Linear Dynamical Systems
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Week - 3 and 4
Controllability and State Feedback
Lecture – 14
Introduction to Controllability

So, hello everyone today we will be starting with the week 3 and week 4 combined of the course Linear Dynamical Systems. So, in this course we will cover both these topics Controllability and State Feedback. So, in this module basically we have combined both the weeks week 3 and week 4, because the idea is that most of the concepts which we will study for the controllability they would be directly applicable for designing the controller particularly the state feedback controller right.

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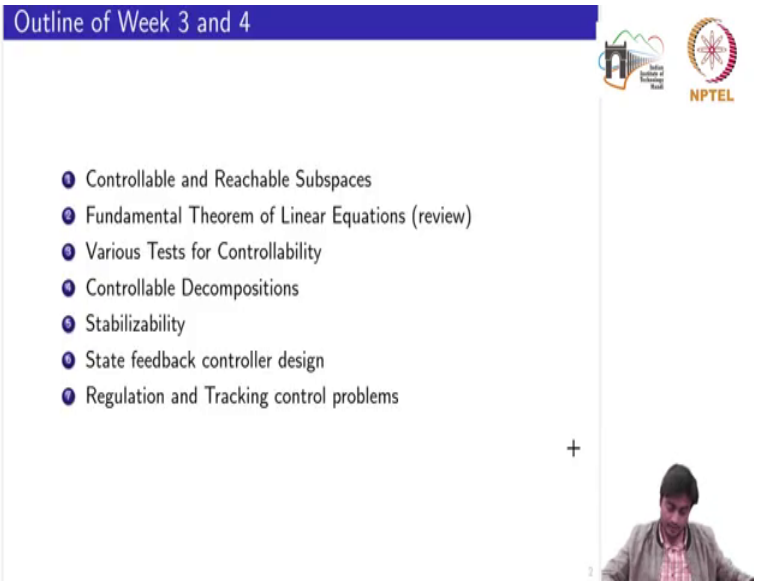

So, this would be the outline of the overall week 3 and week 4. We will start with the controllable and reachable sub spaces with a brief introduction about the controllability. Then we will review the fundamental theorem of linear equations, which would play a key role in the results we were going to discuss in the later topics. Third is we will see the various tests for controllability followed by the decompositions and then the concept of stabilizability.

So, the first five bullet points basically talks about the controllability in overall. The last two points 6 and 7 where we study the problem of feedback controller design and also we would see the regulation in the tracking control problems as a part of the week 4. So, starting with the simple illustration of controllability, so the idea is let us say we let us say we know the how we define the state of the systems.

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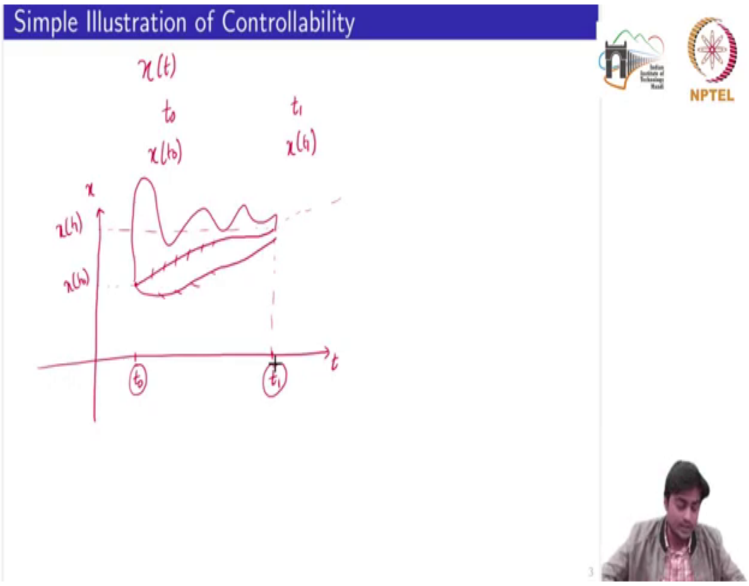

So, from the beginning we have been using this variable x either we are dealing with the linear time invariant system or the time varying system. So, the idea is let us say we pick 2 times t is equal to t naught and t is equal to t 1 and at these times we compute the value of the state, let us say x t naught and x of t 1 ok.

So, suppose these points are marked at t naught and t 1 ok. So now the idea here is that given the value of the state at time t naught which is x of t naught, we want to take this state trajectory or the signal to a point which is defined by x of t 1. Now, the idea here is that to go from t naught to t 1 if there exists any control signal, because with the help of the control signal we want to take this trajectory from the point x of t naught to x of t 1 right. So, this determines that whether my system is controllable in general or not ok.

Now, in the last week we have studied a concept of the Stable systems. Now, in the stable system we studied two concepts one is the asymptotic stability and one is the exponential stability or the marginally stable systems. So, in the marginally stable systems we have understood that the signal or the state trajectory would not go towards the zero, but it would be bounded or it would settle down to some value. Now here the control the concept of controllability is a bit different from the stable or the stability, it should not be confused with the stability first of all why?

So, here the idea of controllability once again that to go from x t naught to x t 1; now after this t 1 how the trajectory or where the trajectory will go this is not in our control and in fact to go from t naught to t 1 there could be number of paths. So, let us say this is one path this is another path or this could be another path, because we want to reach from x of t naught to x t 1 anyhow.

Now, what would happen after this, this is not considered in the concept of controllability. So, these took because in the stability concept we want to ensure that the system. If we are talking about asymptotic stability should still down to the zero value. So, these two times are particularly important for the concept of controllability. So, after seeing this simple illustration of the controllability, let us try to put all these things in the formal framework.

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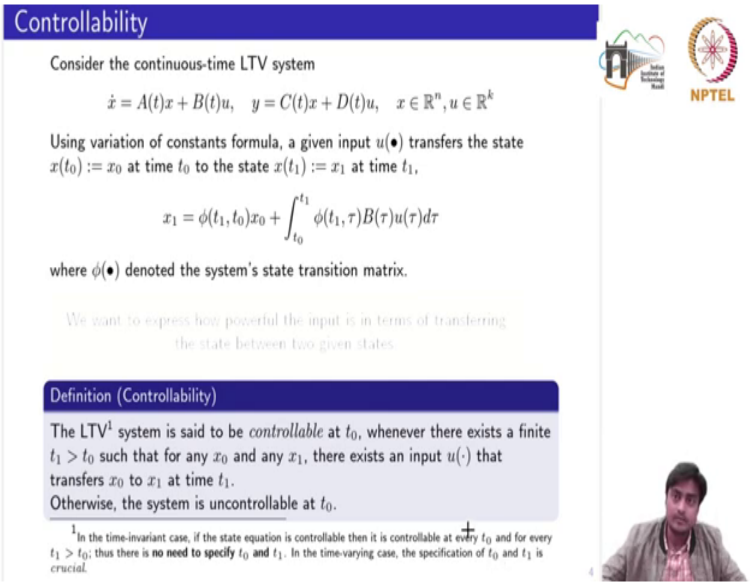

So, we will start with the linear time varying systems and then we again we would tailor all the results for the LTV systems. So, consider the continuous time LTV systems, where the A t B t C t and D t are the time varying system parameters. So, we know that in the first week we saw that using the variation of constants formula a given input u transfers the state x of t naught which we have defined by x naught initial condition at time t naught to the state x of t 1 at time t 1. And, this is the formula we have used to compute the signal x 1, here we have specified at t is equal to t 1 ok.

So, where we also know that this phi defines the system state transition matrix. So, here we want to express how powerful the input is in terms of transferring the state between the given two states. So, let us see the definition of the controllability. So, the LTV system is said to be controllable at time t naught, whenever there exists a finite time t 1 which is greater than t naught. Such that for any initial condition x naught and given any x of one there exist an input u that transfers x naught to x 1 at time t 1.

So, we this is the same thing what we had seen in the previous slide and otherwise if the system is not if or if there does not exist any control signal which could take the state trajectory from the value x naught to any x 1, then we say that the system is uncontrollable at t naught. So, this definition is particularly for the LTV system. Now if we want to tailor this definition for the LTI system.

So, we say that the LTI system if the system is controllable, if we somehow proves or shows the existence of the control signal that the state equation is controllable in the time invariant case. Then it would be controllable at every t naught and for every t 1 which is greater than t naught. But, for that time varying case these two specification of the times t naught and t 1 is pretty much crucial. And, once we develop the concrete results to test the controllability of the system, you would notice that how these times t naught and t 1 are paying the key role ok.

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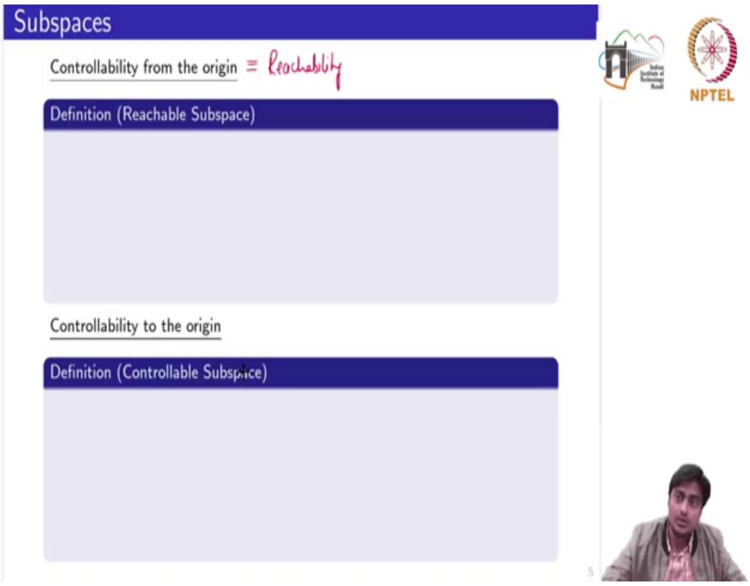

So, now we will define two sub spaces. So, the idea here is first basically defines the controllability from the origin, which we also say or which is also equivalent the same the reachability and another one is that controllability to the origin. Now, the overall the broad concept of the controllability is to stear the trajectory from x of t naught to x of t 1. Now when we speak about the controllability from the origin now here the x of t naught is basically 0.

So, we want to go from 0 to some given x of t 1; now, the controllability to the origin that given x naught we want to go to x naught. Now, the second definition should not be confused with the concept of stability. The stability says the first of all the controllability to the origin is not at all equivalent to the stability. Let us recall the recalling the concept of stability that we want the state trajectory to reach to the origin when t tends to infinity ok. For the asymptotic stability but controllability to the origin says that the state could reach to the origin in some finite time t 1.

But, whether it would stay after there this is another problem. So, here we would define reachable space that what on state variables at time t 1 could be reach could be reached starting from the origin and controllable subspace which defines that, what are the all the initial conditions from where I could reach to the origin in some finite time t 1.

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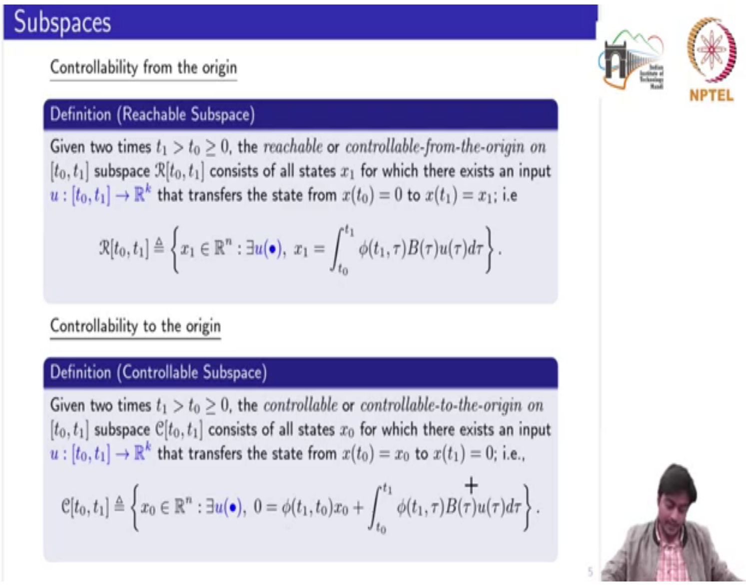

So, given two times the t 1 which is greater than t naught greater than or equal to 0, the reachable or controllable from the origin on this time exists or time interval t naught and t 1. We define the subspace are t naught comma t 1 consist of all states x 1 for which there exist an input u within the same time interval, that transfers the state from x of t naught is equal to 0 to some finite value x 1 that is.

So, this is how we define the reachable sub space R t naught comma t 1, which contains all x 1 which belongs to the n dimensional subspace such that there exist u. So, that I could obtain this of which follows this one. So, this x 1 and u pair should satisfy this equation. Now for the controllability to the origin we defined by this (Refer Time: 11:35) environment consists of all the states x naught for which there exist an input u, that transfers the state from x naught to 0 ok.

So, here it would so the subspace consisting all those x naught in the set of n dimensional or in the n dimensional space. We say that there exist a u such that x of t 1 is equal to 0 is equal to this one. So, this part was not included in the first definition of the reachable space because x naught is equal to 0.

So, when we put x naught is equal to 0 we get we did not have anything here and the rest of the part remain the same. So, here this x naught and u this pair should satisfy this equation ok. Now let us see a bit detail into characterizing these spaces.

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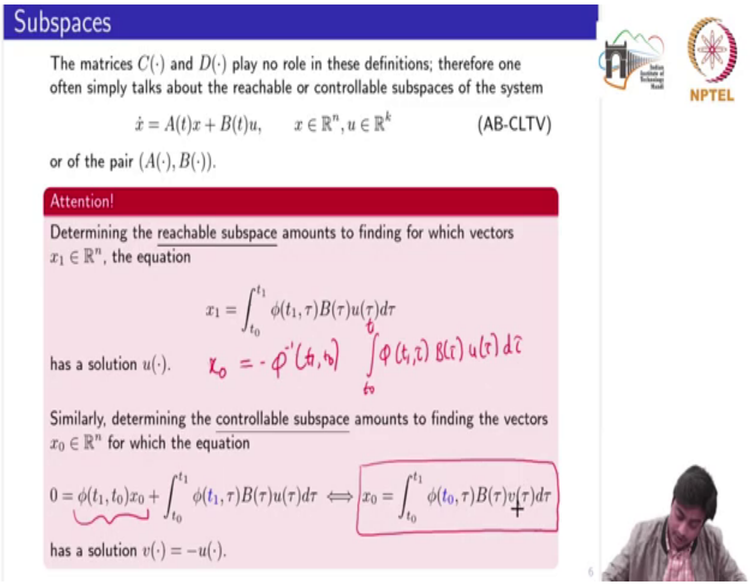

Before that we will we see that the matrices C and D play no role in these definitions ok. So, generally we talk about the or one talks about this controllability only of this state equation or we say the pair a comma b ok. Now, we for the controllability we most of the time we will not be including the output equation ok but, it does not mean that the output equation is not there.

So, we can see some equivalents or some similarity and differences between the two sub spaces what we had just introduced. So, determining the reachable subspace amounts to finding for which vectors x 1 the equation is satisfied for some u. This is the same equation we had seen earlier which consists the or which comprises the reachable subspace.

Now, similarly determining the control subspace amounts to finding the vectors x naught for which this equation is satisfied for some u ok. Now, I use the same equation here, now if I rewrite this equation let us see by taking this part onto the left hand side and then taking the inverse of the state transition matrix or pre multiplying by the inverse of the state transition matrix both sides. I can take the inverse because we knew from the first week that the state transition matrix is non singular ok.

So, in this way I would obtain x naught is equal to minus phi inverse we missed something ok. So, this plus sign won't be there sorry. So, this equation I would obtain. Now we since this integral is over tau I can take this part inside the integral and then using the property of the state transition matrix. I could replace the multiplication of these two state transition matrix by this. Where the final time has been change from t 1 to t naught and we got this negative sign and this negative u or minus of u I have replaced by another signal v.

So, that I could write this compact expression in terms of x naught only which is quite similar to x 1 with a slight difference that in the state transition matrix. The final time here is at t 1 while here it is t naught, the state transition matrix if we recall that the input has been applied at time tau and the response we are getting at time t 1. Now, here since in the continuous time we would replace or we either we could go forward or we could go backward for representing the state transition matrix ok.

So, both the expression almost looks similar with a slight difference, that the state transit that that time t 1 in the state transition matrix here has been replaced by time t naught with u being changed to v, where v is basically the negation of the actual control input.

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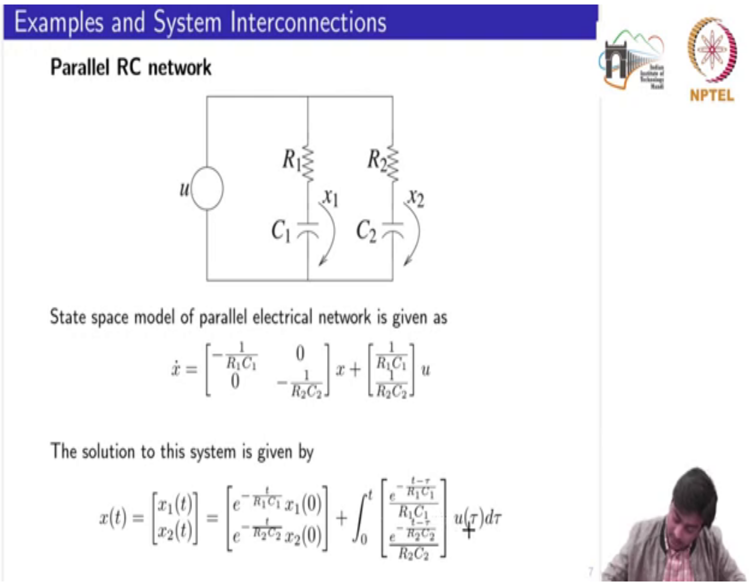

So, let us consider a some examples to better visualize these two sub spaces. So, here we are seeing a parallel RC network, where these in one branch we have the series combination of R1 C 1 connected in parallel to another series combination of R2 C2 and both of them are being powered by a common voltage input by u.

So, if we write the state space model of this parallel electrical network we obtain this state equation where this is the a matrix and this is the b matrix. Now the solution by using the variation of calculus formula, we obtain the solution independently for the two states. So, first of all visualize these state space equations.

So, in the a matrix the elements are only present at the on the diagonal, while the off diagonal elements are 0. So, either I could consider them two independent systems or as one in one system where it is already diagonalized ok. So, writing the solution to this system this is the solution which you can obtain readily.

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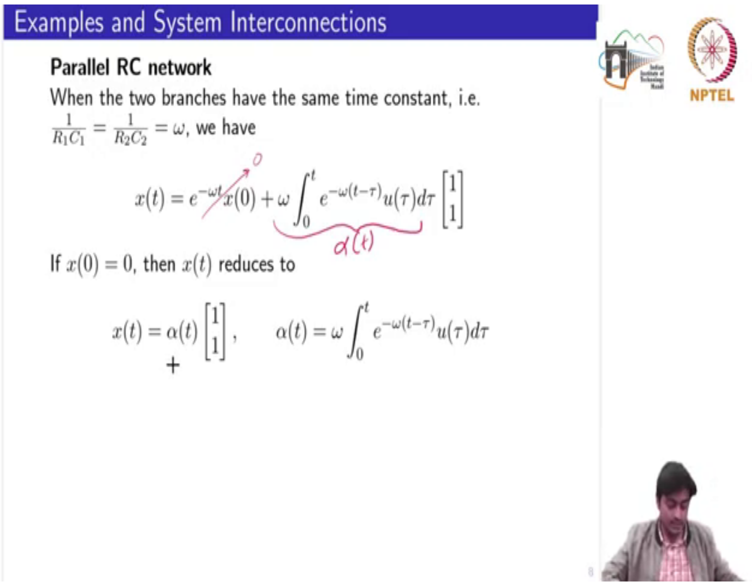

Now, in this formula if we have a couple of condition, that is to say that when the two branches have the same time constants that is 1 by R1 C1 or R1 is equal to 1 by R2 C 2. Let us specify them by some other variable omega. So, these two time constants are basically appearing here in they need with this in the state transition matrix ok.

So, if I replace them by omega I get this simplified equation, because this e to the power minus omega t would be a matrix. And since they were the two independent solution I could express see if this R1and C1and R2 and C2 both are similar, I can take the common part R1 C1 from this matrix and the rest of the matrix would be a vector of one and one. This is what I have expressed it is just a simplification version of the previous equation.

So, now if we apply that formula that is to say the if I put x naught is equal to 0, then x of t basically reduces to this equation where alpha of t is defined by this. So, here I put x of 0 is equal to 0, this part goes to 0 and the remaining one by specifying this whole part by some help of t we get this simplified expression.

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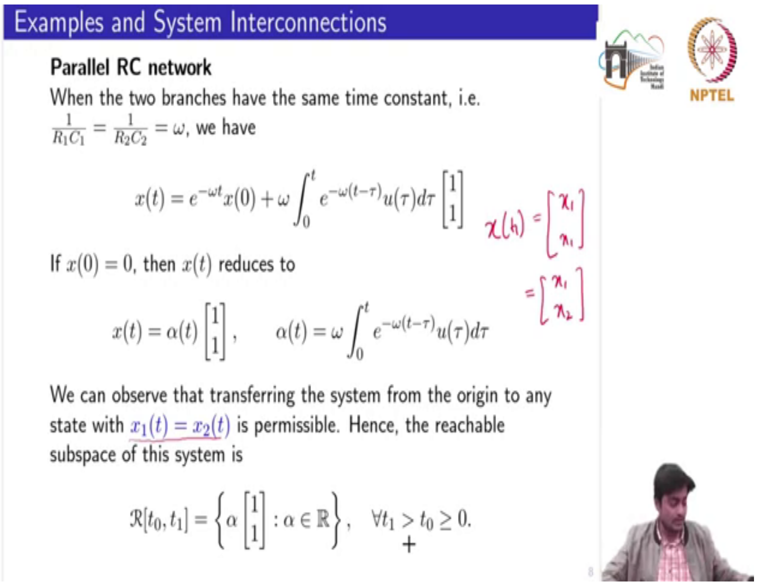

So, what it says that that we that they there could exist input u, which could transfer the system from origin by putting x 0 is equal to 0 to any other state with the. Where the state has to satisfy these constraints x 1 is equal to x 2. Now, if I want to reach some what does it mean that let us say if the state x t 1 they should be same, if I am having the value x 1 it should be x 1.

Now, if I want to reach to another value let us say x 1 and x 2 which are defined by 2 at the times then I cannot reach. So, this is controllable under this condition only. So, we define the reachable sub space of the system is this 1 alpha 1 1 such that alpha belongs to a set of real numbers. When we put t is equal to t 1 here this becomes a proper definite integral that is why it belongs to a set of real number ok.

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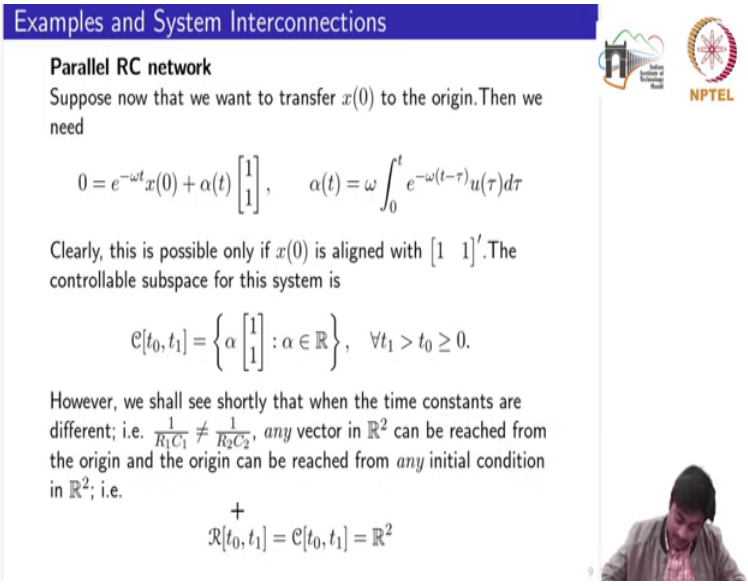

Now, in the similar way we can compute the controllable subspace, which says that we want to transfer from x 1 x x of 0 to the origin. So, we put x 1 is equal to 0 with a non-zero x of 0 ok. Now, again this part would appear as it is this alpha of t, now the simplification what we had seen by when we had seen the some similarity and differences between both the sub spaces I could again x.

So, I can again express this controllable sub space as this one, there this alpha could be a another value. But, the space would remain the sub space would remain the same. So, here we obtain that when one by R 1 C1 is equal to one by R2 C2, we obtain a control constraint sub space the reachable sub space and the controllable subspace.

So, now if we see that if this condition is not satisfied that 1 by R1 C1 is not equal to 1 by R2 C2, then you could also test by yourself that any vector in the two dimensional space can be reached from the origin and the origin could also be reached from any initial condition in the two dimensional space ok. So, that is to say that R t naught comma t 1 is equal to C t comma t 1 is equal to the entire two dimensional space ok.

So, this you can verify by yourself, if you have if you have any parallel network and the time constants of that parallel network are the same. Then the system is not completely reachable or controllable, but if the parameters are not equal then it is completely controllable ok.

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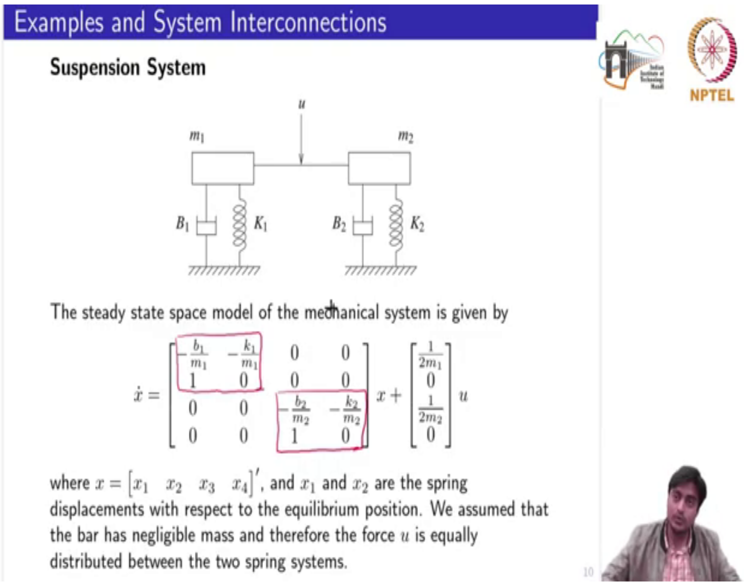

So, let us see another example or another application to visualize or characterize this two sub spaces. So, we have this setup where some mass M 1 is connected to the parallel combination of the damper and the spring a similar system has been shown here with M 2 B 2 K 2 and these two systems are connected by a road on which the control or the u is being applied ok. Now, if we try to model the system we obtain this a b matrices.

Now, if you give a close look to this particularly a matrix. So, I can combine this block and also try to visualize this block ok. So, again we see that either I could consider both the systems separately, it is its shows that both the systems are connected in parallel with the common input u and the off diagonal block matrix is a 0 matrix ok.

So, again if these parameters M 1 B1 K 1 are equal to M 2 B2 K2 you would notice that the system is reachable and controllable under the constraints, that x 1 x 2 is equal to x 3 x 4 ok. Now, if these two parameters are different, then the system is completely controllable. So, we want to generalize this particular scenario.

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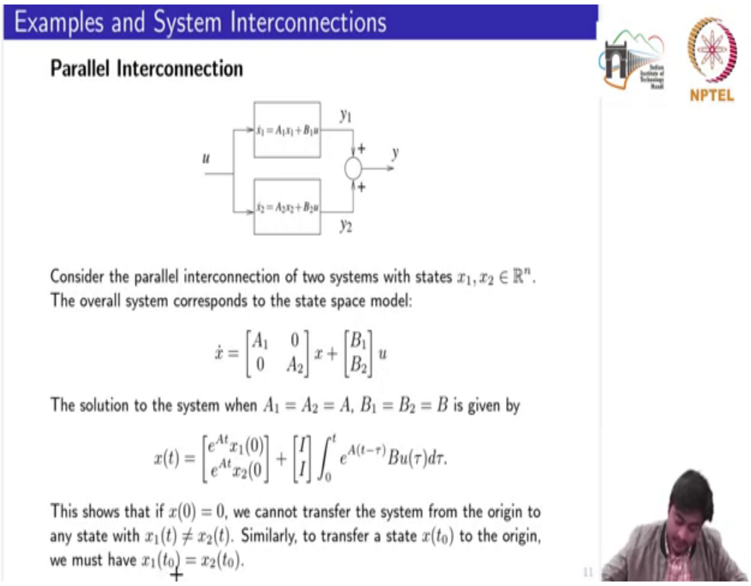

Let us consider the summation of or that parallel interconnection of two subsystems, where these matrices A1 B1 and A2 B2 producing the output y 1 y 2 and sum them up by y ok. Again we would express the state equation using this a matrix where A1 A2 are the block matrices and similarly the 0 matrices and they are corresponding input distribution matrix B1 B2 with the multiplication. Now, again if A1 or A2 are equal and B1 and B2 are equal, again you would see that the system is controllable to the origin and controllable from the origin under the constraints that x 1 should be equal to x 2.

So, they showed that if x 0 is equal to 0 we cannot transfer the system from the origin to any state with x 1 is not equal to x 2, this is what we have also seen in the previous examples. Similarly to transfer a state x t naught to the origin we must have this constraints, that both the state would either reach or would the initial condition would remain the same.

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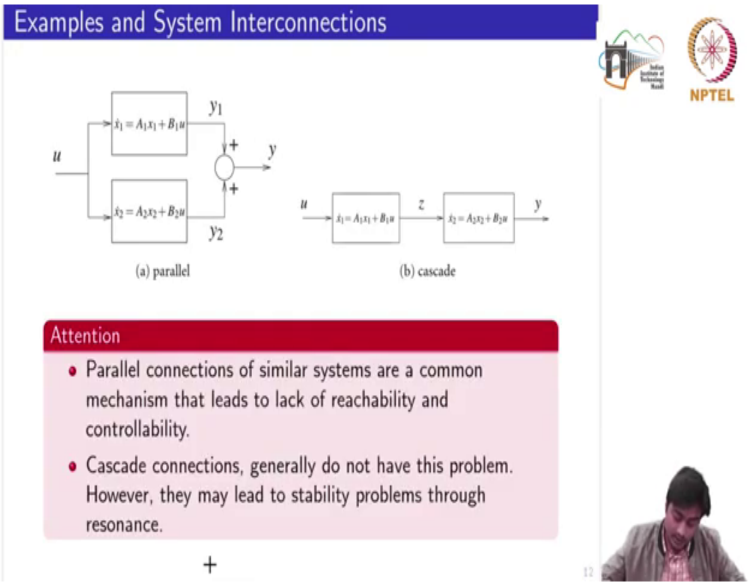

Now, talking about the cascade connection in general so; we see that parallel connections of similar systems are a common mechanism that leads to a lack of reachability and controllability. So, this is what we have seen in the previous examples. Now, we talk about the cascade connection they generally do not have this problem, but they may lead to stability problems through the resonance this is what we had seen in that week two slides or week two lectures ok.

So, this parallel combination is you could see in many industrial processes. Where particularly in chemical process industry where you have the multiple tanks are being supplied to the common control input. So, there could be this would lead to the lack of reachability and controllability.

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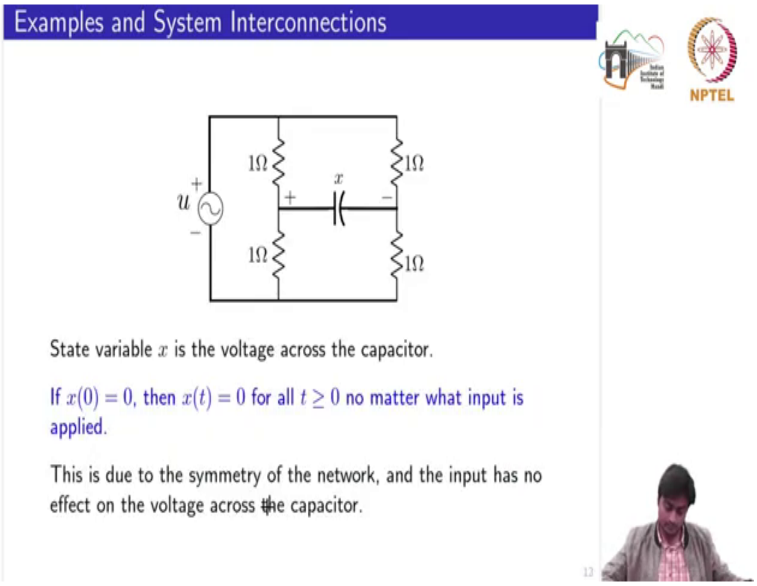

So, one more example we could consider here. So, let us see we are given this electrical network, where we see the four resistance are connected when which are connected by this capacitor. Now, in this electrical network we define the state variable x which is the voltage across the capacitor. Now, if x of 0 is equal to 0, if the voltage across the capacitor at initial time 0 is equal to 0. Then the trajectory x of t would always remain zero no matter what input you apply to the system right.

So, this should be careful because when we speak about the two sub spaces reachable subspace and controllable subspace we particularly, the reachable sub space where we want to go from 0 to some finite value in finite time. But, here depending on the circuit, if you are starting with 0 it might be possible that you will never reach to finite value. So, this is particularly due to the symmetry of the network and the input has no effect on the voltage across the capacitor. If you recall your basic measurement course this is a bridge network, where the current will never flow in this middle branch.