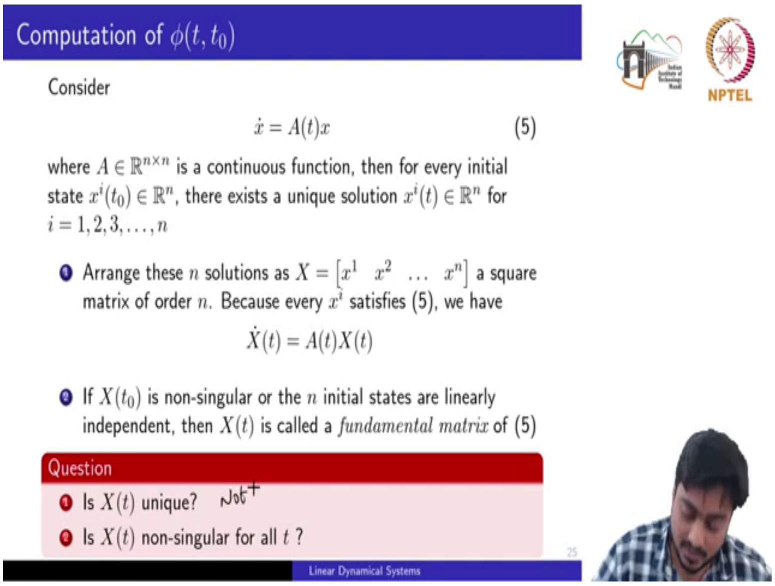

Now, if X of t naught is non singular meaning to say that if I arrange if I club all these X at initial conditions is non singular or we say that these n initial states are linearly independent, then X of t is called a fundamental matrix of 5; basically the fundamental matrix of 5.

Now, here two question raises; whether my X of t is unique and second is X of t non singular for all t? Now, the uniqueness property is straightforward. Why? We will come to these equations later on, but first we will see one simple example which demonstrate the computation of this fundamental matrix.

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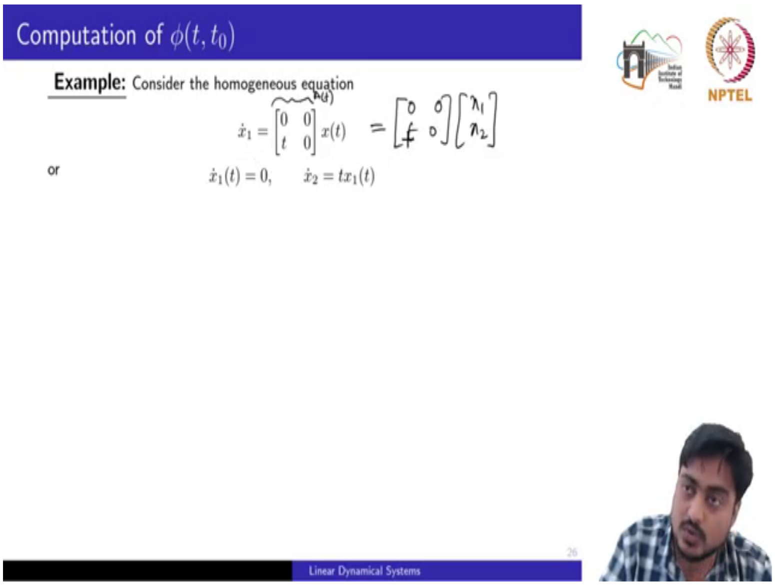

Let say we have this homogenous equation, where this is the A matrix ok. Now, I can write these complete set equation in do into two differential equation that is x 1 dot is equal to 0 because the first row is equal to 0 and x 2 dot is equal to t into x 1. So, we can see in detail let us say like this x 1 and x 2. So, this says that x 1 dot is equal to 0 and x 2 dot is equal to basically t into x 1 right.

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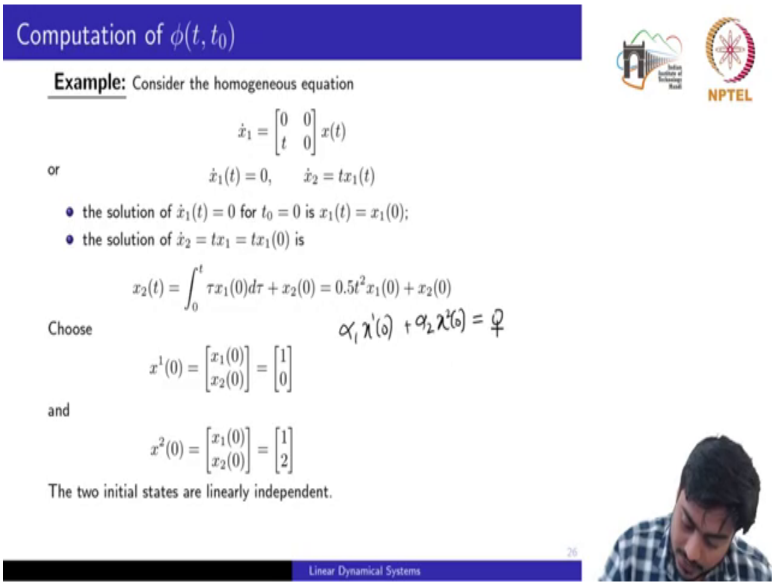

Now, we can compute the solution of this equation and the solution of this equation is x 1 of t is equal to x 1 of 0 and this is only driven by the initial conditions ok. Now, I put this x 1 of t into the second differential equation for the state for the second state, I would get x 2 dot is equal to 2 x t x 1 which is finally, equal to t into x 1 of 0.

Now, I can compute the direct integral because it is quite simple integral and finally, I would have this solution. So, I can compute the solution of x 1 and I can compute the solution of x 2. Now, we chose two different initial conditions. Let say one initial condition is 1, 0; another initial condition is 1, 2.

Now, these two initial conditions should be linearly independent because of the property we saw in the last slide that if the these two initial conditions are linearly independent, then that matrix would be a fundamental matrix right. We could quickly verify in the sense that if I multiply alpha 1 of into x 1 of 0 plus alpha 2 x 2 0 equal to 0, then alpha 1 and alpha 2 should be 0 for the linearly independents right, then in that only this equation can be satisfy.

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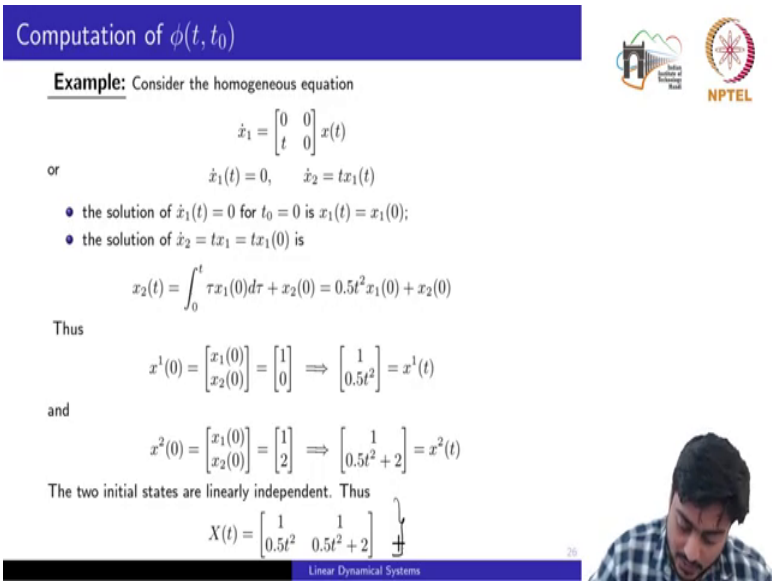

So, if I compute the solution for these two different initial condition, we get one solution this and another solution is this one. Now, if I club both these solution into one, which would give me this X of t. This is the fundamental matrix. This is how can compute the fundamental matrix. Now, we see now to answer these two questions. So, we can take the first one about the uniqueness. We know that there could be infinitely number of initial conditions and we know for every initial condition there would be a unique solution.

So, X of t is basically not unique. For different initial condition, we would have different solution and if the initial conditions are linearly independent, then X of t being the fundamental matrix would not be unique right.

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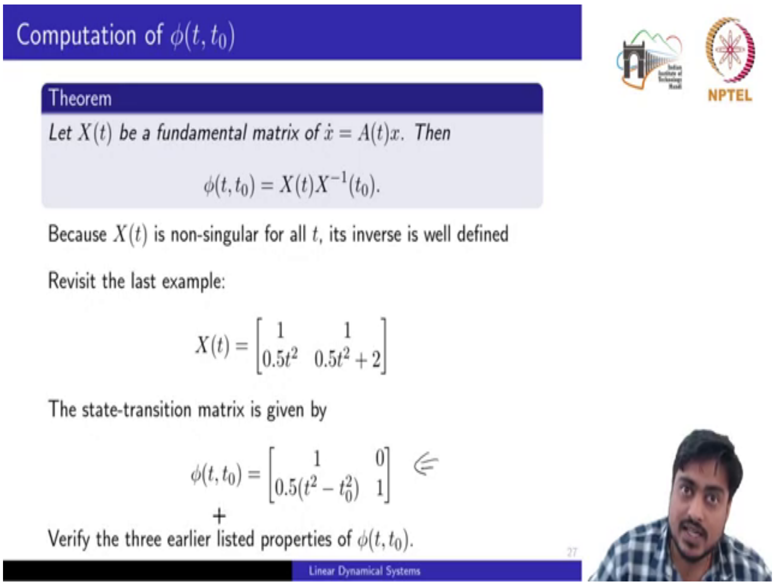

About the singularity, we will see this result in the next result which say let X of t be a fundamental matrix of x dot is equal to A t into x, then the state transition matrix is given by the multiplication of the fundamental matrix and the inverse of the fundamental matrix at time t is equal to t naught.

Now, because X t is nonsingular, for all t its inverse is well defined. Let see. So, this is the example in the last example, we computed the X of t. If I compute this X of t at t is equal to t naught and multiply by this matrix with the inverse of the fundamental matrix at t is equal t naught, I would get this state transition matrix.

So, instead of computing the Peano-Baker Series, we can compute the close form solution of the state transition matrix in terms of the fundamental matrix ok. So, once we computed this phi t comma t naught, you can verify by yourself all the three properties which we had discussed initially that the solution of the state transition matrix, the semi group property and also the non singularity and the inverse property ok.

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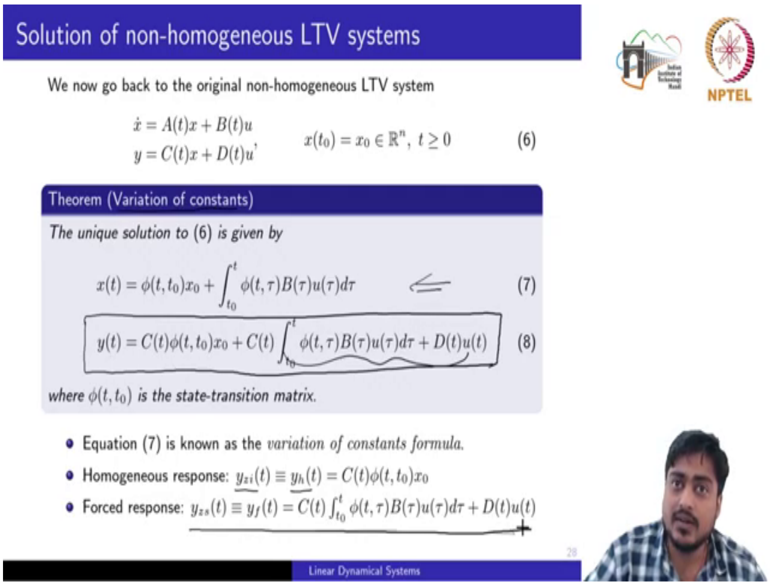

Now, see the we can go back to the original non homogenous LTV system, there now we are included the u as well ok. The rest of the thing remains the same that we have the initial condition at t is equal to t naught which is given by x naught.

Now, in this result, we also called this the variation of constants the unique solution to 6 to the system is given by X of t is equal to the state transition matrix multiplied by the initial condition plus the integral from the initial time to the current time multiplied by the state transition matrix, the input matrix and the input signal itself ok. Now, if I put X of t back into this equation because the second equation is basically the algebraic equations.

So, second equations says y of t is equal to C of t multiplied by whole this equation X of t plus D t u T ok. So, we know that phi is the state transition matrix right. So, there are certain properties of this result that the equations have a basically this equation is call the variation of constant formula by which it gets its name variation of constants ok.

Now, this is the overall response of the y t and in the last lecture, we saw that the total response is basically driven by 2 individual responses; one is the zero-state response and another with the by the zero-input response.

So, here if we put the u of T is equal to 0, then both this part both this part will go away and the solution, we would have only this one which we saw in the last result also. So, if we call either by the zero input response or the homogenous response. Now, if we have the initial state x naught is equal to 0, then the only response is this one because of the input signal and we call it either the zero-state response or the force response which is given by this part.

So, if you see some similarity from the previous lecture basically in the last lecture, we discussed only this response ok. We will see that how what is the relationship between emphasis forms and the state space representation of the time varying system ok.

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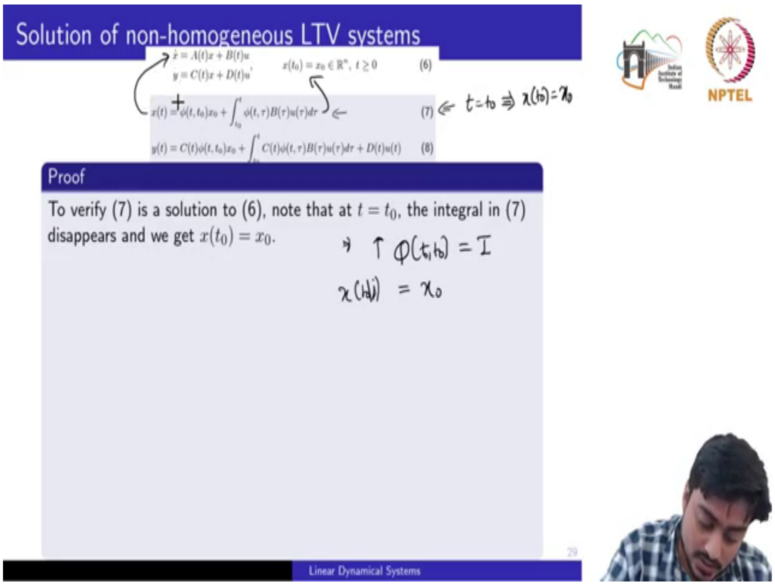

So, here we will because this is an important result, we will go through very quickly the proof of the proof this result that whether this x of t and y of t is a solution of this LTV system. So, we just a recall the previous equation because we would going to use these three equations into the proof.

So, there are two parts of the proof; first that this parts x of t should satisfy the initial condition, the second part is this one meaning to say that because this is the x of t. Now, if I now in this equation 7, if I compute at t is equal to t naught, the solution should be x of t naught is equal to x naught right.

Now, let us put t is equal to t naught into this equation. So, it would give me x of t naught which is equal to phi t naught comma t naught and from the property of the state transition matrix phi t comma t is basically equal to the identity matrix ok.

So, the x naught would remain. Now, if I put t is equal to t naught here, this integral would vanished right. So, this solution we would have that x of t naught x of t naught is equal to x naught. Now, the second part of the proof, I need to put x t into this state space equation.

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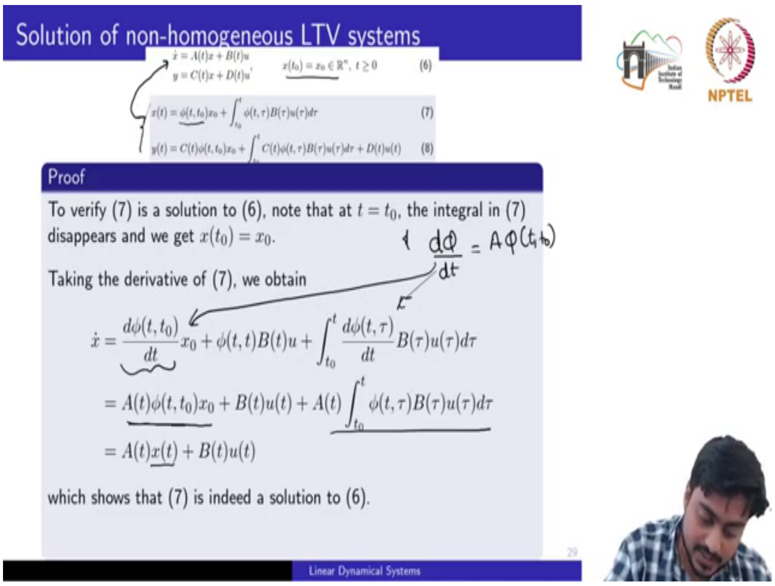

So, I need to take the derivative of this 7. Let us, we take the derivative x naught is equal to because this is the only part which is a function of time. So, we take d phi by dt into x naught plus by using the differentiation rule of a integral, if because inside the integral we have only the function phi which is a function of t.

So, we take the derivative of this function phi; B and u would remain as it is. We would not be taking the integral plus the final value of this integral at time t. So, tau basically would be become t phi t comma t; all the tau's would become t right.

So, we would have phi t comma t B t into u ok. U basically u of t because we are computing the solution at time t is equal to t. Now, we can replace this part, d phi by dt with by using the property that which is given by d phi by dt is basically the solution of A phi t comma t naught.

This was the first property of the state transition matrix. So, if we put this here we get this if A t phi t comma t naught into x naught, phi t comma t is basically the identity matrix. So, it would go away and B t u t plus similarly we could replace here as well and then this part would remain.

Now, if we take A of t common from this term and this term we would have phi t comma t naught x naught plus this part and basically this is the solution x of t right. So, we could replace this whole term by x t. So, we have x naught is equal to A t into x t plus B t into u t which is this one.

So, when we put this solution into the left hand side of this equation, it becomes equal to the right hand side meaning to say that this equation or this solution satisfy this equation plus it satisfy the initial condition as well ok. And simply if I put this x of t is here in the right hand side, it could become equal to the left hand side ok.

So, we can say that this x t and u T is basically the solution of this linear time varying system. This is straight forward that a t of tan by direct substitution of x t in y of t ok.

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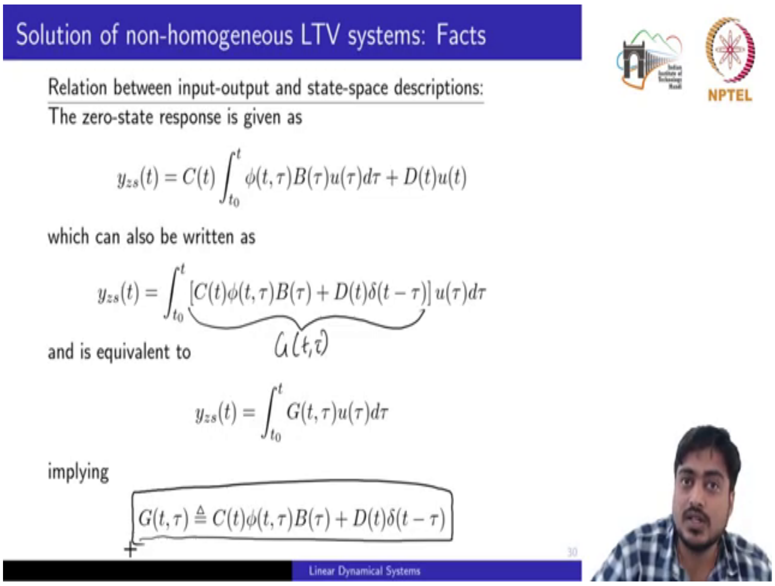

Now, let us see some interesting fair scrub out the solution because in the last lecture, we studied the impulse response, now here we would try to relate the impulse response with the state space representation of the time varying system. So, we saw in that one of the previous result that the zero-state response is given by this part ok, the zero-state response.

Now, I can write this last part by this equation meaning to say because this part d t into u t was outside the integral. Now, to take this inside the integral what I tell I multiplied by an impulse function. So, I know that at t is equal to tau this would be equal to this one.

So, I can write this zero-state response by taking this part inside the integral inside the integral and which is equivalent to the G t comma tau into u tau basically I can replace this whole part by G t comma tau and this is the impulses response. So, this is the relation between the impulse response of the time varying system and the state space solution.