Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture – 18
Tests for controllability - I

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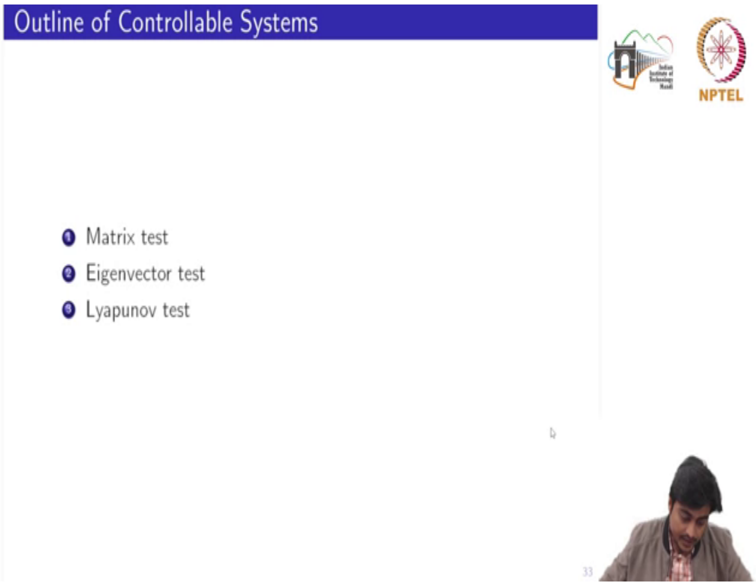

So, this is the next section of this week Controllability, where we would be starting with the Controllable systems. And these three main topics we would cover in this subsection. First is the matrix test, second is the eigenvector test and third is the Lyapunov test. So, so far we had discussed mainly about the subspaces and their characterization.

Now, we will be discussing more about their applications two systems and see how we can determine whether the system is controllable or not.

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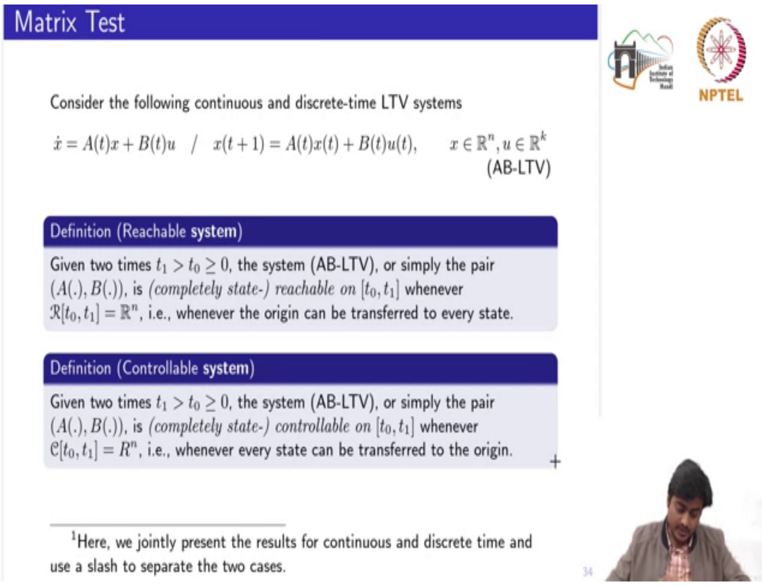

So, first is the matrix test. The continuous and discrete time LTV systems can be represented by these pairs A of t and B of t. So, it should be implicitly clear whenever we are discussing about the continuous time system or the discrete time systems and most of the time we would be using the slash if we are giving the parallel results for the continuous time and discrete time, but the matrices would remain the same and also the denotion of the time operator.

So, talking about the reachable systems, so given two times t 1 greater than t naught the system; now, we are formalizing all those subspaces in terms of the systems. The system or simply the pair AB is completely state reachable on this time interval t naught to t 1 whenever the origin can be transferred to every state. So, note this point every state, we will make use of this in our later results. The controllable systems, the pair is controllable on this time or completely state controllable on this interval t naught to t 1 whenever every state can be transferred to the origin, ok.

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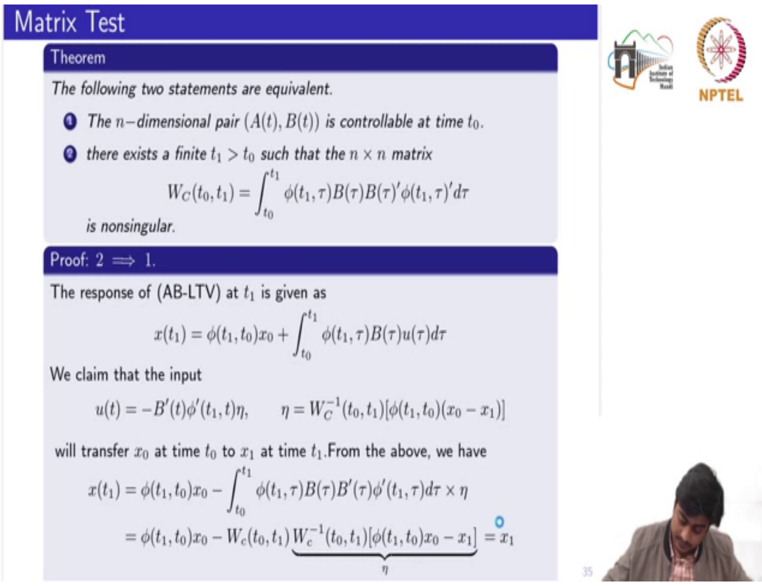

So, the first result says that the following two statements are the equivalent the n dimensional pair A and B in. For the time variant case is controllable at time t naught there exists a finite time t 1 greater than t naught, such that the n cross n matrix W c given by this which is basically the controllability Gramian is non-singular, ok.

So, if you recall that when we had introduced about the controllability Gramian when we were discussing about the subspaces, we make an assumption that the controllability Gramian is symmetrix and positive, semi definite when a matrix is positive semi definite, then it is a singular matrix. Now, it gives us a strong result that that controllability Gramian if that Gramian is non-singular or positive definite, then the system would definitely be controllable.

So, since this is one of the important results. We will go through proof of this and along and while going through the proof, we will explore other minor results which might be helpful also. So, first we will see the proof of these two implies 4 that is to say if this controllability Gramian matrix or this n cross n matrix is non-singular, then the system is controllable. So, this information is given to us that this matrix is already a non-singular matrix.

The response of this AB-LTV at time t 1 is given as this by the variation of constants formula. So, we claim that the input, so recall that this input is nothing, but quite similar to the input we had specified while discussing about the characterization of the controllability subspace with the controllability Gramian. So, we claim that this input where eta is specifically given by this now. If you recall when we have specified this u, we had taken some arbitrary vector eta. Now, here we had specified eta to be this one meaning to say that. So, first recall that the controllability says starting from x naught, we want to reach to some to some finite or some x 1 value.

Now, this result is a bit broader in the sense that we want to transfer x naught at time t naught to x 1 at time t 1, right. So, here either of that could be 0, because we have already seen that in the continuous time case, both these definition reachability and controllability coincides, ok. So, we claim that this input can transfer this from x naught to x 1, where ets is given by specifically this one; so now what we will do. We will apply this u here on the right hand side, and then we will show whether the value of x at t 1 is basically coming equal to x 1.

So, applying this value it would be this one sense eta is a constant vector. It is not a function of time it is a function of specific time. So, all these entries in this ets would have a definite value. So, that is why I can take this eta outside this integral which I am integrating over tau, ok.

So, now see some similarity this part this whole part is nothing, but your W c t naught comma t 1 and eta is given by this one. So, we would have this part would come as it is, this is we have W c t naught comma t 1. This is the inverse of the controllability Gramian which comes from the eta and phi and x naught minus x 1.

So, now this part and this part would become an identity phi t 1, t naught, x 1 x naught would cancel out from here, this part and the rest part remains negative of negative x 1 meaning to say that x at t 1 is basically equal to x 1.

Now, the point of having that assumption that the controllability Gramian is a non-singular matrix because if it is not, then I could not express eta as like this because in that case W the inverse of W c does not exist. So, this is the proof of the part that if this matrix is non-singular, then the system is controllable and the controllability of the system ensures that starting from some point x naught at m t naught I could reach to x 1 at time t 1 is equal to x 1.

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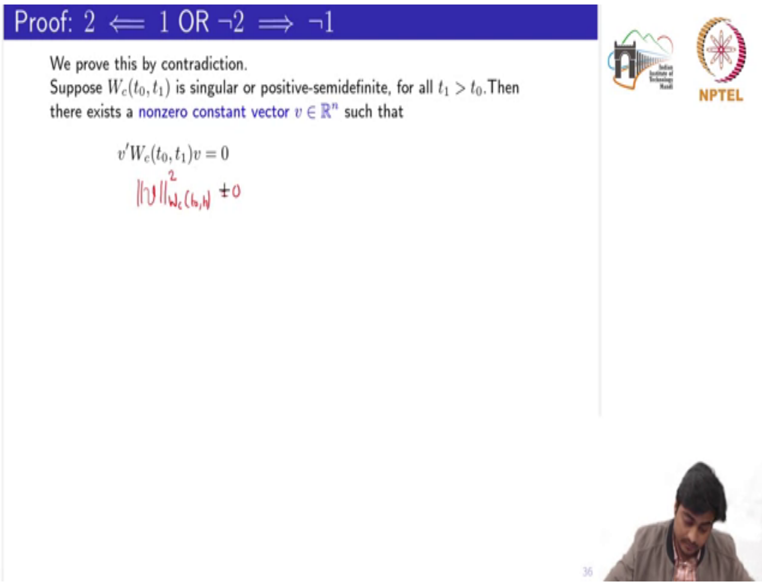

So, now for doing the proof of the one implies to meaning to say that if it is given that the system is controllable, then it implies that the matrix W c is non-singular. As we have seen earlier, we could also show that if the system is not controllable, then it implies that the matrix W c would be singular, right.

So, we will prove this by contradiction. So, suppose W c t naught comma t 1 is singular or positive semi definite for all t 1 greater than t naught, right. What does it mean that there exist a non-zero vectors v, such that v transpose W c into v is equal to 0. This is nothing, but the quadratic form we had studied in the week of stability, because if this matrix is positive semi-definite, then there exist some non-zero vectors. B is not necessarily 0, but still it could the quadratic form could be 0 or I could write this as norm v square with a weighted matrix W c t naught comma t 1 equal 0, ok. W c is basically the weight.

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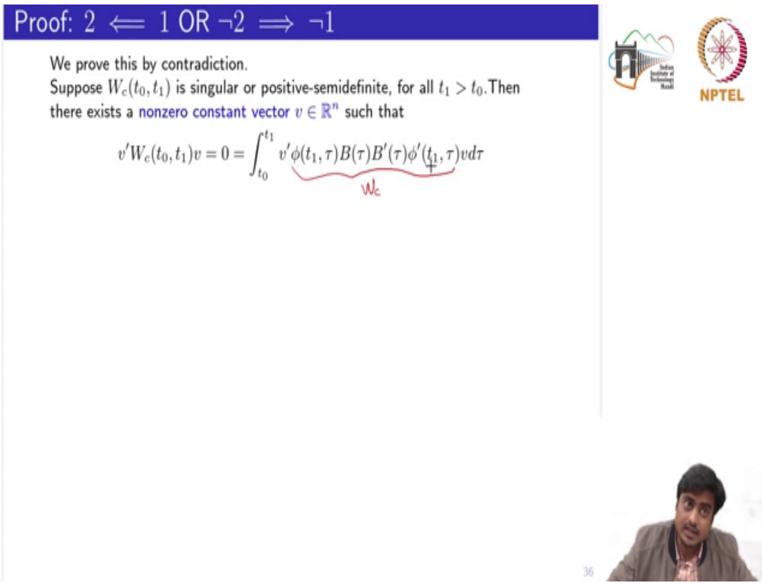

Now, if I put W c from the result here, I would have this entire part is basically W c and I can take this constant vectors v inside the integral, ok.

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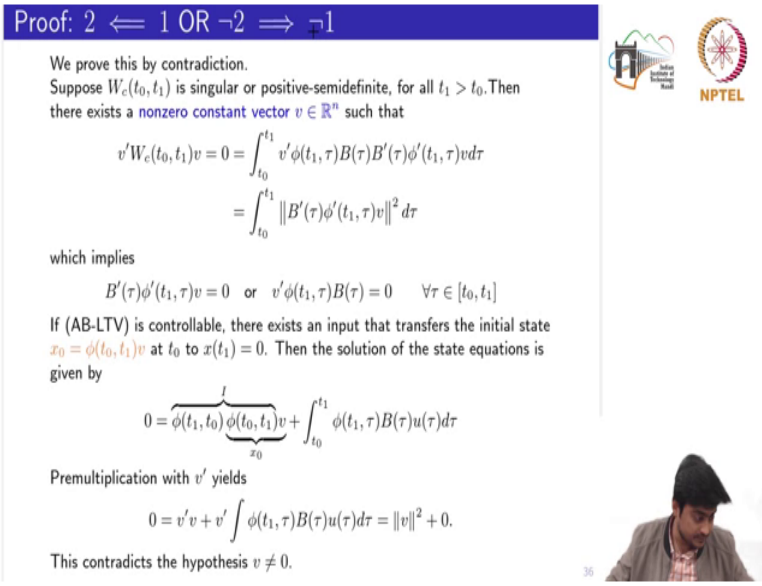

Now, see some similarity that again I could write this as a norm which is given by this one. The norm of B transpose, phi transfers and v, the squared norm and this integral, ok. This is the property of using equivalence between the quadratic form and the norm itself.

Now, using the one of the property of the norm that if it is equal to 0, it will only B equal to 0, if this the matrix or the vector inside the norm itself is 0 which implies this is 0 or I could take the transpose of this part and conclude this one. So, I would have B transpose phi into B equals to 0 for all time tau inside the integral t naught comma t 1.

Now so, suppose if the system is controllable, then there exist an input that transfers the initial state x naught which is specified here at t naught to x t 1 equal to 0. So, the idea here is we had given that the matrix is a singular matrix, ok. Now, we and we came to one result. Now, we want to prove this by contradiction saying that if the system would have been controllable meaning to say that given any initial state which is specified here in the terms of the state transition matrix and the non-zero vector. So we know already that this phi is not 0, this v is not 0.

So, we are not actually starting from 0 because it might be also possible that starting from x naught is equal to 0. We are reaching to x at t 1 is equal to 0 and this is what we had seen in the one of the examples, that starting at some 0 initial conditions if the value of the state at other time t 1 is also equal to 0, then it might be possible that the input does not help any influence over it. So, first of all we need to ensure that this x naught should be non-zero ok.

So, if the system is controllable, then there exists an input which could transfer from x naught to x of t 1 equal to 0. Then the solution of the state equation is given by this one. So, x of t 1 is equal to 0. This part phi t 1, t naught if the state transition matrix this part is the, my initial condition x naught plus the rest of the part, ok. And this part by using the property of the state transition matrix, we had studied in the first week that the multiplication of the state transition matrix t 1 comma t naught, and t naught comma t 1 is basically equal to identity matrix, ok.

So, I would have so from here what we would have this part v plus the rest of the part. Now, if I pre-multiply that equation by v transpose I would have v transpose v plus v transpose the rest of the part, ok. Now, this part v transpose v is nothing, but equal to the norm square plus this one is already 0 from the above.

So, I would have the square of the norm is equal to 0 and this is only possible if the vector itself is 0 and from beginning we had assumed that the it is a non-zero vector. So, this was the contradiction which means that if the system or sorry if the controllability Gramian matrix is singular, then it implies that the system is also not controllable, ok. So, this completes the overall proof.

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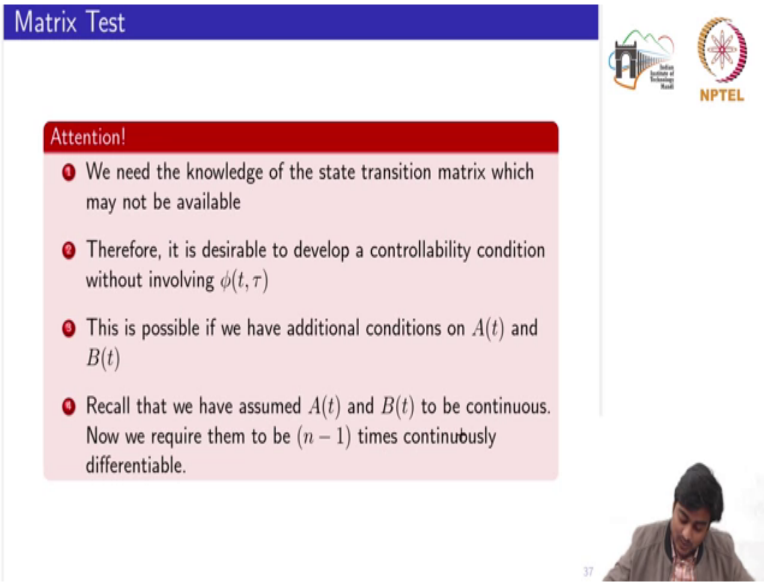

So, moving forward in this direction; so, we had seen that for computing the controllability Gramian we need the knowledge of the state transition matrix which may not be available. In fact, we had seen in the first week that there are certain restrictions in computing the state transition matrix and only for specific structure of the a matrix sometimes it is possible. But for a generic matrix it might not be possible to compute the state transition matrix.

Now, if it is not possible to compute the state transition matrix, you cannot compute the controllability Gramian. So, we need to find some other conditions such that we could escape the computation of the state of transition matrix, ok. So, to do this we require some additional conditions on the matrices A and B. What we require that we had already assumed that the AB matrices are continuous first of all, but now an additional condition we required that these two matrices to be n minus 1 times continuously differentiable, ok.

So, to in order to relax the computation of the state transition matrix, we need some additional condition on the AB matrices to be n minus 1 time continuously differentiable.

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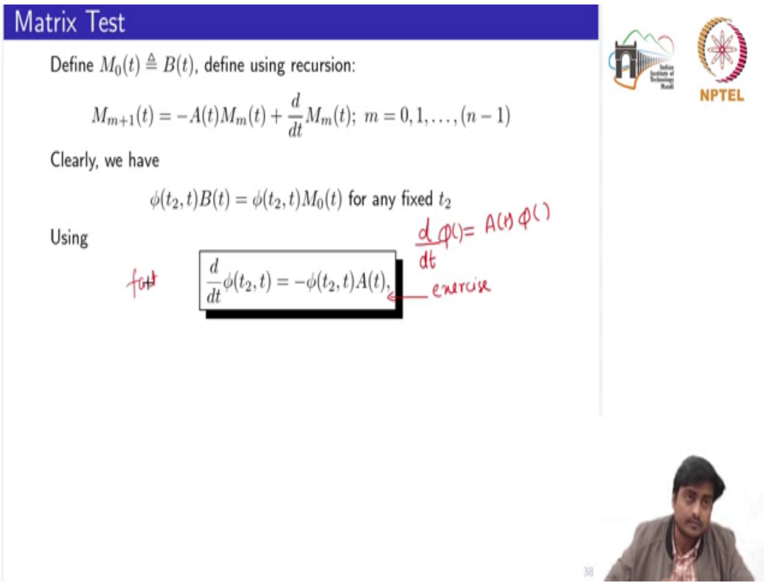

Now, before looking at that result we will do some book keeping, so that we have clear proof by construction itself. So, here we define two matrices; one is the M naught which is given by the B of t itself and another using the recursions M naught or starting from M 1 to M n minus 1. So, we compute n matrices basically starting from 0 to n minus 1 and using this equation minus A into M m plus the derivative of M m. And using these two equations we would going to state the result which does not require the computation of the state transition matrix.

So, let us say if I write phi t 2, t for any fixed t 2 and multiply with the B matrix, it is nothing but equivalent to this. Here, I just replaced B t by M naught which was the in the definition initially. Now, I use this result that the derivative of the state transition matrix is equal to the negative of the state transition matrix into A of t.

Now, recall from the first week the original result says that if I take the derivative of the state transition matrix, then it is equal to A of t into the state transition matrix, because this infect one of the property of the state transition matrix that it should satisfy the homogenous equation for the time variant case. But this stander property is different from this one.

So, I suggest that you take this as an exercise and show this that the property of the derivative of the state transition matrix is this one starting from here come to this part that this is now becomes a fact, which you can prove by yourself ok.

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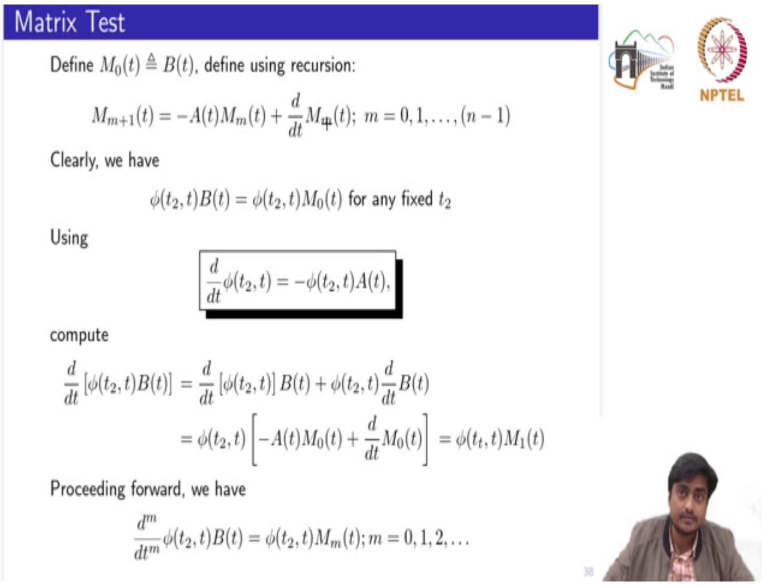

So, now we compute the derivative of this term which was given on the left hand side. So, the derivatives by using the chain rule we have the derivative of the first into B t plus the first term into the derivative of the second term.

Now, here this part I would replace by this one. So, it would give me minus phi t 2, t into A t here multiplied by B t and B t is nothing, but my M naught. So, using this above I substitute all those in this equation and finally, write this simplified one where I have taken the state transition matrix common out of this standard term, ok.

Now, notice here that this term, this whole term is nothing, but your M 1. If I put M is equal to 0 into this equation, I would have M 1 is equal to minus A t M naught plus d by d t of M naught. So, this becomes nothing, but equal to phi sorry, it should be you should treat this as t 2 phi t 2, t into M 1 t.

Now keep taking the derivatives. So, proceeding for what I take the M times derivatives of the left hand side and finally, I would see that this part is nothing, but it every time is equal to the multiplication of the state transition matrix for any fixed time t 2 multiplied by this M m which we have defined here by using recursion, ok.