Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture – 15
Reachability and Controllability Gramians

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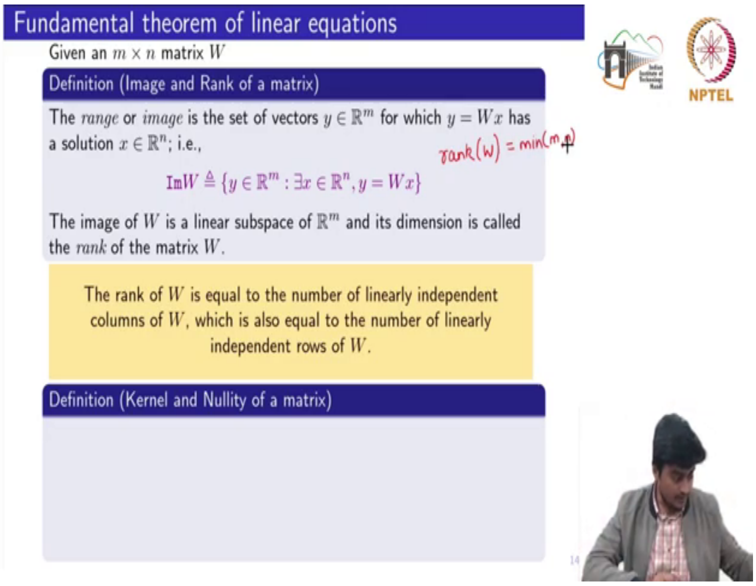

So, now we will start with the Fundamental Theorem of the Linear Equations. So, here we will do a brief review of this theorem because this theory we would going to use it later for deriving the results relate to the controllability and also of the observability. In this controllability week, we would also use some of the tools we had discussed in the last week related to the quadratic forms and the norms as well, ok.

So, for a given m cross n matrix W, we define two first of all we will defined two subspaces, one is called the range subspace and the second is called the null subspace. So, the range or image is the set of vectors y for which y is equal to W x has a solution x has a solution x in an dimension space, that is to say the image so we would represent whenever we are talking about the image of any matrix by this symbol image of W is defined as the set containing all the y's in the m dimensional space such that there exists x into the n dimensional space following this linear equation y is equal to W x.

Now, the image of W is a linear subspace of R m. So, this is a straightforwardly clear and its dimension is called the rank of the matrix W, ok. So, another point here to remember is that you can also write the rank of the matrix W is equal to the number of linearly independent columns of the matrix W, which is also equal to the number of linearly independent rows of W. Say for example, because we already know that the matrix is of m cross n, so the rank of the matrix W is basically first of all the minimum of either m or n ok, or maybe it could be lesser then the m and n if the matrix is not full rank.

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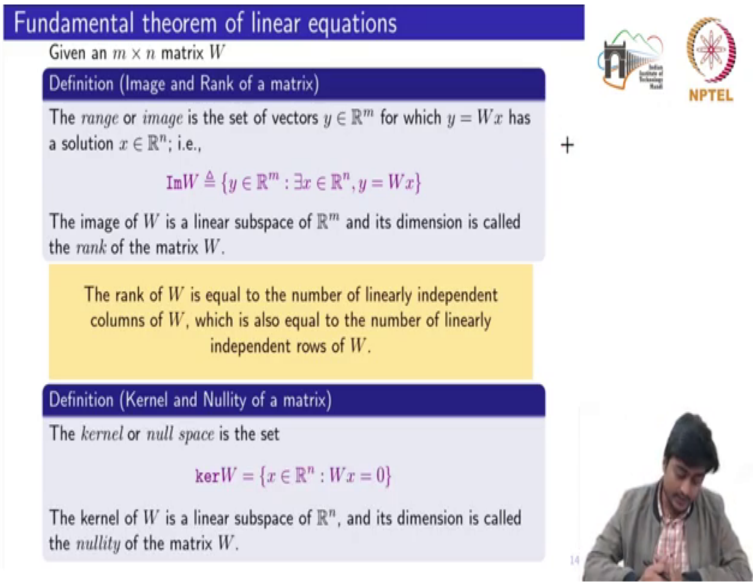

Another space is called the null space or the kernel is the set we use this symbol kernel of this matrix W, kernel of W is contains all those vector x which satisfies this equation that is multiplying that vector with the given matrix W it yields a 0 vector, ok. So, the colonel of W is also a linear subspace of an n dimensional space and its dimension is called a nullity of the matrix W. So, we are talking about the dimension of the image of the matrix it is called the rank and the dimension of the kernel the matrix is called the nullity.

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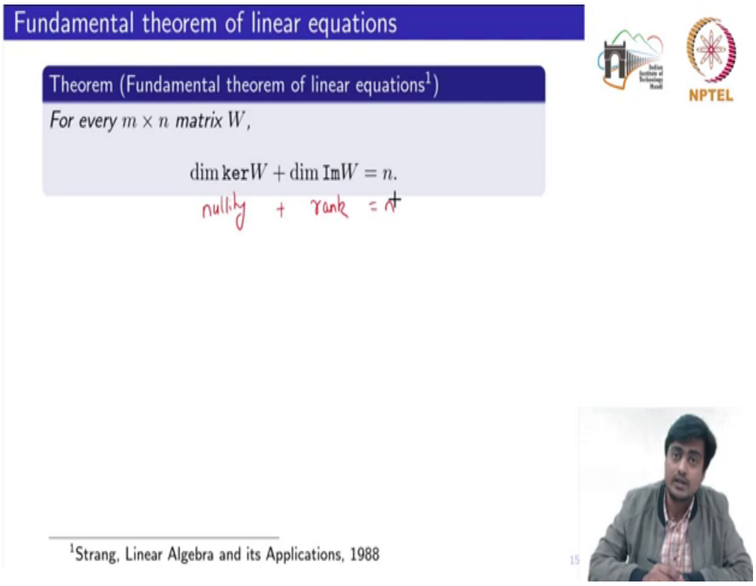

So, after introducing these two subspaces we are now in a state to a state the fundamental theorem linear equations which is given by here that for every m cross n matrix W. The dimension of the kernel of the matrix W plus the dimension of the image of the matrix W is equal to the number of columns of this matrix W, right.

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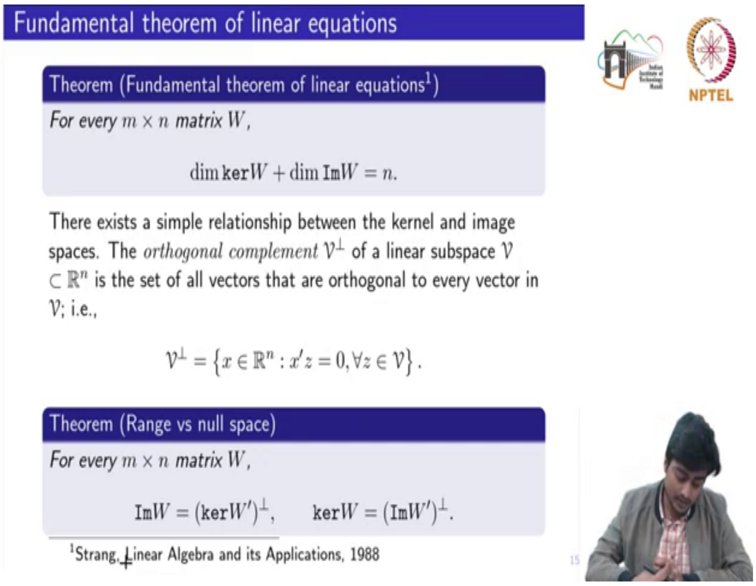

So, dimension of the kernel W we have defined as the nullity. So, you can write this in this way also and the dimension of the image we have defined it by the rank, so the nullity plus rank should be equal to the number of columns of the matrix W. So, there also exists a simple relationship between the kernel and image subspaces, but before that we will introduce one definition of the orthogonal compliment of a linear subspace V which is defined by a perpendicular sign on the subscript of the space V.

So, it contains all vectors that are orthogonal to every vector in V, that is so, if we have this we define this V perpendicular is equal to all those x, such that the multiplication of all those z belonging to the subspace V multiplied with that transpose of the vector x yields 0, ok. So, this would be a scalar quantity, right.

So, we have the next result which says the range versus null space for every m cross n matrix W, the image of the matrix W can also be represented as the orthogonal compliment of the kernel of the transpose of that matrix and vice versa that kernel of W is equal to the orthogonal compliment of the image of that transpose of that matrix, ok.

So, these two results are pretty much standard results which you can see in the book of strength linear algebra and its application. We will not be carrying the proofs of these two fundamental theorems, but we will be using this to derive the results regarding to the controllability and observability.

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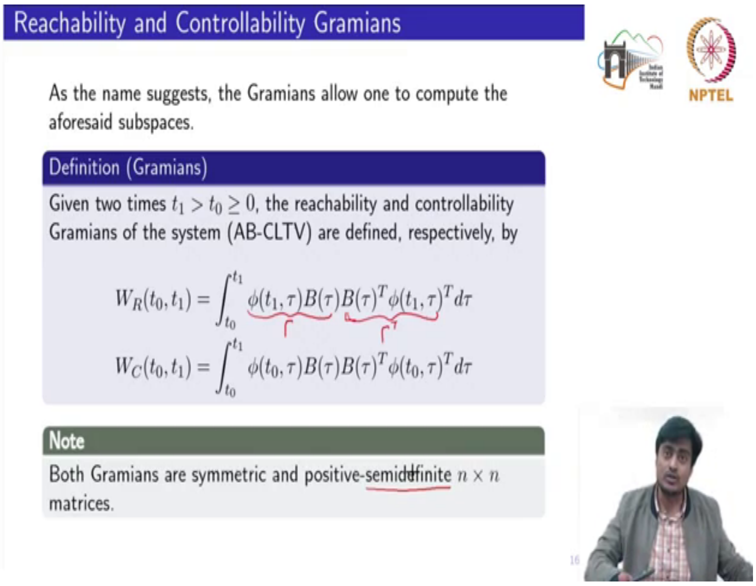

Next we see the reachability and controllability Gramians. So, as the name suggests that Gramians allow one to compute that aforesaid subspaces. So, now we are pretty much clear about that what are the different two subspaces for studying the controllability and reachability. Now, we want to compute those subspaces. So, to compute directly those subspaces we will introduce this Gramians.

In fact, in the example what we had considered previously we saw a bit of glimpse of computing the subspace particularly for the parallel interconnection of the systems, but now we will a uniformly define that how you can use the Gramians to actually compute those subspaces.

So, given two times t 1 which is greater than t naught or greater than equal to 0, the reachability and controllability Gramians of the system of the pair AB of continuous linear time varying system are defined by. So, we use this symbol W R reachability Gramian and W C controllability Gramian, so W R t naught comma t 1 is defined by this integral t naught to t 1 phi which is the state transition matrix B is the input distribution matrix and their transposes, ok.

Similarly, W C is state transition matrix with t naught. So, we already had seen the slight difference between the reachability and controllability subspaces. Then we had specifically seen that there is merely a change of t 1 and t naught by computing the reachability and controllability subspaces. So, there are two properties associated with it, that both Gramians are symmetric and positive semidefinite. So, note this point positive semi definite at this moment because we will define one key result later on with respect to linear time varying systems.

So, this W R and W C are would be the square matrix or dimension n cross n now it is straightforward to see that if I denote this complete thing by a matrix gamma big gamma, so this part is nothing, but gamma transpose the transpose of the gamma. So, similarly goes for the controllability Gramian.

Now, if I take the transpose of this matrix W R it again I would going to have the integral of gamma into gamma transports. So, this matrix is symmetry. Now, since this put if we talk about the if gamma being the scalar then basically it would be the square. So, it can never go to or it could definitely be a non-negative number or a matrix, so that is why we have defined positive semi-definite, because either it could be positive or 0.

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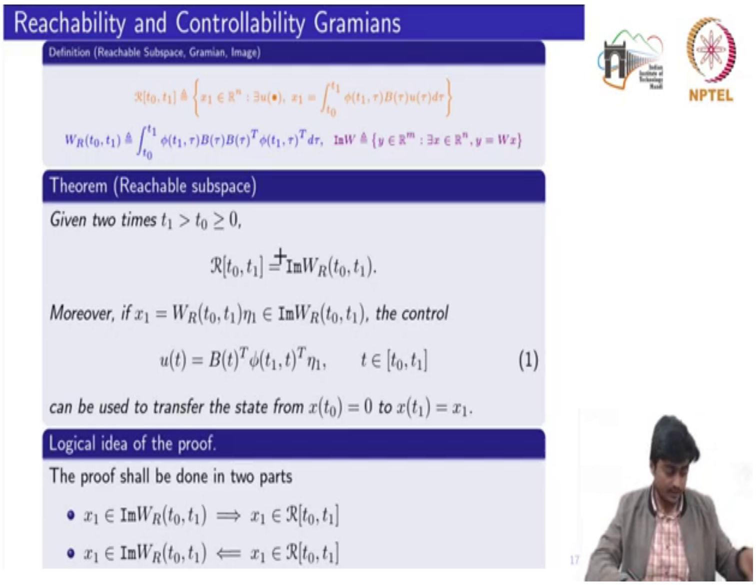

Now, before stating the results let us recall the 3 important terminology I have introduced so far. This is the reachability subspace given by the set x 1 for which there exists a u such that x 1 is given by this one, ok. We have also defined the reachability Gramian by this equation and the image of an matrix W by this set.

So, now we were going to set up some relationship between these 3 important terms or 3 important equations such that we could compute the subspace in terms of Gramian. So, the key result here is given two times t 1 greater than t naught greater than equal to 0, the reachability subspace and t naught to t 1 is basically equal to the image of the reachability Gramian matrix. This is the first reason.

Second is moreover if x 1 is equal to W R into eta 1 for some eta 1 vector which is equivalent to saying that x 1 belongs to the image of W R, because if you see this definition image of W it is all those y such that y is equal W x. Now, here y is x 1 W is W R and x is some eta, ok. So, we can say that if x 1 belongs to the image of the matrix W R, the control or the specific control signal given by B transpose and phi transpose into eta 1 for all t belonging to t naught to t 1 can be used to transfer the state from x from the origin to some finite value x 1, some finite vector x 1, ok.

So, this is a like I said this is one of the key results, so we will see we shall see that detailed proof of this important theorem. So, the proofs would be done in two parts, because both these first of all you should see clearly that this one and this one are two spaces. This is a reachable subspace and this is also a subspace a image subspace of the matrix or the for the matrix and W R. So, if we are able to show that for some x 1 belonging to this space implies that x 1 would also belong to this space, ok.

The second part is x 1 belonging to this space implies that x 1 will also belong to this space. So, basically we want to close the loop once we close the loop it becomes an equivalent sign, ok, it becomes a if an only if condition what we had seen so far, right.

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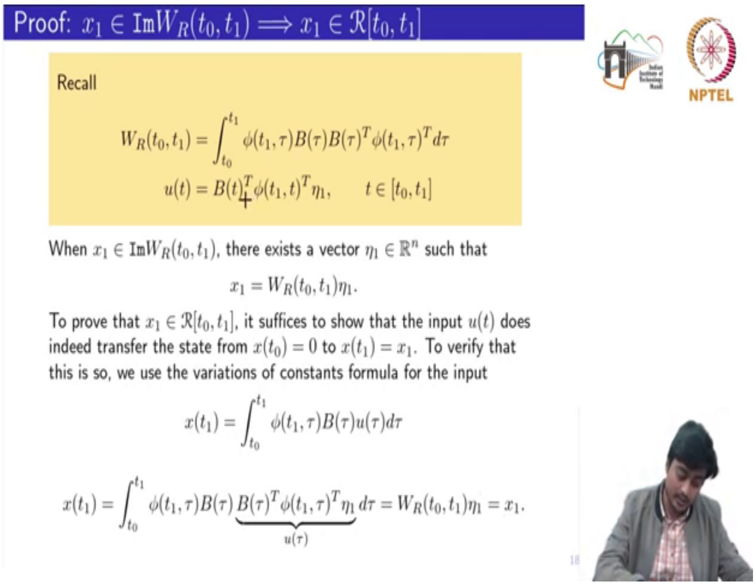

So, let us see the proof of the first part that x 1 belonging to this space implies that x 1 also belongs to this space. So, we here we recall two equations, one is the reachability Gramian this is the equation what we had seen and this u of t which we want to prove, for some specific input. So, when x 1 belongs to this space that is image of W R matrix, we could say that there exists vector eta 1 in the n dimensional space such day x 1 is equal to W R into eta 1 and this is what the definition of the image of any matrix.

Now, to prove that x 1 also belongs to the reachable space its surfaces to show that the input u t which is given here or which is given in the theorem does indeed transfer the state from the origin to some x 1 at t 1, ok. So, to verify this we use the variation of constants formula for the input which is given by this one, x of t 1 given by the integral t naught to t 1 state transition matrix B into u. Now, if I put this u here which is given in the theorem I would get phi into B into B transpose and phi transpose and eta 1, ok. Now, this is the u which I have taken from the above.

Now, see it clearly that this whole part this whole part is nothing, but your W R the reachability Gramian. So, I could write is equal to W R into eta 1 and W R eta 1 is nothing, but x 1, right. So, this is the overall proof that first part that x 1 belonging to the image of W R implies that x 1 would also belongs to the range subspace for this specific u, ok. So, meaning to say that there exists some u.

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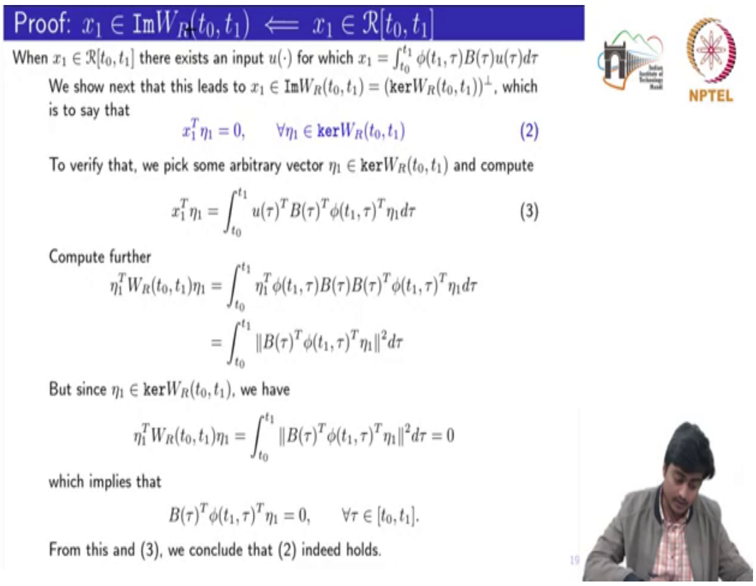

Now, going for the second part x 1 belonging to this space implies x 1 would also belong to this space to see the proof. So, what we want to show that when x 1 this which is given condition that x 1 belongs to the reachable space. We know that there exists some input u for which this equation is satisfied, right.

Now, we show next that this also leads to x 1 belonging to this image of W R and using the one of the previous results I could replace this W R or this image of W R by the orthogonal component of the kernel of the transpose of that matrix. Now, since W R is symmetric already I have directly written W R otherwise it should be the transpose of the W R, which is to say, so if x 1 belongs to here meaning to say that for all etas belonging to the kernel of the W R it should satisfy x 1 transport eta 1 is equal to 0.

We have directly borrowed this from the given definition and the fundamental results of the relationship between the kernel and the image, ok. So, now, instead of so now, given this we need to prove this and proving this is equivalent to showing that there exist or this equation is being satisfied for all eta belonging to the kernel of W R, ok. So, let us see.

So, to verify that we shall we will pick some arbitrary vector eta 1 which belong to this space and compute directly the left hand side and the left hand side is given by this x 1. If I take the transpose of this x 1 I would get I would get u transpose, B transpose and then phi transpose; u transpose, B transpose, phi transpose and multiplied by eta 1 because eta 1 is some constant vector. So, I can take this inside the integral.

Now, we will compute further the quadratic form of this form where eta 1 transpose W R into eta 1, ok. Now, here I have directly written W R as it is. So, we would have eta 1 again is a constant vector. I can take eta 1 transpose and eta 1 both inside the integral and this part is basically W R, ok. Phi into B into B transpose and phi transpose.

Now, note here that this complete part now this complete part is the transpose of this one, meaning to say I could write it as a square of the norm of this entry W transpose phi transpose eta 1, the square of that norm, ok. So, this is I have directly used the property of the norm.

Now, but we know already that eta 1 belongs to this kernel of W R, so which which implies that W R into eta 1 should be equal to 0, right. So, here we have this W R eta 1. So, this part would definitely be equal to 0, right. So, we have integral norm square is equal to 0 and this implies and by or let us say by using the property of the norm this could be 0 only if the matrix in or the vector inside the norm itself is 0 and otherwise it cannot be 0 for all tau belonging to t naught to t 1.

Now, from this from this equation and the third equation we conclude that two will definitely hold because we started from the left hand side started from the left hand side we have proved that, right hand side is nothing, but equal to 0 which implies that x 1 belonging to the reachable space would also means that x 1 would also belongs to the image of the matrix W, ok. So, this completes the overall proof of the reachability Gramian.

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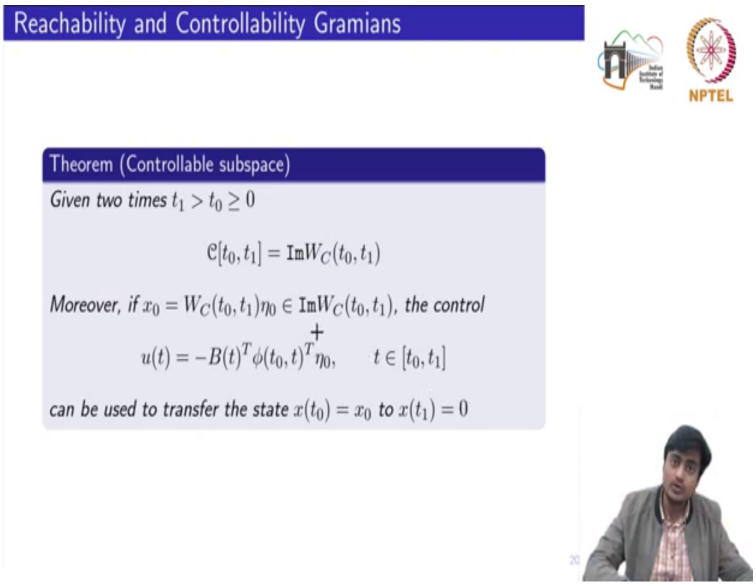

Now, similarly we could use the controllability Gramian to compute the controllable subspace given two times t 1 greater than t naught greater than equal to 0. So, we the control well subspace is given by the image of this controllability Gramian.

Moreover, if x naught is given by this one which is equivalent to saying that x naught belonging to the image of the matrix W C, the control this is specific control can be used to transfer the state from some nonzero initial condition to 0, ok.

So, the proof of this part would also would almost remain the same of the part we have seen for the reachability Gramian, ok. So, we would not discuss in detail the proof, but this result is also important because once we have computed the Gramians you could directly compute the subspaces, ok.