Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture - 16
Controllability Matrix

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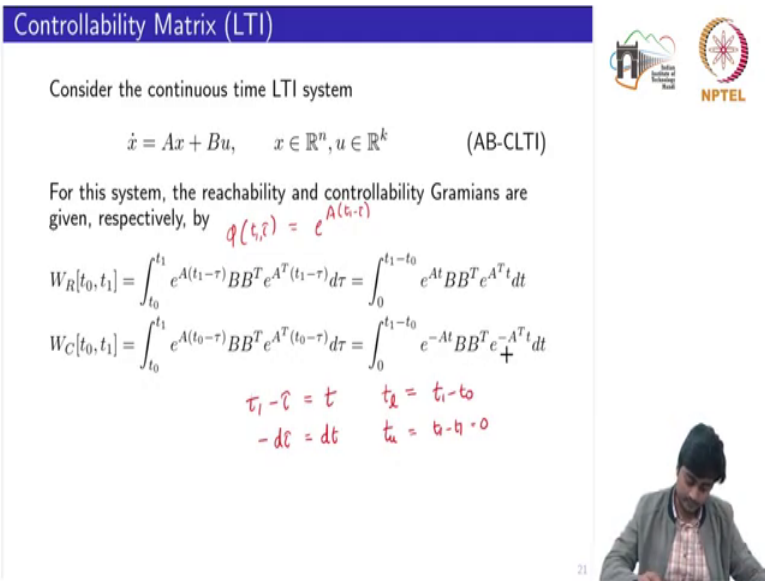

So, now we shall see the results for the LTI System. So, for the LTI system as the title of the slide says the controllability matrix. So, we will going to define a controllability matrix and we will construct some space, some subspace out of there controllability matrix and then we will try to compute the reachable subspace and the controllable subspace. So, consider the continuous time linear time invariant system given by this A B because we are only concerned with the A B matrices not with the CD matrices, where x is an n dimensional vector and u is a k dimensional vector.

So, for this system the reachability and controllability Gramians are given respectively by this. So, we have already defined the Gramians in terms of the state transition matrix. Now, for the LTI system we already knew the computing the state transition matrix is pretty much difficult, but for the LTI system we have some flexibility that we can express the state transition matrix in terms of exponential matrix, ok. So, this is what we have done.

We just replace all the state transition matrix from phi to the exponential matrix with a single argument t and tau, t minus tau, ok. So, here we would have. So, if you recall we would have phi t 1 comma tau as the state transition matrix. So, it would become for the LTI case, it would become e to the power a t 1 minus tau, ok. So, this is what I have written here similarly it would be the transpose. Now, and similarly for the controllability Gramian we could replace with the exponential matrix with a slight change from t 1 to t naught, ok.

Now, see here if I replace t 1 minus tau let us say we replace t 1 minus tau by another variable t. So, changing this t tau would yield minus of d tau is equal to dt, ok. Now, computing the lower and the upper limit of this new integral, let us say t lower t lower would be I would put t naught, so it would become t 1 minus t naught and t upper limit would become t 1 minus t 1 equal 0, ok. So, it could start from t 1 minus t naught to 0 with a negative sign because of this part.

So, use, so replacing that negative sign, we just changed that lower and the upper limit from t 1 minus t naught 0 to 0 to t 1 minus t naught, ok. Similarly, we have done here, so that we could compactly write the integrand e to the power At B and its transpose similarly e to the power minus At B into its transpose, ok.

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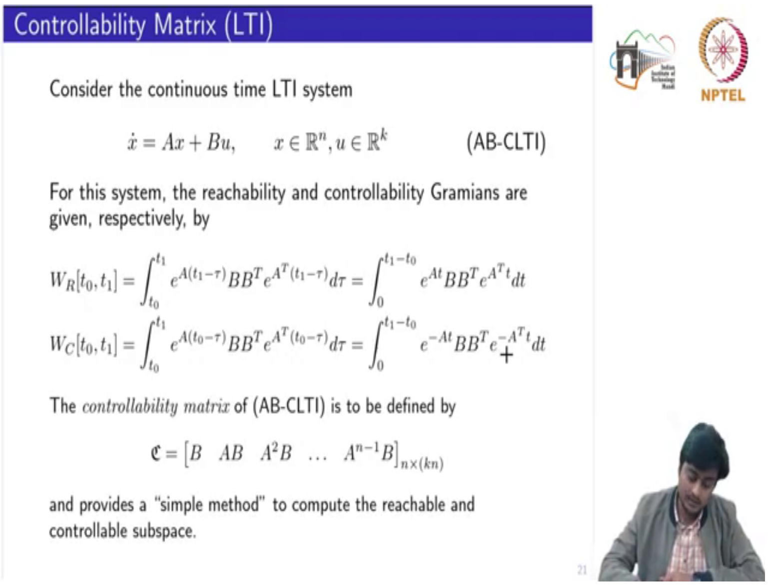

Now, in addition to that we define another matrix called the controllability matrix of the pair AB which is defined by this symbol which is in the (Refer Time: 03:41) environment B, AB, A square B up to A to the power n minus 1 into B, where n is the dimensional of the x. And the dimension of the controllability matrix would be it would contain n number of rows n k into n number of columns, the multiplication of this k n. So, this provides a simple method to compute the reachable and controllable subspace.

Now, we already knew from the LTV systems that we can or we can compute the reachable subspace and the controllable subspace by computing the images of these matrices, ok, but having an LTI system gives us more flexibility in representing those result in terms of this specific matrix.

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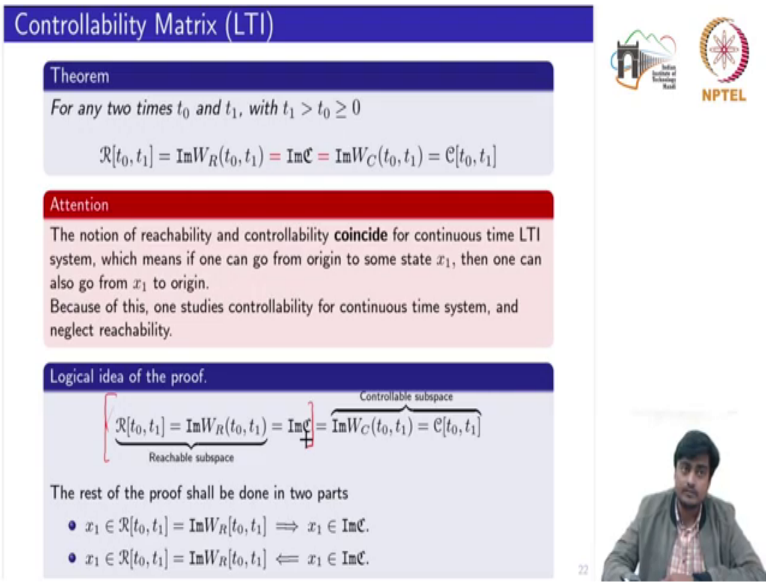

And this result is given here, that for any two times t naught and t 1 with t 1 greater than t naught greater than equal to 0 we have reachable space is equal to the image of the reachability Gramian which is again equal to the image of the controllability matrix, equal to the image of the controllability Gramian equal to though a controllable subspace, ok. This is a very very important result. Why?

So, it tells us many things. First of all you see these two equal signs which are here now denoted by red. So, this notion of reachability and controllability at least for the LTI system coincide with each other, meaning to say either I can use the either I can show the reachability or the controllability both remains the same, ok.

So, if one can go that is to say if one can go from origin to some state x 1 then one can also go from x 1 to origin which is not or generally true for the LTI systems that is why we carry out the analysis of both the subspaces, ok. So, because of this one studies controllability for continuous time systems and neglect reachability. So, for the LTI system we will mostly study about the controllability, but whenever we are discussing for the time varying systems we will see the analysis of both the subspaces, ok.

So, as I said this is an important result. So, we will go through a detailed proof of this results. So, this part we have already proved while having a result on the reachable subspace and this part is for the controllable subspace. So, we need to prove either of this equality because we know at the outset this is equal to this for the LTI systems, ok.

So, again this proof will be done in two parts because we are here again we are talking about the equivalence of two subspaces, meaning to say if there is any element belonging to one subspace, if that element also belongs to another subspace meaning to say both subspaces are equivalent and this is valid for all the points in that one particular subspace, ok.

So, again x 1 belonging to the reachable subspace which we have already shown is equal to the image of the matrix W R, also implies that x 1 also belongs to the image of this controllability matrix. Again, just since it is an equivalence we will close the path by having a implication otherwise in the another direction, that x 1 belonging to this also implies that x 1 also belongs to the reachable subspace, ok. So, either of this equality you can prove. We are going to show this part of the proof because this would be automatically equal, ok.

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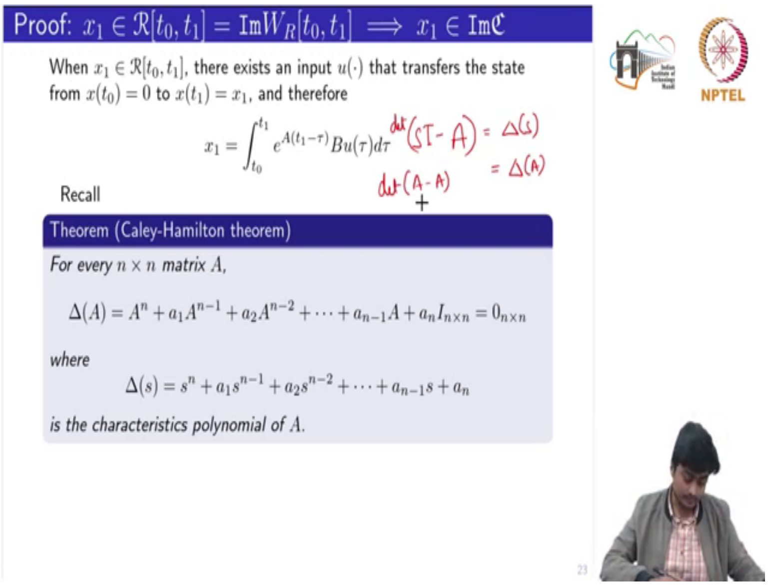

So, let us see the first part. So, again starting along the same lines that when x 1 belongs to this reachable subspace there exists some input you that transfers the state from 0 to x 1 and therefore, x 1 is given by this one. This is basically comes from this solution of the LTI systems, ok. So, we already know that there exists some u, ok.

Now, quickly we will recall the Caley-Hamilton theorem because we were going to use this result. That for every n cross n matrix A delta of A is equal to this one should be equal to 0 matrix, where this delta of s is basically the characteristics polynomial of A, ok. So, given any matrix A; given any matrix A we have been computing the characteristic polynomial by computing the determinant of this matrix basically this is delta of s, ok.

Now, we already know that delta if I put if I replace s by A, here I would have determinant A minus A, so it would be equal to 0. Now, this comes from the Caley-Hamilton theorem we want to see the proof of this theorem, but using that theorem I can write this part this exponential part or any exponential matrix by this matrix.

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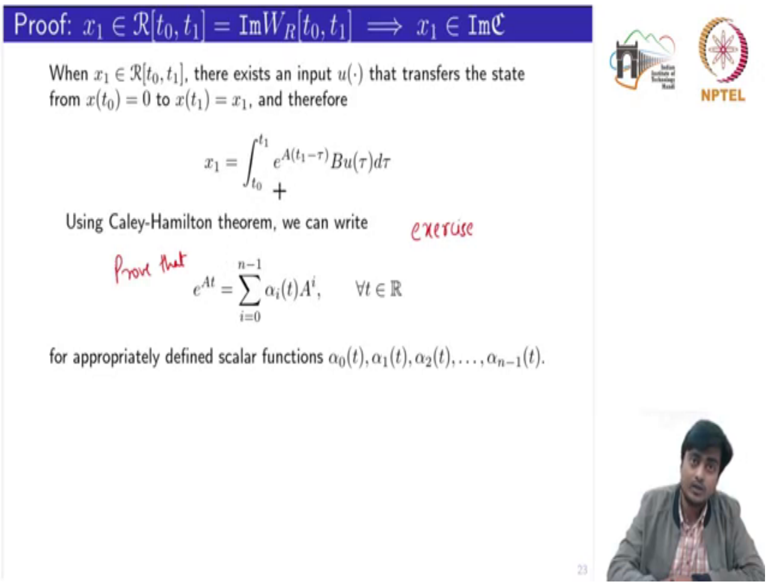

So, you can take this as an exercise if you have any difficulty to see this equation coming by the application of the Caley-Hamilton theorem that what you need to take this as an exercise that prove or you need to prove that I can express e to the power At as this, ok.

So, by the application of the Caley-Hamilton theorem we can write this. For some scalar functions alpha naught to alpha n minus 1, ok. Now, simply I will put e to the power At here, where t would change from t 1 minus t to t 1 minus tau, ok. So, here I would have x 1 this summation i 0 to n minus 1.

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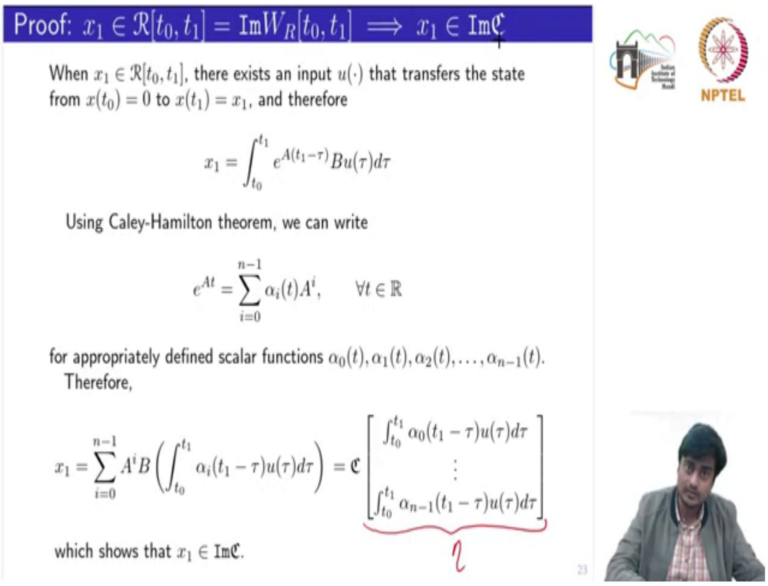

All these are scalars alpha e. So, I can take them on the right hand side also, ok. So, I brought A i forefront and then B matrix coming from here, this whole part which includes u and these alpha i, ok. So, I directly replace this exponential term here by this term while introducing some scalar functions from alpha naught to alpha n minus 1, ok.

Now, see that this part I can write if I open this summation this whole part A to the power i B would appear as the controllability matrix and this whole terms would becomes a vector individual vector of this alpha functions. I just wrote this summation part into form of a matrix, ok.

Now, if I represent this as some vector eta, so now, we have this x 1 is equal to sum matrix into eta which means that x 1 belongs to the image of the controllability matrix, ok. So, starting from x 1 belonging to the reachable subspace we have shown that it also implies that x 1 would belong to the image of the controllability matrix, right. Let us see the otherwise the other direction implication.

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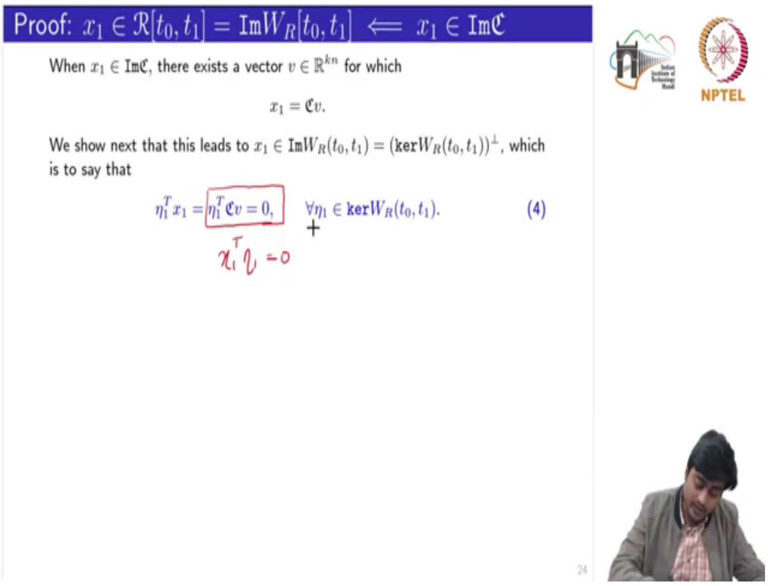

So, now we are given that x 1 belongs to the image of the controllability matrix. Now, we know that if x whenever x 1 belongs to this image of this matrix that there exists a vector v for which I can write this equation x 1 is equal to some matrix C into v, ok.

Now, we show next that this leads to x 1 belongs to the image of W R which is equivalent to saying that x 1 belongs to the orthogonal compliment of the kernel of the reachable Gramian, reachability Gramian matrix, ok. Again we know that W R is a symmetric matrix. So, we I have to remove this transpose which is to say that for all eta 1 belonging to this kernel would satisfy eta 1 transpose into x 1 is equal to 0, right.

So, in the later proof we had seen x 1 transpose into eta 1 is equal to 0, right. So, if, so just to be clear in the earlier proof we have x 1 transpose into eta 1 is equal to 0, if you recall this, ok. If we just take the transpose of this equation you would get this, eta 1 transpose into x 1 is equal to 0, right. Now, x 1 I know already is equal to this matrix into v. So, I replaced this x 1 by this. So, I just need to prove that this equation is satisfies for all eta 1 belonging to the kernel of the W R matrix, ok.

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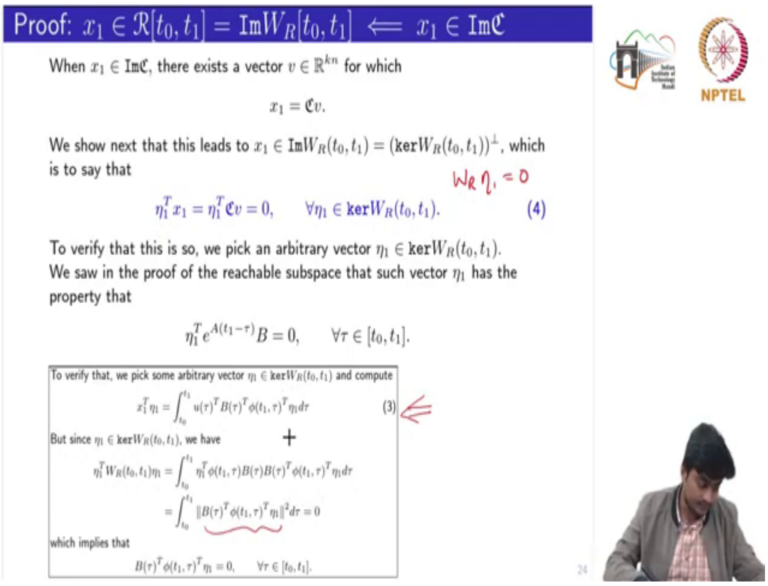

So, let us see. So, to verify this, so we pick an arbitrary vector again eta 1. So, in the proof of the reachable subspace we had seen, so from that proof we will borrow one property here we will not be showing that property.

So, this is just a snapshot, here this is just a snapshot which I had taken from the other proof where we started with x 1 transpose into eta 1 and then we wrote one quadratic form and then we showed that this quadratic form is nothing, but equal to 0 because eta 1 belongs to the kernel of this W R, right, which means that W R eta 1 should be equal to 0. By using that property we have implied that this is equal to 0 and this is only possible if this matrix is 0. So, I will borrow this part.

So, this part I have written here only, by taking the transpose. If you take the transpose of this equation you would get eta 1 transpose phi into B transpose sorry; B only without the transpose. So, here we have eta 1 transpose phi for the LTI system is the exponential matrix into B is equal to 0, ok. So, I knew this already, so I just skip all this part which we have already proven, proved.

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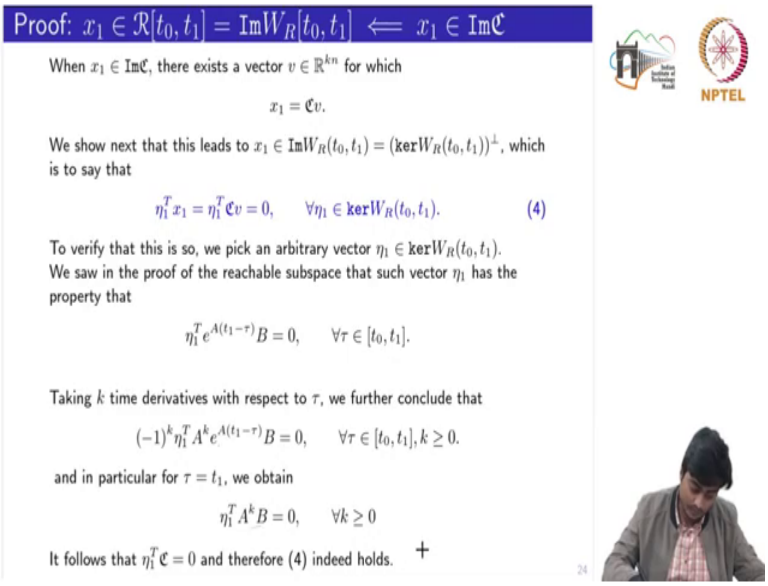

So, starting from this part I will take k time derivatives with respect to tau of this equation. So, we further conclude that if we take k times derivative I would have because of this minus tau minus 1 to the power k, eta 1 transpose I would also have a to the power k by taking the k time derivative of this exponential, this exponential itself and the B matrix. And on the right hand side it should also be equal to 0.

Now, if I put t 1 equal to tau, at tau is equal to t 1 I would have eta 1 transpose A to the power k this part would be e to the power 0 which is nothing, but an identity matrix into B, all right. So, the negative sign does not make any sense because we have equal to 0. So, I can simply write it eta 1 transpose A to the power k, B is equal to 0, ok.

Now, if I open this all case starting from 0 to n minus 1 this will become the controllability matrix. Meaning to say that eta 1 transpose into the controllability matrix is nothing, but equal to 0, right, which implies that x 1 belongs to the image of W R, ok. This is what we want to prove that eta 1, C v is equal to 0 and in this equation we already have eta 1 transpose C is equal to 0. So, although we have the nonzero vector v, but this equation is always satisfied. For v is equal to 0 it would definitely satisfy, right, but for nonzero v, it will also satisfy and this is the complete proof, ok.

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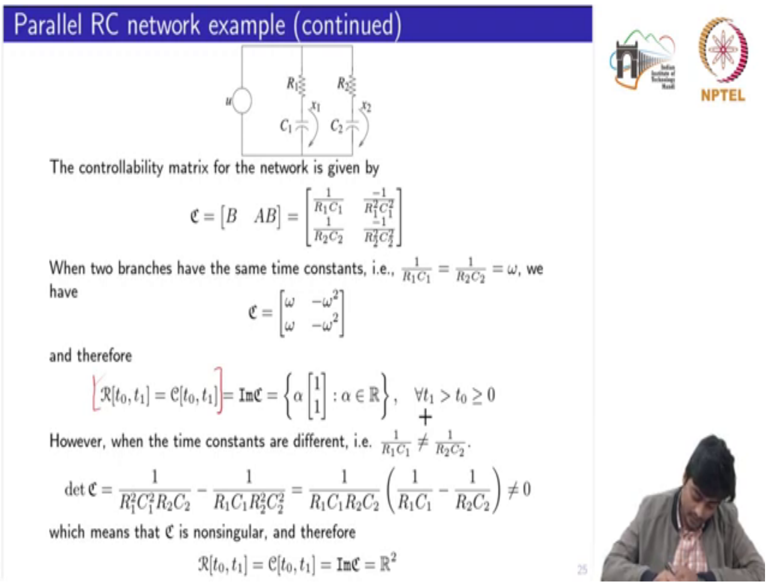

So, this completes the overall proof of the equation. We can recall the example with which we have started as a motivation. So, this is the same example where we have two branches R 1, C 1 in parallel with R 2 and C 2 supplied with some voltage u, ok.

So, in the previous example we have made a conclusion that the reachable subspace and the controllable subspace are basically the set of alpha times 1 1, one matrix one vector of two dimension, ok, which means that the system is not completely controllable. So, again if which compute the controllability matrix for that particular example we would obtain this.

Now, that those subspaces we have computed when R 1 C 1 is equal to R 2 C 2. So, if I put R 1 C 1 is equal to R 2 C 2 here we would have controllability matrix is equal to this, right and the controllability matrix and therefore, we would have the reachability subspace is equal to the controllability subspace which is this is the conclusion we had seen in the previous example. Now, which is again the image of C which is again given by some alpha times 1 1, ok.

However, when the time constants are difference 1 by R 1 C 1 is not equal to 1 by R 2, C 2 we had seen earlier that both then the system is completely controllable and reachable. Why? Because the both the subspaces were equal to R square. Now, if I compute the determinant of this equation we would see that the determinant of this is not equal to 0, right. So, the image of c is the complete space. So, either, so these matrices the Gramian matrices for both; for both the cases LTV and LTI and the controllability matrix particularly for the LTI system we could compute both these spaces subspaces which are quite important for the analysis of the controllability system, ok.