Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture -19
Tests for controllability - II

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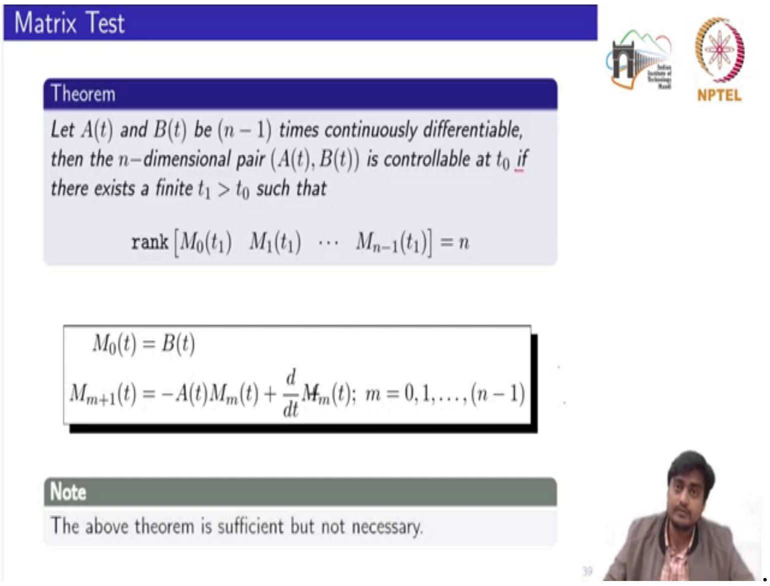

So, once we have done the bookkeeping of computing those matrices this m matrices particularly, now we could state the result that let A and B be n minus 1 times continuously differentiable, then the n dimensional pair A comma B for the time bearing case is controllable at t naught if there exists a finite time t 1 greater than t naught such that the rank of the matrices formed by all these M matrices is equal to n which is the dimension of the state ok.

So, here you would notice one thing that so far we have discussed the results which were necessary and sufficient both. There we have studied if and only if right but here you would notice that we have only if we do not have the only if right; meaning to say that, this above theorem is only sufficient but not necessary ok. Necessary in the sense that the system is controllable; if this rank condition is satisfy right. But if this, what the rank of this matrix is equal to this if the system is controllable, we cannot ensure that the second statement the otherwise the other implication.

So, here there are couple of things to be noted. So, far if we are able to compute the state transition matrix, then we could have computed the controllability Gramian. Now the non singularity of the controllability Gramian is a strong result because it was necessary and sufficient both to prove the controllability. But now if it is not possible to or if it is difficult to compute the state transition matrix, we have another result by which we can escape the computation of the state transition matrix.

But this is a weaker result because it is not both necessary and sufficient. And in addition we require that if n minus 1 times continuously differentiability of the matrices A and B which was not required in the previous stronger result ok. So, recall these two matrices the M basically the M matrix at 0 and at m is equal to 1 to n minus 1. So, now, we will see the quick proof of this.

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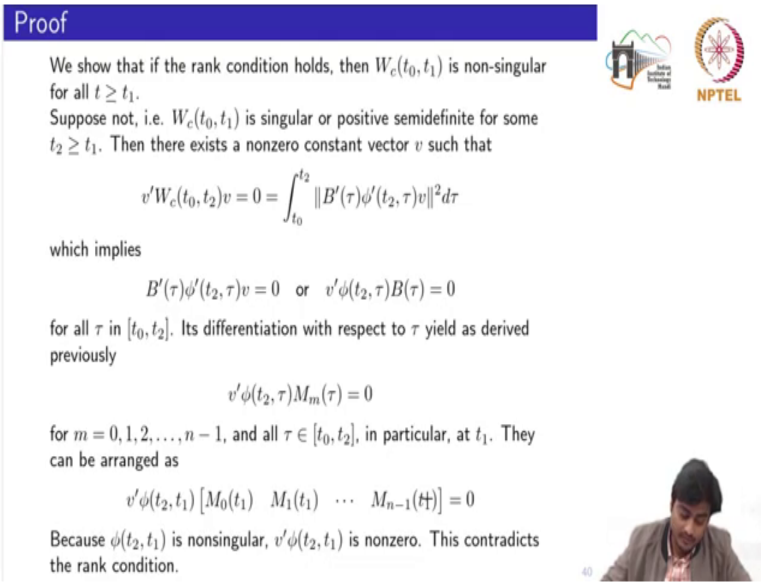

So, we show that if the rank condition holds, that the rank of that matrix then W C t naught comma t 1 is non singular right, because we have we already knew that the controllability of the pair A and B is basically equivalent to saying that the controllability Gramian is non singular ok. So, now, instead of showing that if the rank condition hold then the system is controllable, it is equivalent to saying that if the rank condition holds then the W C matrix is non singular.

So, suppose not, meaning to say that W C t naught comma t 1 is a singular or positive semi definite for some t 2 greater than or equal to t 1 right. Then again, there exists a nonzero constant vector v such that I could express this quadratic form equal to 0 because of the singularity of this weight matrix. Now we had seen in the earlier proof, that this the simplified form of this quadratic form I can write this one, which implies that this is only possible if the vector inside the norm is itself equal to 0 or its transpose equal to 0 right, for all tau in t naught comma t 2 right. Now it is differentiation with respect to tau yields as derived previously. In the sense that if phi take the m times derivative of this part. Basically I would come to this one, this is what we had done while having some bookkeeping of the defining the matrices m matrices and also their m times derivative, for all m is equal to 0 to this one ok.

Now, if I arrange all these values starting from m 0 to m minus 1 I could arrange this one also because this is 0 for all m; meaning to say it would be 0 for m is equal to 0 m is equal to 1 up to m is equal to n minus 1. So, I can arrange all these equations into a matrix equation which would become like this ok; which is nothing but equal to 0.

Now note one point here, initially we had assumed that W C is non singular sorry singular which ensures the existence of a non zero constant vector; meaning to say v is non zero, phi being the state transition matrix is non zero.

So, this would become equal to 0 only when this matrix is equal to 0, right this matrix is equal to 0 which contradicts the rank condition. Because we started we wanted to prove this but we prove the negation of that implication. That if the matrix W C is singular then the system is not controllable or the rank condition would not satisfy sorry; because the matrix is itself is 0. So, the rank of that matrix could not be equal to n right.

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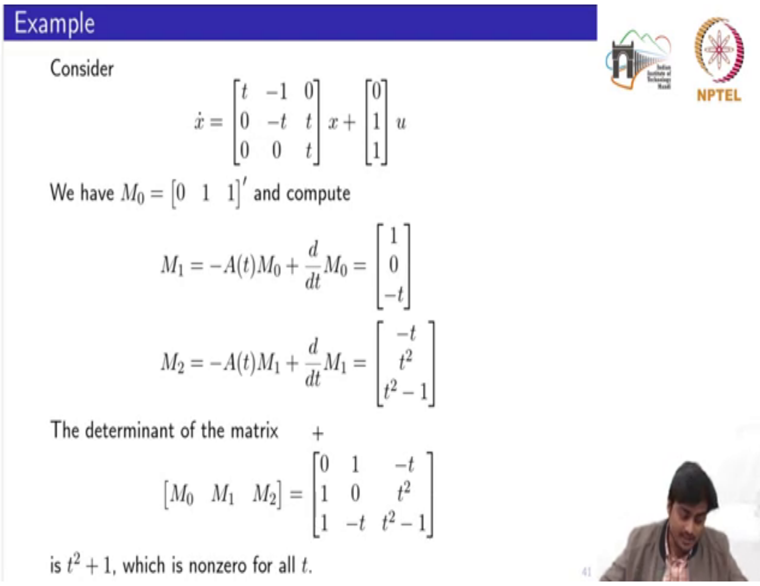

So, this was the weak results. Let us see one quick example so, that you it is pretty much clear of computing the state transition matrices and the additionally defined matrices. So, for example, we are given this A and B matrix where we have specified M naught is equal to basically the B matrix and looking at the dimension of this matrix which is 3; we need to compute 2 matrices M 1 and M 2 by using that formula we have introduced.

So, A of t is already given M naught we have taken from the B matrix and then the derivative of this M naught ok; which would which would again be 0 matrix because B is not time varying ok. So, I computed this matrix M 1 and M 2 which are time varying matrices. Now if I arrange them as a matrix, the I arrange them in the columns and then compute the determinant; it determinant come t square plus 1 which is non zero for all t.

Meaning to say that, this matrix is a positive definite matrix, if it is a positive definite matrix then the rank condition would always be satisfy ok; you should not forget that in the week of the stability lecture, we have shown that you cannot compute the. So, first of all note that to show that the matrix is a positive definite matrix, in the stability week we have computed the eigen values. We had shown that if the eigen values of the matrix are positive then the matrix is a positive definite matrix.

Now we had also shown at that time that you cannot use this computation of the eigen values on to the time varying matrices ok. Now here M is a time varying matrix. So, you in order to show that this matrix is positive definite you cannot use the eigen value test because it will not satisfy right. So, that is why we are computed determinant and I have shown that this determinant is non zero and positive for all t ok.

So, this is a pretty much faster result. Now since this is a sufficient condition you should not forget that if the determinant happens to be 0, it does not mean that the system is not controllable. The system may still be controllable this is because of that this is a sufficient condition not the necessary condition.

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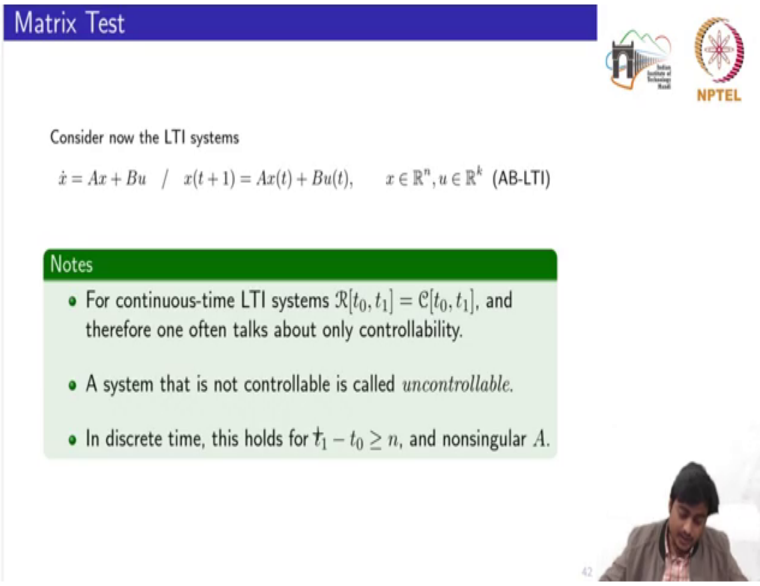

So, let us see the matrix test for the LTI system. Again we have used slash operator to denote the A, B matrices for the continuous time case and for the discrete time case. So, for the continuous time LTI system we have already seen that both the subspaces are equivalent. And we do not need to talk about separately about the reachability and the controllability.

But also we should not forget at the same time that in discrete time this holds this equivalence holds only when the matrix A is non singular ok. In one of the examples in the week 1, we had seen that this A matrix is basically could be a singular matrix.

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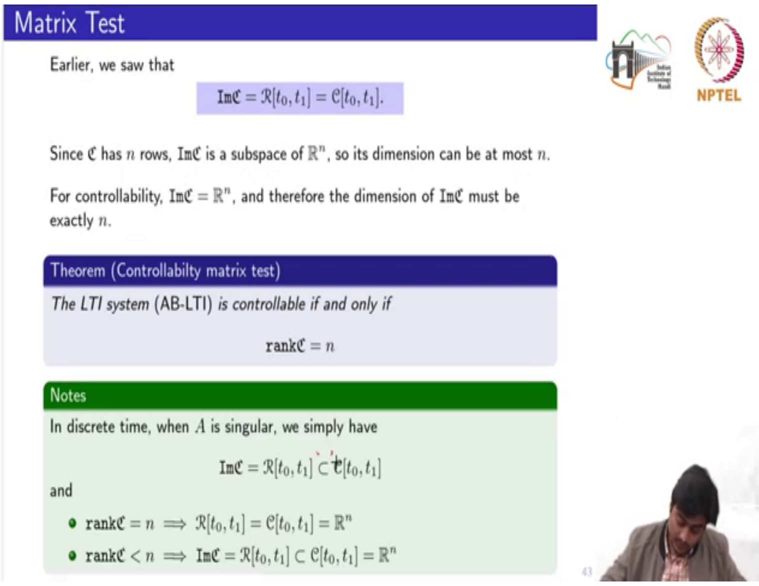

So, earlier we have seen this, the equivalence of all these three subspaces given by the controllability matrix the reachable subspace and the controllable subspace. So, since this controllability matrix has n rows; the image of that controllability sub controllability matrix is a subspace of an n dimensional space right. So, its dimension can be at most n, it cannot be greater than n since because it is a subspace of an n dimensional space ok.

So, for controllability if you remember that this equivalence we had proved equal to the n dimensional space and therefore, the dimension of image of the controllability matrix must be exactly equal to n right.

Based on this we have the next result that the LTI system AB pair of the LTI is controllable; if and only if the rank condition of the controllability matrix is satisfied for the n dimensional pair ok. So, in the discreet time when A is singular, we simply have this intuition, but not that equivalence. So, for the proof we would see that the rank of C is equal to n implies that the reachable and controllable subspaces are equivalent to the n dimensional space for the discrete time space.

Now if it happens that the rank of the controllability matrix is less than n, it means that this subspace is equal to this subspace which is a subset of this subspace of the controllable subspace. And this rank of C mathcal C is less than n one of the possible causes is that the matrix A is a singular matrix ok.

So this was all about the matrix test where we have a separate test for the time varying systems and a separate matrix test for the time invariant system. For the time varying system we had two tests, one is the non singularity of the controllability Gramian which was a strong result a weaker result. We had in terms of the computation of the additional matrices M matrices.

And for the LTI we have the rank condition on the controllability matrix which should be equal to n. Now the next test is the eigenvector test. So, before stating the main results of the eigenvector test, we will introduce a couple of important definitions and results which would help us later to state the result of the controllability matrix in terms of the eigen vectors.

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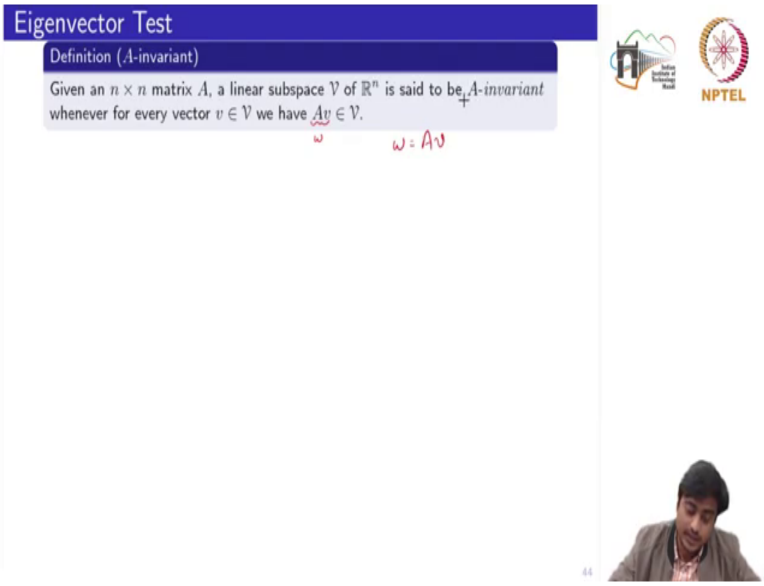

So, first definition is the A invariant spaces. So, given an n cross n matrix A, a linear subspace V of the n dimensional space is said to be A invariant, whenever for every vector v belonging to this space V we have this vector A into v should also be belonging to V ok. So, you can see this in terms of the transformation also, that if we are given with any vector v belonging to the subspace V; this I can define by another vector w and w we have the transformation w is equal to A into v by this transformation matrix A ok.

If that vector is formed by a transformation matrix given by A and that vector w also belongs to that subspace; meaning to say that subspace is A invariant. That under the transformation of the A matrix A the property of that subspace does not change or the characteristics of that subspace do not change ok.

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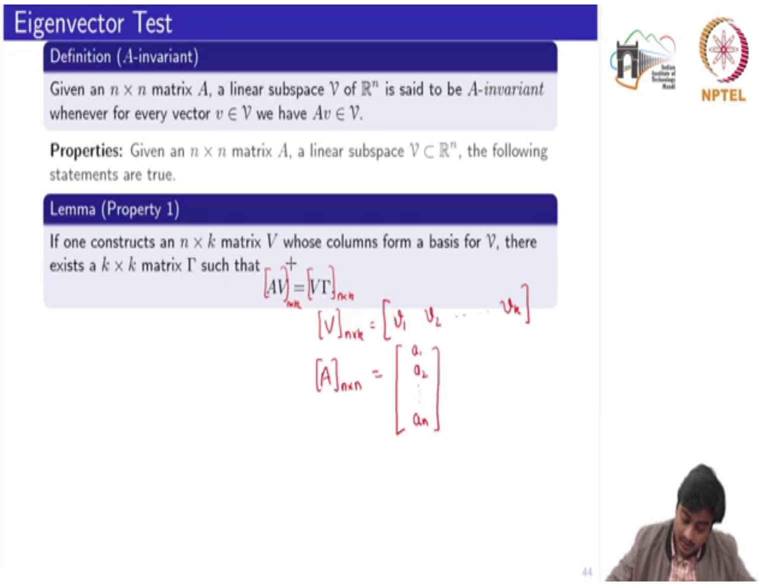

So, there are couple of important properties of this A invariant subspace which we are going to use to state the eigenvector test for the for determining the controllability. So, given a matrix n cross n of dimension n cross n a linear subspace V, the following statements are true. So, I wrote it in terms of the lemma because we are going to see a proof of it as well.

So, the property 1 says that if one constructs an n cross k matrix V ok, whose columns form a basis for that subspace V mathcal V then there exists a k cross k matrix big gamma such that AV equal V into gamma.

So try to first visualise, this lemma what it says. That if one constructs I will repeat once again that if one constructs an n cross k matrix V whose columns forms a basis for V ok. So, let us say I define a V which is of dimension n cross k ok. Now one way of writing this matrix is that I have k number of elements or k number of vectors and each vector is an n dimensional vector ok. I can write this which means that v 1 is an n dimensional vector up to vk.

Meaning to say this complete matrix would be an n cross k matrix. Now this subspace V which is a subspace or which is a subset of an n dimensional space this can easily forms a basis meaning to say that if I have this A matrix which is a n cross n matrix ok. Now, I can write this n cross n matrix as a 1, a 2 up to a n; where all a i's, i starting from 1 to n is row vector of n dimension ok.

So, every element here is a row vector of dimension n; which would give me the overall matrix of n cross n ok. So, this elements of the matrix V forms a basis for V which is an n dimensional space right which I can write with the existence of another k cross k matrix gamma is equal to this.

So, this A into V you can imagine this into this. So, A into V would be n cross k overall dimension and for this V cross V into big gamma would also be n cross k because v is n cross k. So this in fact, I just gave you the logical idea behind the proof.

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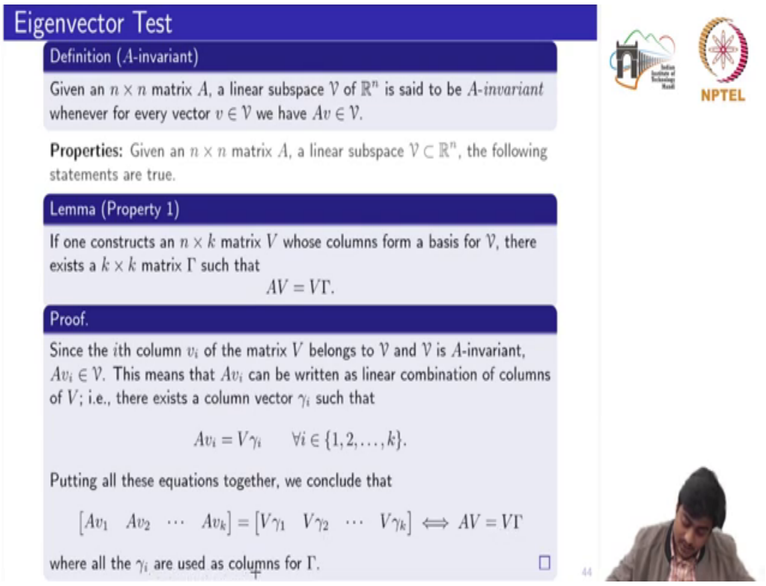

So, we can see a formal proof. So, since the i th column v i of the matrix V belongs to V and V is invariant ok. If V is invariant I would have A vi should also belongs to V coming from the definition; that if any element belongs to V then the space being an A invariant would also satisfy this one ok. So, this means that I can write A v i the multiplication of the matrix a into the i th column of the matrix V as linear combination of columns of V and linear combination meaning to say that there exists a column vector gamma i such that A v i could be written equal to V into gamma i, for all i belonging to the set 1 to k ok.

Now if I just put all these equations into the form of a matrix starting from 1 to k, I would have A v 1 up to A v k equal to on the right hand side starting from V gamma 1 up to V gamma k. So, I could take A matrix out of common here which is equivalent to saying that

AV is equal to V into gamma ok; where all gammas are used as columns for big gamma ok. So, this is the first property.

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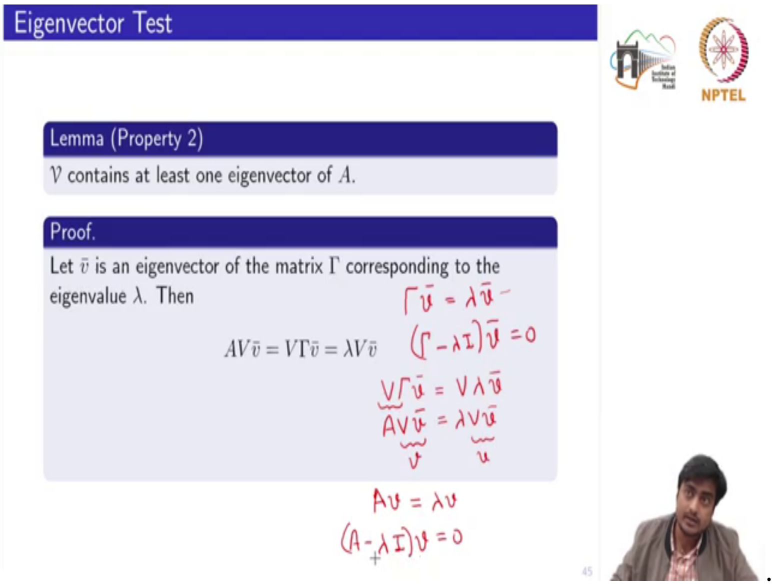

The second property says that, subspace which is an A invariant subspace contains at least one eigen vector of A ok. So, let us see the proof of it. So, suppose let v bar is an eigenvector of the matrix big gamma corresponding to the eigen value lambda ok. So, I can write this, I can write one equation based on the first statement that big gamma into v bar is equal to lambda v bar.

In fact, this equation you have been used you have been using to compute the eigen values which you can write gamma minus lambda I v bar equal to 0. Which means to say that for this matrix big gamma there is an eigen vector v bar associated to the eigen value lambda such that this equation is satisfied ok. So, starting from here I pre multiply this first equation by v.

So, I would have V big gamma v bar is equal to V lambda v bar right. Since lambda is a scalar I can commute this with the matrix V. So, I can write this lambda V v bar ok. Now in the first property we saw that this part V into big gamma is nothing but equal to A into V and this v bar ok. So, this v bar comes from here and this comes from the upper part.

Now let us define this V v bar by another vector v similarly here. So, I could write this equation I could write this equation as A v equal lambda v which I can write as A minus lambda I v equal 0. So, starting from this one that v bar is an eigen vector of the big matrix gamma corresponding to the eigen value A; now corresponding to the eigen value lambda we found another vector V for the matrix A ok.

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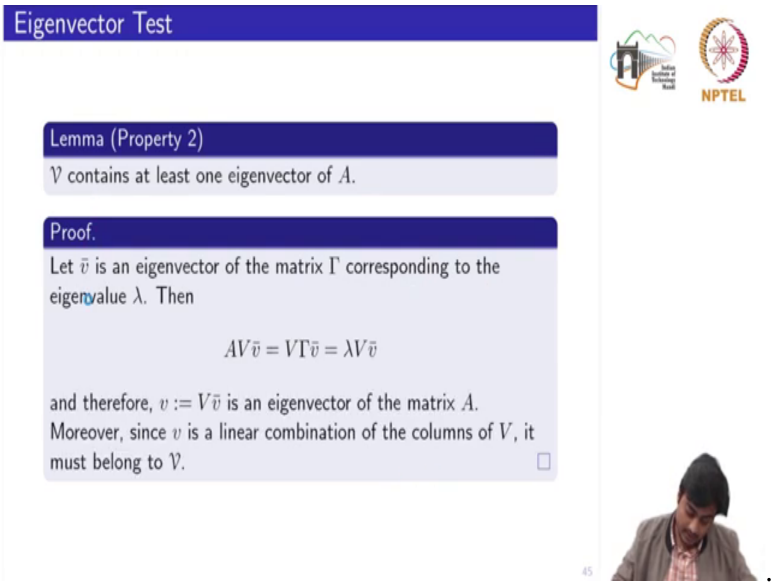

Meaning to say that V the subspace V contains at least one eigen vector of A; it might contain more than one eigen vector also ok. So, this is the proof of the part 2 and both these properties. So, this is the sketch of the definition and the properties that first of all we have defined the A invariant subspaces. And then we have defined we have defined the property 1 of the definition. And the property 2 was defined based on the results we had obtained in the property 1. Now we would going to use this definition and these properties to give the eigen vector test ok.