Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture - 17
Discrete-time Reachability and Controllability Gramians

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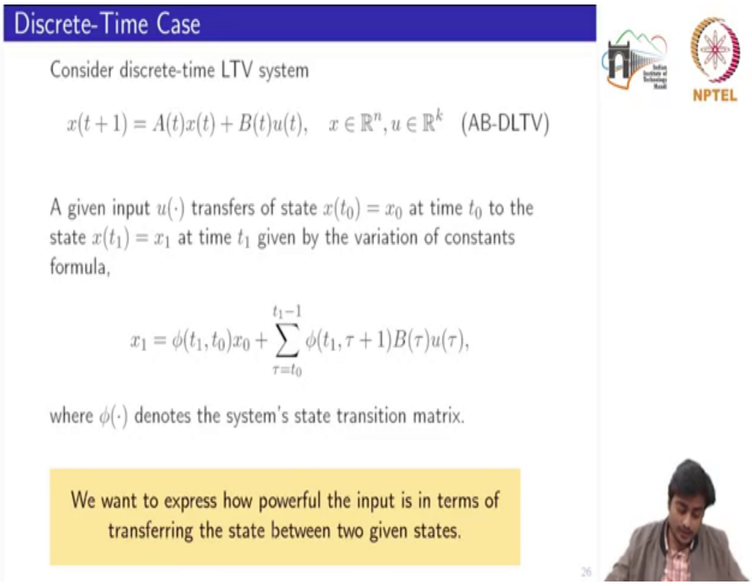

So, moving forward into the Controllable and Reachable Subspaces Sections; so, far we have discussed about the LTI systems and in fact, the continuous time, linear time varying and linear time invariant systems. So, one of the important conclusion we reached that for the continuous time systems, the controllable subspaces and reachable subspaces basically coincide.

Now, the natural question arises whether they also coincide for the discrete time scenario. So, today we will start with the discrete time systems. So, consider the discrete time LTV system given by these A t and B t pairs. Now, t is belongs to a set of integers. So, a given input u transfers the state x naught at time t naught to the state x 1 at time t 1 given by the variation of constants formula.

So, we would be using the similar concepts that with which we started by doing the analysis for the linear systems. So, again we want to express how powerful the input is in terms of transferring the state between the two given states.

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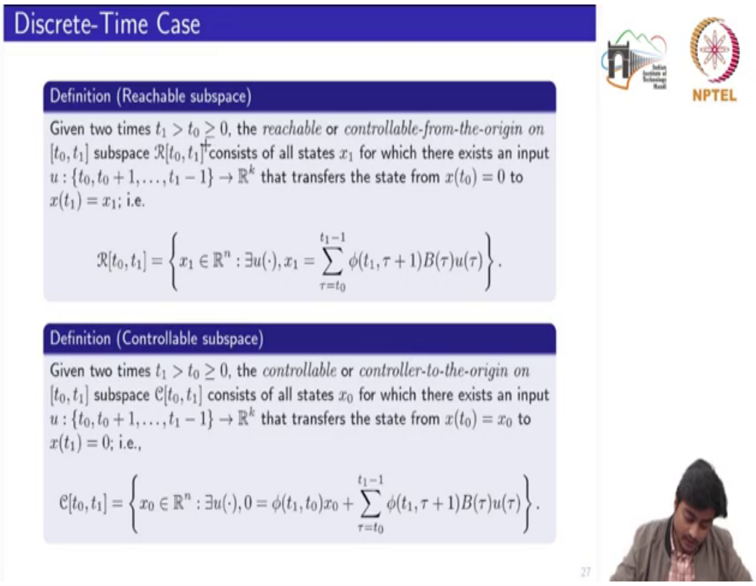

So, these were the two subspaces we have discussed for the continuous time systems. So, again recalling them for the discrete time systems; given 2 times t 1 greater than t naught greater than equal to 0. So, note that again that do not forget, in fact, the t here is belongs to the set of integers. So, we are talking about the discrete time systems.

So, the reachable or controllable from the origin on within this interval t naught to t 1, consist all those states x 1 for which there exists an input starting from, so it would be sequence of inputs starting from u t naught to up to t 1 minus 1, that transfers the state from x t naught equals 0 to x 1 which is given by this set. So, this is the reachable subspace.

In a similar, way we define the controllable subspace. There we want to see or we want to form a set of all states x naught for which we could ensure that there exist an input u that transfers the state from x naught to the origin. So, again there are two concepts one from the origin we want to take the state to some value, so let us say x 1 which is the which would give me the reachable space the controllable space, we I am looking for all those points where, starting from where I can reach to the origin.

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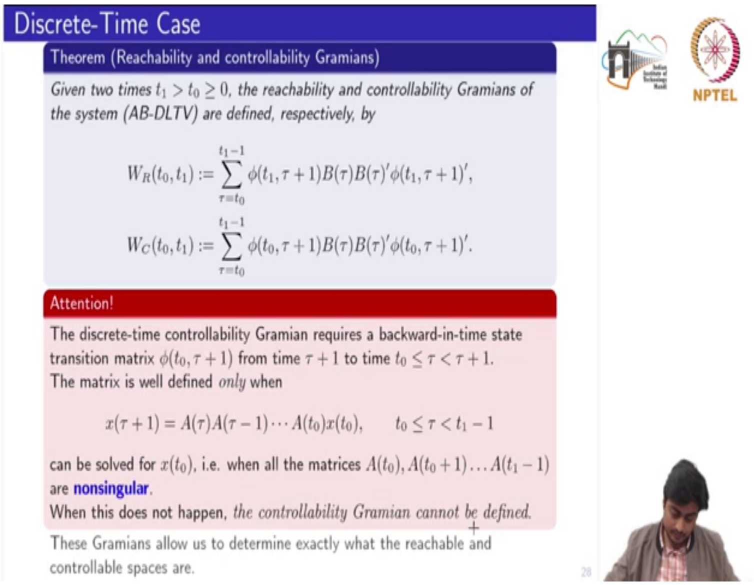

So, talking about the Gramians in a similar way what we had discussed for the continuous time systems. So, we defined two Gramians, one is the reachable Gramian and or rechability Gramian, another one is the controllability Gramian. So, the idea of introducing these Gramians is to characterise the complete subspace.

So, for the reachability Gramian it is given by this summation starting from t naught to t 1 minus 1, the state transition matrix from t 1 comma tau plus 1 into B tau and it their transpose. Now, note here about the controllability Gramian. So, here the summation here is taken from t naught to t 1 minus 1 and phi is t naught comma tau plus 1 and B tau and their transpose.

So, here you need to make a bit of attention there that the discrete time controllability Gramian requires a backward in time state transition matrix because say for example, if starting with tau is equal to t naught it would give me phi t naught comma t naught plus 1 and the state that two are (Refer Time: 04:26) in within the state transition matrix says that starting whenever the input is applied at this time I am collecting the response at this point.

So, that I need to compute the backward in time which is quite visible in the state definition of the state transition matrix; so, it imposes some restrictions in the sense that this matrix is well defined the state transition matrix is well defined only when all these matrices the matrices at different times are non-singular. Say for example, if we see the response of the discrete time systems this is how we computed the response by using the advances in the time because in fact, solving or computing the solution of a discrete time state space system is basically solving algebraic equation. So, we use some recursions to compute the solution.

Now, if I want, so this controllability or controllable subspace is basically we want to characterise all these points starting from where I can reach to the origin. Now, if I want to compute this point x of t naught, what I need to do? I need to pre multiply both sides by the inverse of these matrices. So, I need basically I need to compute this one. So, to compute this one and for the solution that all these matrices should be non-singular, ok; so, when this does not happen the controllability Gramian cannot be defined.

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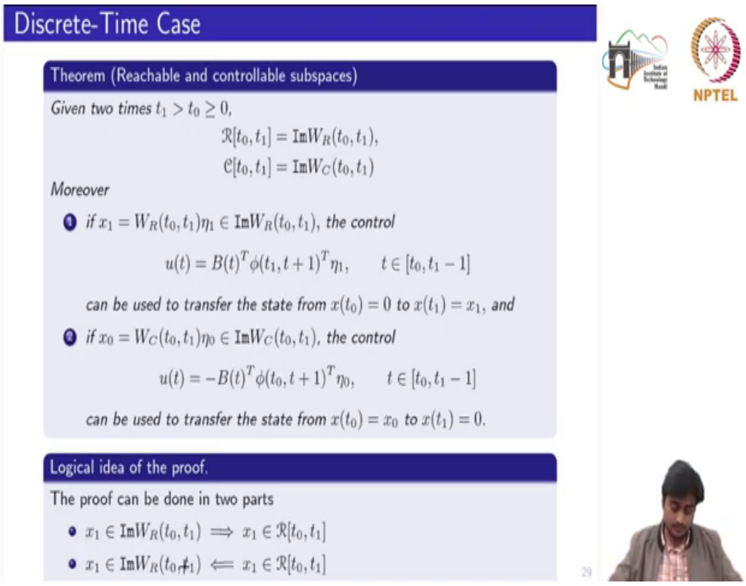

Then the characterization of the subspaces in terms of the subspaces given by the Gramians; so, given 2 times t 1 greater that t naught greater that equal to 0, the reachable subspace is basically equivalent to the subspace formed by the image of the reachability Gramian. Moreover, if x 1 belongs to the this subspace is equivalent to saying that I can write x 1 is equal to reachability Gramian into some vector eta 1 this control which is given by here can be used to transfer the state from the origin to some point x 1, In a similar way, we give the result that the controllable subspace is equal to the image of the controllability Gramian.

Moreover, if x naught belongs to this image of or belongs to this space this specific control for some eta naught can be used to transfer the state from x naught to the origin. So, we will not be taking the or going to the proof for this because we have already seen the proof for the for the continuous time system, but which you can do by yourself.

So, the idea here we need to do the proof in two parts, basically both ways implication this that is to say if x 1 belongs to this subspace implies that x 1 also belongs to this reachable subspace, the other and the other side implication. So, once both these implications are shown for the reachability Gramian and controllability Gramian then you could have the proof for this Gramian.

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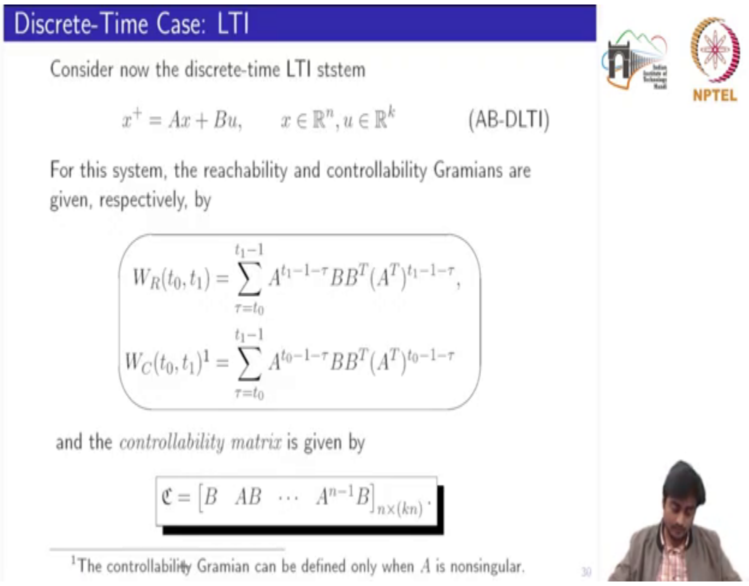

Now, for the LTI system when we discuss about the for the continuous time continuous time scenario, we introduced one metrics which is which we called the controllability matrix. So, again the controllability matrix can be computed in a similar way. So, there are three important parts, one is the discrete time system itself, the controllability and the reachability Gramian, in addition we define the controllability matrix and we set up one relationship between the Gramians and the controllability matrix, right.

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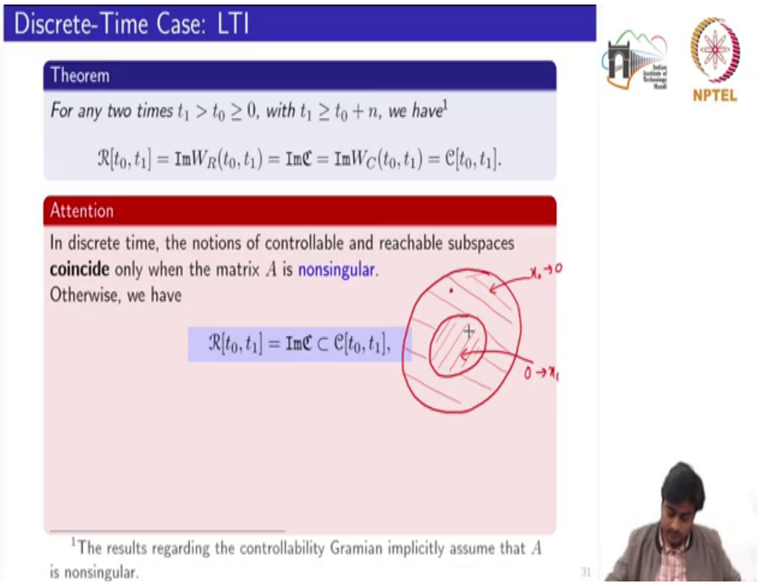

So, this was one of the key result where we have shown the equivalence between all the subspaces formed by the reachable subspace its characterization with the help of the Gramians and for the LTI case we use the controllability matrix and we have shown the equivalence of for of all the subspaces. So, in the discrete time case, the notions of controllable and reachable subspaces coincide only when the matrix A is non-singular. One of the reasons we had seen there because you will not be able to compute the solution or you will not be able to compute the x naught.

Now, if it does not happen it might be possible that at certain x m point, x m time this matrix A could be non-singular or could be sorry singular. In this case, the reachable subspace would be equal to this space given by the image of the controllability matrix and this both these spaces would be subset of the controllable subspace, ok. So, try to visualise what is happening here.

Now, if the matrix a in non-singular it is pretty much clear that we will have both sides equivalence, ok, but now let us say we have this bigger space and subset or sorry the subspace. Now, this we here we have all the points the which is the reachable subspace where from origin we want to go to x 1, right. Now, this subspace says starting from x naught we want to go to origin, right. So, this in inner space is basically the reachable subspace and this is the outer one is the controllable subspace.

So, what does it tell? Let us say if I pick any point, if I pick any point here then it says that there are some points or there are some states from starting from where I can go to the origin. But, these points start or starting from origin I cannot go to these points because these points are out of this subspace, ok.

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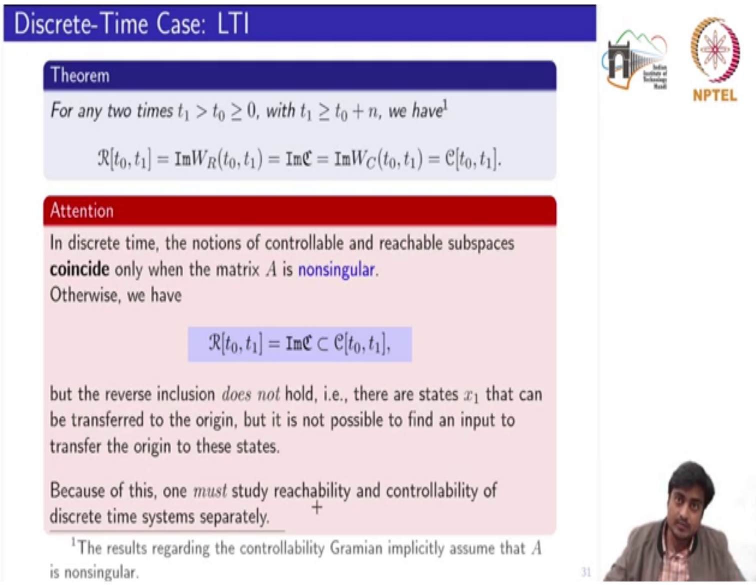

So, this is the geometrical meaning the reverse inclusion does not hold because if the reverse inclusion holds then it means to say we would be having the equal sign with which we were having in the main result. So, that there are states x 1 that can be transferred to the origin, but it is not possible to find an input to transfer the origin to the states to these states, ok. So, because of this one must study reachability and controllability of discrete time systems separately.

Now, for the for the continuous time system we could study either of them and proved our result prove our results. Now, for the discrete time if the matrix A is non-singular, oh, I am sorry if the matrix A is singular then we need to study separately both these concepts the reachability and the controllability.

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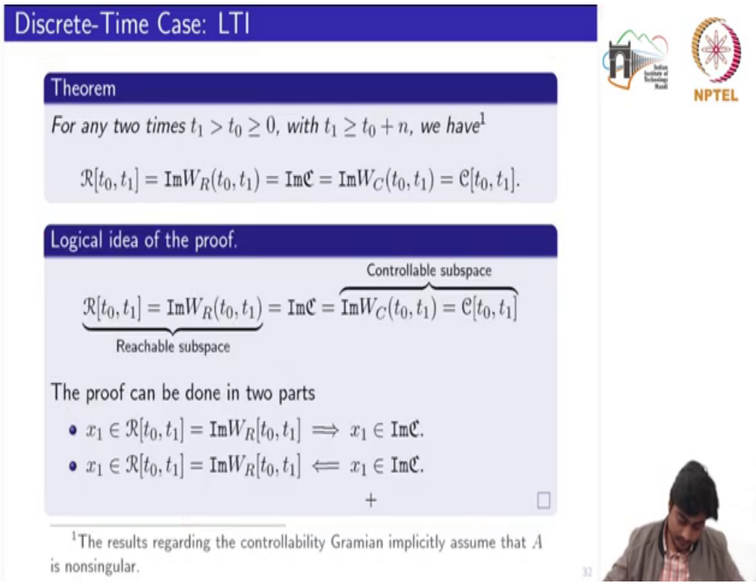

Again, the proof could be done in two parts, similar to what we had done previously. This equivalence comes from the controllable sub space and its characterization and this reachable subspace, we need to only show these two equivalence both sides which you can do either starting from reachable subspace result or from controllable subspace result ok.

Note that here whatever the results we are talking about where we have considered the controllability Gramian it implicitly assume that the matrix A is non-singular, ok. Because why, because the controllability Gramian includes the backward computation of the state transition matrix and the backward computation is only possible if the matrix a is non-singular for all time t in case of time varying and in case of time varying it should be only non-singular, ok.