Linear Dynamical Systems

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Tutorial on Stability
Lecture - 13
Tutorial - 2

So, today we will see the Tutorial for the Stability week.

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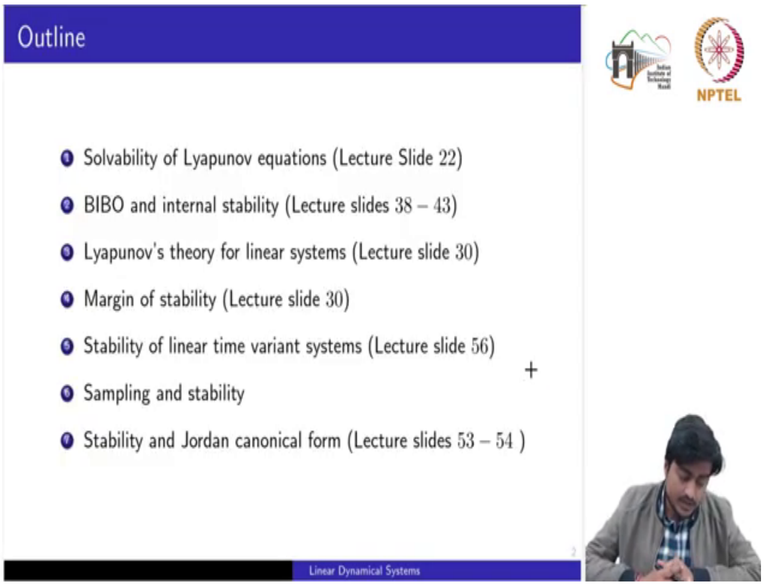

So, again the outline of this tutorial is arranged in such a way so that whatever the theoretical research we have discussed during the whole week, we will see the direct application of those detailed results. In addition, we have also written the slide number as well. So, that you can refer to the theoretical results while applying those results to solve some problems.

So, we will start with the solvability of Lyapunov equations; BIBO and internal stability; questions related to the Lyapunov theory for linear systems. We will also see the margin of stability which we have not discussed in detail during the lecture.

Then, we will see the stability of linear time variant systems followed by the sampling and stability. This topic was also not covered explicitly during the lecture week. And finally, the stability or some or knowing some relationship between the stability and the Jordan canonical forms.

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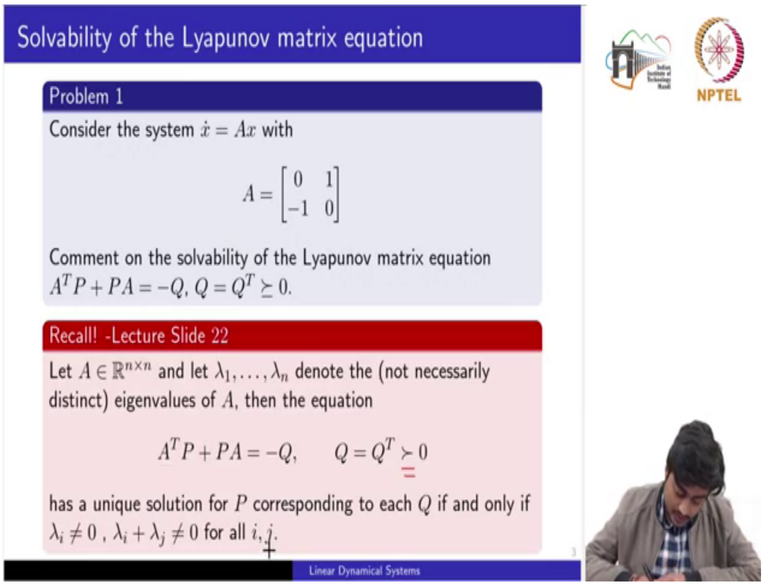

So, the problem first deals with the solvability of the Lyapunov matrix equation. So, while one of the key results in the stability, we had seen that Lyapunov equation played a key role and it hinges on to compute the matrix P given the matrix Q. So, here we will see that how you can compute that matrix.

During the lecture slide we had seen one formula which requires the computing one integral. So, if we do not want to compute the integral and we just want to comment on the solvability of the of the equation or computing some P matrix for a given Q. So, this problem deals with that issue.

So, here we consider a linear system which is a homogeneous with the state matrix A 0 1 minus 1 0. So, here we need to comment on the solvability of the Lyapunov matrix equation which is given by this one.

Now, note that here that in the stability result concerning with the Lyapunov equation Q was chosen as a positive definite matrix. Now, here we have chosen Q as semi definite. So, we will see in the solution that what is the significance of selecting the Q as a semi definite matrix.

So, here recall the lecture slide number 22; therefore, squared matrix of dimension n. We, an having the eigenvalues lambda 1 to lambda n, eigenvalues of a; then, the equation has a unique solution of P corresponding to each Q. Now, note that here Q is against selected as a positive definite.

So, in the slides we discussed or we focus mainly on the positive definite matrices. So, corresponding to each Q if and only if, the eigenvalue of the matrix A is not equal to 0. In addition, the summation of that 2 distinct eigenvalues or in fact, 2 similar eigenvalues should not be equal to 0 for all i comma j.

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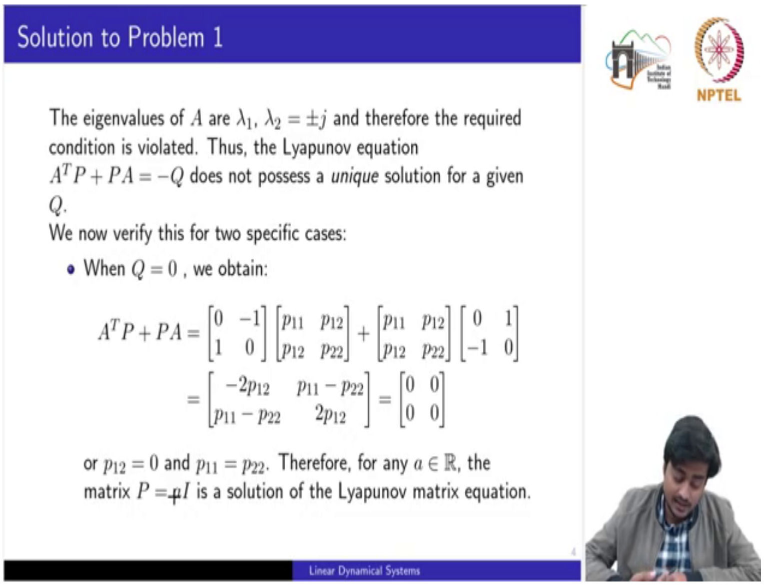

So, if you compute the eigenvalues of the matrix A directly, you would see that it is on the imaginary axis plus minus j. So, therefore, the required condition is violated. So, we can say that the Lyapunov equation which was given by this does not possess a unique solution for a given Q.

Now, we wanted to comment on to the solvability only for a semi definite matrix; but we had seen that if Q had been chosen as a positive technique matrix and if the eigenvalues are lying on the imagery axis, then it does not possess a unique solution.

Now, let us consider a case where Q is a semi definite. So, we directly select Q as a 0 matrix and if we start solving the left hand side of this Lyapunov equation, we would have A transpose P plus P into A.

So, all the matrices a matrix and its transpose are written in or a substituted in the into this equation and for computing the P matrix, we have used ah symbolic variables P 1 1 P 1 2 and this P 1 2 as well because it is a symmetric matrix and P 2 2. So, substituting these values, we obtain that we can compute the solution for P's which are given by P 1 2 is equal to 0 and P 1 1 is equal to P 22; meaning to say they exists a solution for a semi definite matrix.

Now, suppose therefore, for any a belonging to a set of real numbers the matrix P is equal to a is the solution of the Lyapunov matrix equation. So, the consequence of choosing Q as semi definite matrix, we see though it does not possess a unique solution because for different a s, we will obtain different P, but being a semi definite matrix, we can ensure that there exists at least 1 solution ok.

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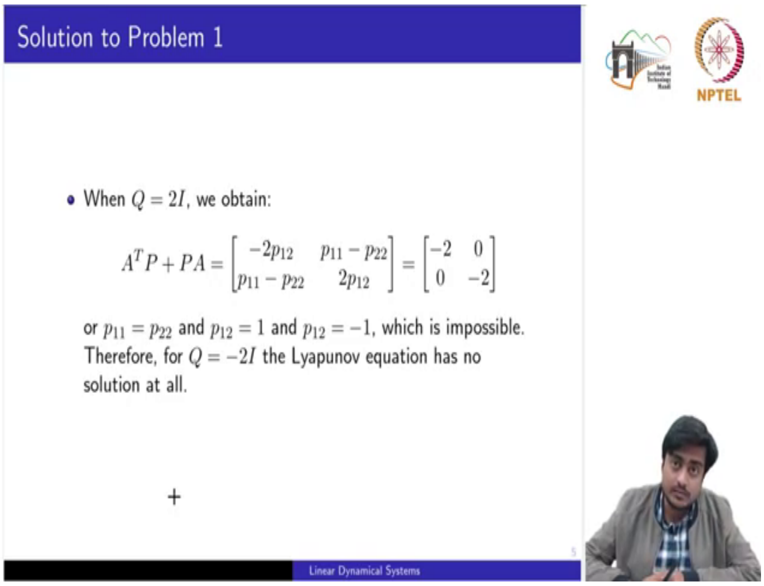

Now, choosing Q as a strictly positive definite matrix, you would see once you start solving for all the P i j's you would see that you obtained P 1 1 is equal to P 2 2 and P 1 2 is equal to 1 and P 1 2 is equal to minus 1 which is not possible. So, for Q is equal to 2 I should be 2 I; the Lyapunov equation has no solution at all right.

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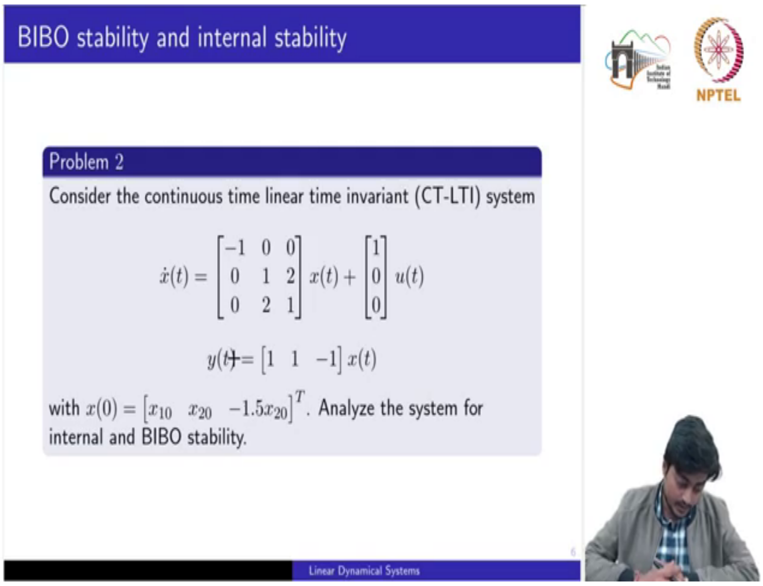

The problem 2 deals with that given continuous time linear time invariant system, where the A matrix is a square matrix or dimension 3, similarly the B matrix is 3 cross 1 and the C matrix is 1 cross 3. So, again it is a single inputs single output system with initial conditions given as this. So, note that here that x 3 is given in terms of the initial condition of the second state. So, we want to analyze the system separately for the internal stability and also for the BIBO stability. If we write explicitly the dynamics of that state specific equation, we can write 3 different equation.

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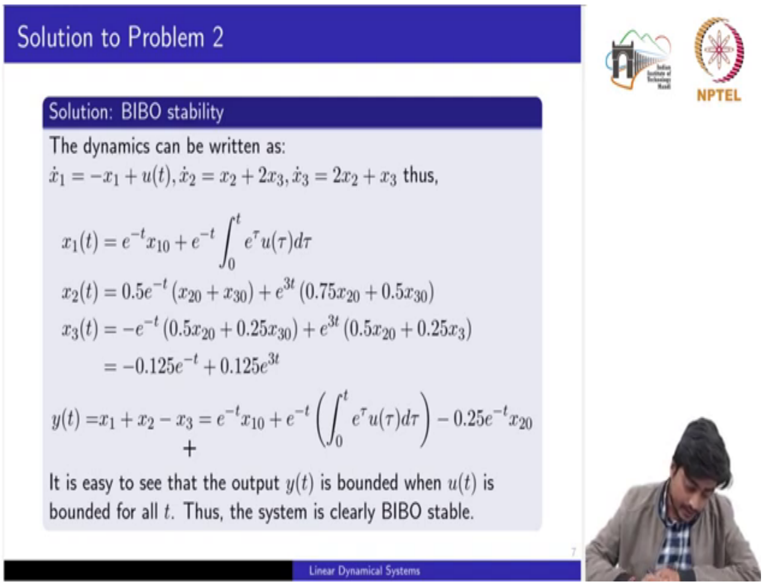

One in each for the different sets; x 1 dot, x 2 dot and x 3 dot. So, if you see the first equation first the state of or the right hand side does not contain any x 2 and x 3. So, we can separately solve this equation for x 1 which has been done here by using the variation of constants formula; then, putting those x 1 or using those x 1, we can compute this x 2 and x 3 separately for that given initial conditions.

Now, right computing y t which is nothing but the summation of the first two set minus the third set, we obtain final this result. So, you would see that we can show that given bounded input signal, you being a bounded signal, we would obtain a bounded output for all time t hm. So, this system is clearly a BIBO stable system.

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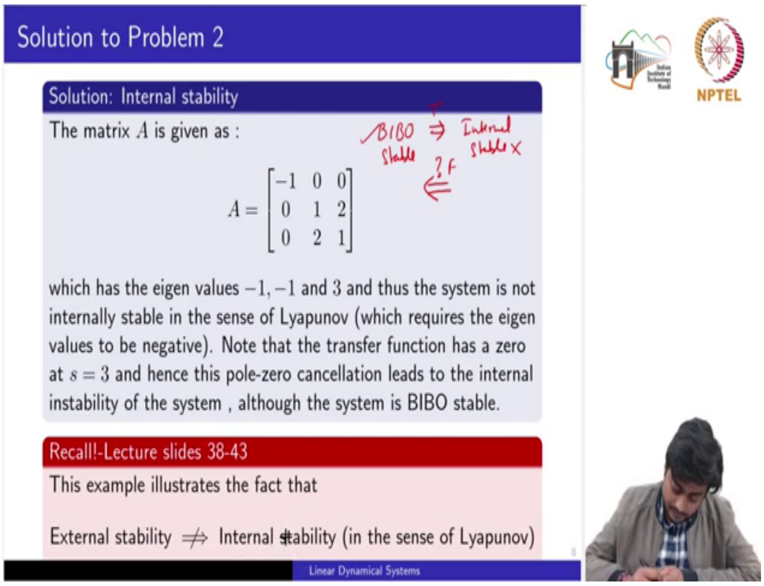

Now, checking for the internal stability which requires to compute the eigenvalues, you would notice the eigenvalues of the a matrix computed as minus 1 and minus 1 and 3 ok. So, it is clearly visible that because of one of the eigenvalues being on the right hand side, the state space system is not at all stable in the sense of Lyapunov is one of the definitions we had introduced during the lecture week.

Now, one important conclusion we also made during the lecture week while discussing the relationship between the BIBO stability and the Lyapunov stability, whether the both ways implication holds that if my system is BIBO stable, does it implied that it would be internal stable or if my system is internal stable my system would automatically be BIBO stable.

Now, with this example we have see that the state space equation is BIBO stable, but it is not internal stable. So, this implication would hold true while this implication won't hold. So, which is a false.

So, here we have written this external stability. By external stability, we actually mean the BIBO stability because we also call this external stability because we are dealing it, we are dealing with the external signals u and y not with the internal signals which is the state. So, that is why the we call it the internal stability ok.

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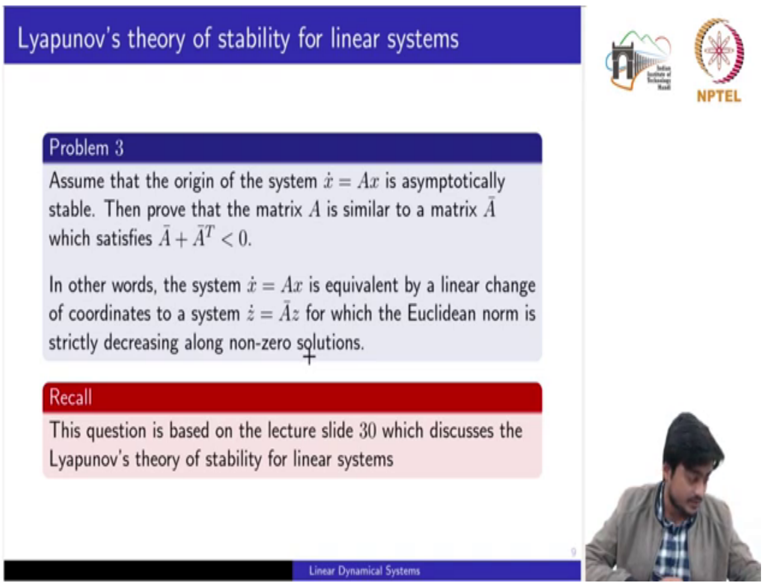

The problem 3 deals with that the assume that the origin of the system which is a homogeneous system given the state matrix A; A is asymptotically stable. So, we know that all the eigenvalues of this matrix lie on the left hand side, then we need to show that the matrix A is similar to another matrix A bar which satisfies that A bar plus A bar transpose is a negative definite ok.

So, this question can also be formulated in another words that is to say with the system is equivalent by a linear change of co-ordinates to a system in terms of the state denoted by z with the A bar state matrix for which the Euclidean norm is strictly decreasing along non zero solutions.

So, if you recall that the Euclidean norm, it is equivalent to computing the 2 norm of the matrix or the vector ok. So, let us see the solution to this problem. So, here you should recall the slide number 30 which discusses the Lyapunov theory of stability for linear systems.

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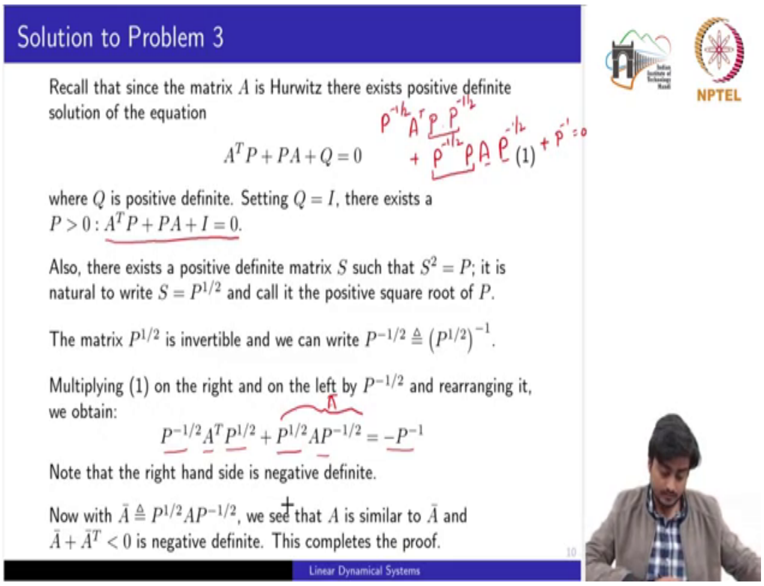

So, recall that since the matrix A is Hurwitz, saying the matrix A being Hurwitz meaning that all the eigenvalues are on the left hand side. So, there exist a positive definite solution to this equation given a positive definite matrix Q ok. Now, this Q can be selected any positive definite matrix.

So, we have selected that I. So, their exist any P which is again symmetric and positive definite satisfying this equation ok. Since, the matrix is already is a stable matrix. So, the Lyapunov equation would hold for the given Q and P.

Now, once we have computed the P matrix, we can define another positive definite matrix let us say S such that S square is equal to P. Now, we can also write S is equal to the under root of P and we shall call it the positives square root of the matrix P. Why the positive square root?

Because we know that it whenever we are taking the square root of any a positive number. So, the square root would be either on the positive side or on the negative side, but we are specifically taking the positive square root and we call it this matrix as a positive square root matrix of the computed matrix P.

Now, using the property of the symmetric and a positive definite matrix we say that this matrix P to the power 1 by 2 is invertible and we can write this matrix is P to the power minus 1 by 2 is the inverse of the square matrix itself ok.

So, multiplying 1 on the right hand side and on the left by P to the power minus 1 by 2 and rearranging it, we obtain this equation. So, see if we can see this step in more detail. So, starting from this equation, it would be I multiply P to the power minus 1 by 2 and here I would have A transpose and P into P to the power minus 1 by 2 right.

This is the first term. The second term is P to the power minus 1 by 2 PA again P to the power minus 1 by 2 plus the multiplication of P to the power minus 1 by 2 by itself would yield P to the power minus 1 is equal to 0 ok.

Now, we combine these two terms which leads P to the power 1 by 2 which is system a to the A transpose would stay as it is; this matrix as well. Now, on the second term, we would have the P to the power 1 by 2 a matrix and this matrix would stay as it is.

Now, we take this P inverse onto the right hand side writing minus P inverse. So, we note here that sense P is a positive definite matrix, its inverse would also be positive definite. So, the right hand side would definitely be a negative definite matrix because of the negative sign..

Now, with A bar defining is this complete part. We see that a is similar to A bar and A bar plus A transpose is less than 0. In fact, you can see that the A bar is nothing but stimuli algebraically equivalent to this matrix A, which is related by a non-singular transformation matrix P to the power 1 by 2 ok. So, this completes the proof of this part.

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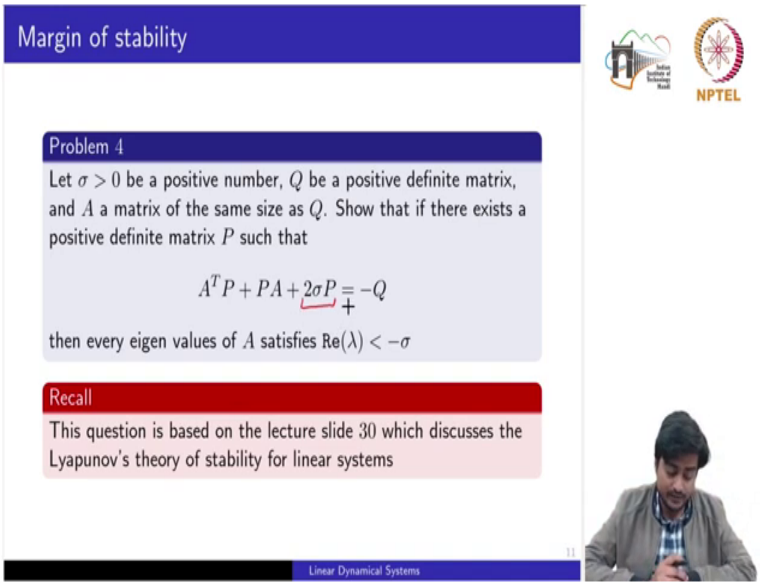

In the problem 4, we see that given a positive number sigma and Q be a positive definite matrix and A a matrix of the same size as Q. So, that if there exists a positive definite matrix P such that it satisfies this equation; then, every eigenvalue of a satisfies that the real part of the eigenvalue is less than minus sigma.

So, you would notice here that if this part is not there, then this equation is nothing but Lyapunov equation. Now, and we know that for a given positive definite matrix Q if there exists a positive definite matrix P, then the A matrix would be a stable matrix that is to say that all eigenvalues would be on the left hand side, if this part is not there ok.

Now, with the addition of this part, what we need to show now or what it implies that the eigenvalues of the A matrix are also shifted towards the left hand side or let us say the in fact, the axis the 0 axis is shifted to the left hand side. So, the results which we discussed in the lecture slide 30; by using those results, we would going to show that the axis has been also shifted towards the left hand side by adding this time into the Lyapunov equation.

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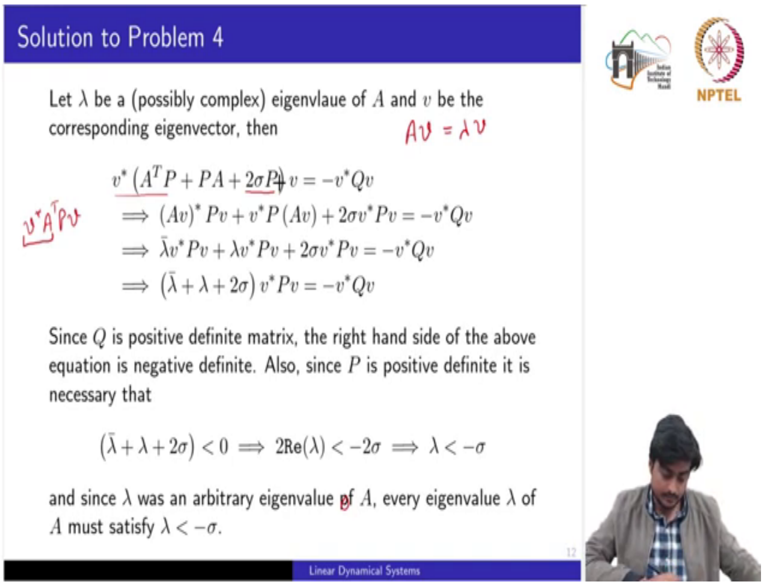

So, here we would going to use the quadratic forms and the matrix norms which we have introduced during the lecture week. So, let lambda be a possibly complex eigenvalue of the matrix A and v be the corresponding eigen vector right. So, we know already that for associated to the eigenvalue A, the eigenvector we can represent as A v is equal to lambda time v, where v is the eigenvector associated with the lambda.

Now, using the equation, we write the quadratic form which is v star; v start as a complex conjugate of the vector v and this complete matrix which is on the left hand side into v. Now, we know that this equation already holds. So, I can write on the right hand side also minus v star Q v. Now, using one of the so, if I write this part only we would have v star A transpose v. Now, another way of writing this part or in fact, I forgot this one.

So, we would have this part also if I open this bracket. So, specifically this part, I can write as the complex conjugate transpose of the matrix of the vector A into v hm. You can see this equalization transiting from this part to this part, the rest of the part would remain same plus v star P A into v plus I k because sigma is a scalar. So, I can commute with the vector. So, I write 2 sigma v star P into v which is equal to the right hand side as it is.

Now, using this equation since it is a complex conjugate transpose, it would become the complex conjugate of the of the lambda as well. Lambda bar v star P v similarly A v could also be replaced by this lambda v. Again, lambda is a scalar. So, I can commute. So, we would have v star P v and similarly, on to the third term. Now, if I take the common part which is v star P v onto the right hand side and group all these scalar values which is lambda bar plus lambda plus 2 sigma right.

So, what do we need for this matrix to be stable that the right hand side of the above should also be negative definite. Now, this would be become negative definite, if this scalar is less than 0 is on the left hand side. Now, if I combine this lambda bar plus lambda, it becomes twice the real part of lambda because lambda bar is a complex conjugate transpose of in fact, the complex conjugate of the lambda only.

So, I would have that lambda should be less than minus sigma and since lambda was an arbitrary eigenvalue of A. So, this is a o and every eigenvalue lambda of a must satisfy that lambda should be less than minus sigma ok.

So, this completes the proof of this part. So, one important conclusion you should note here that if we add this term into the Lyapunov equation, the key implication is that all the eigenvalues of A would also be shift to all would also be shift towards the left hand side of that scalar parameter introduced in the Lyapunov equation ok.

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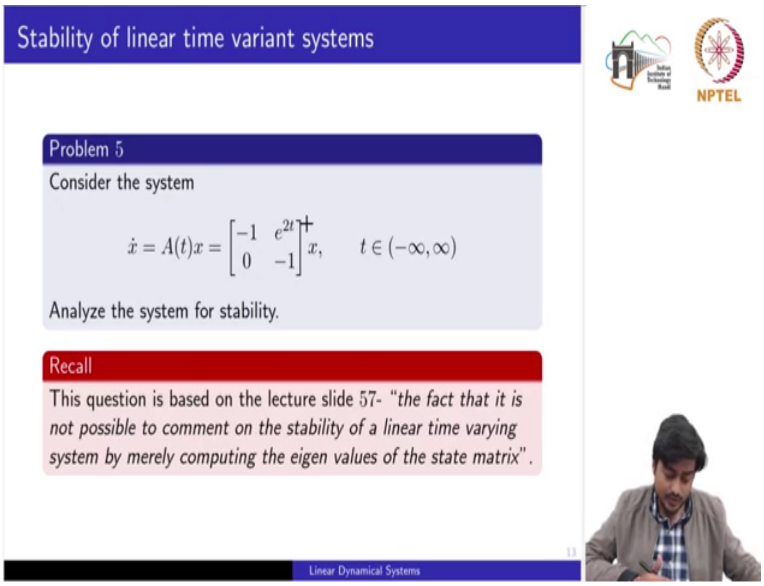

So, the problem 5 deals with the stability of linear time varying systems. So, consider the system now homogeneous system once again, where A of t is given by this matrix which is a time dependent matrix. So, we want to analyze the system for stability.

So, this equation is this question is based on the lecture slide 57, where if I put some of the one of the statements from that slide that the fact that it is not possible to comment on the stability of a linear time varying system by merely computing the eigenvalues of the state matrix. So, so far if my system is an LTI system, we have been determining the stability by computing the eigenvalues.

Now, if we want to applying the same concept to time varying matrix, we had established that using the eigenvalue concept, you cannot determine the stability of time varying system though it is linear. So, we will see that how we can compute the determine the stability.

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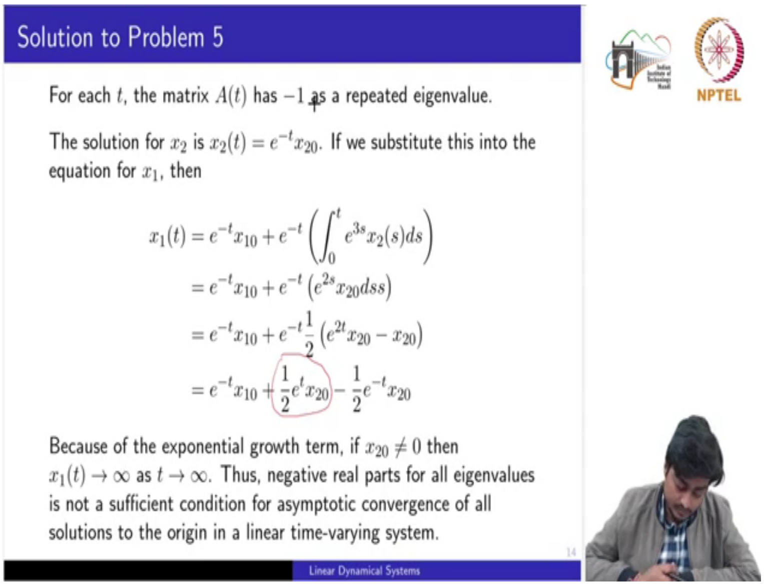

So, for each t, this you can do very a very quick test that using this A of t matrix first of all will compute the eigenvalues. So, you would notice that the matrix has a repeated eigenvalue on minus 1.

Now, according to that concept, we see that the matrix or the homogeneous system is a stable system. But if we pay attention towards its computing the solution, so we see that x 2 can be computed explicitly and by putting x 2 into the first equation, we can compute x 1.

So, x 2 trajectory is a stable trajectory for every initial condition right, but if we pay close attention to x 1 t because of this part, since it involves the positive powers of the exponential, we see that the x 1 trajectory we will reach towards infinity; x t tends to infinity meaning to say that this system is not stable because if the system is happened to be stable.

Both the state trajectory should reach to 0, x t tends to infinity. While performing the eigenvalue test on this, homogeneous system a signifies that this homogenous system is a stable system. So, eigenvalue test is not applicable for time varying systems and we need to or one way is to compute the solution.

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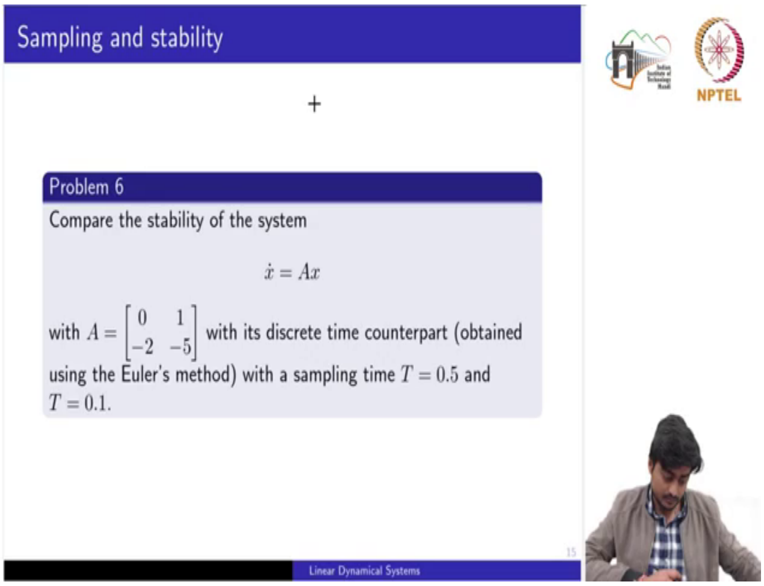

So, this problem is an interesting problem, where we would utilize some concepts which we the discretization concept, we have introduced in the week 1 that we want to compare or first of all we want to determine the stability of the LTI system given this a matrix and we want to compare the stability once the system is discretize.

Now, we had separate test for determining the stability given a continuous time system and at a discrete time system. Now, the problem here is that we are provided with the continuous time system. Now, if we discretized it using the methods we have introduce in the week 1, we want to verify that the discrete time system is also stable or whether the stability depends on the sampling time right.

So, here first of all we will. So, there were 2 methods we have introduced for the discretization; first is the Euler's method and the where we have approximated this day where depart by that method, another method is when we considered the signal u as a piecewise constant signal between the 2 sampling instant and then we computed another discretization form of the continuous time system.

So, here first of all we will do the sampling using the Euler method at 2 different time sampling times 0.5 and 0.1 and we see whether the system is a state stable or unstable given the stability or un stability in the continuous time domain.

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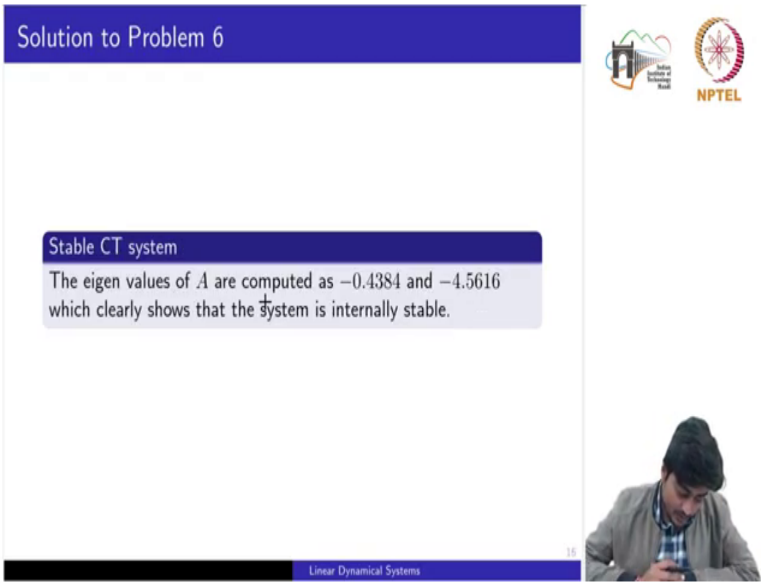

So, it we can quickly verify that the eigenvalues of the continuous time A matrix lie on the left hand side and on the real axis. So, it is clearly an internal stable system right.

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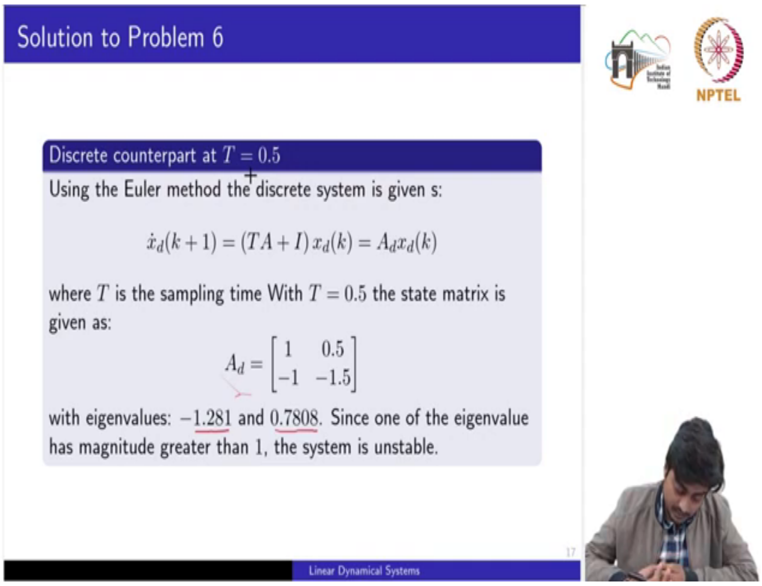

Now, if we sample it using the Euler's method at the sample time T is equal to 0.5, this would be the A matrix in that case sample time multiplied by the A matrix plus an identity matrix of the appropriate dimension. So, we obtain this ad after porting this T is equal to 0.5, the continuous time a matrix and the identity we obtain this A of d ok.

Now, computing the eigenvalues of this ad matrix, we see that one of the eigenvalues is inside the unit circle, while another eigenvalue is outside the unit circle. So, if I sample my continuous time system at 0.5, we notice that the system is no longer a stable system in the discrete time domain.

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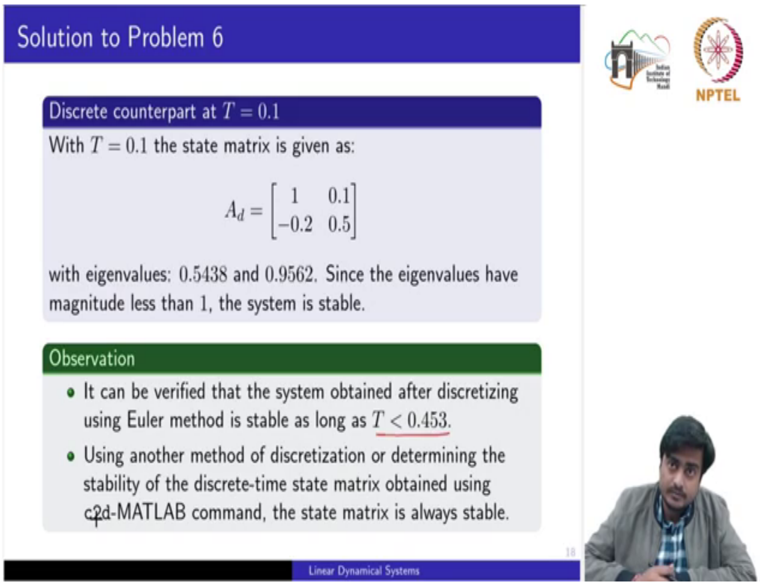

Now, verifying at T is equal to 0.1, we see this A d matrix and computing the eigenvalues, we notice that both the eigenvalues are in fact, inside the units circle. So, at 1 sample time T is equal to 0.1, it is a stable matrix or it is a still a stable system; for another sample time which is a bit on the higher side, we see that the system is no longer stable system and you can also parameterize the stability in terms of the sample time by solving this last equation in terms of T and determining the condition on the T.

So, we see that for all positive sampling time less than 0.53, the discrete time counter part of the continuous time system would always be stable right. This is bit what we have noticed; at once we have chosen capital T is equal to 0.5 which is greater than this value, then the system becomes an unstable system right.

So, now if we use another method of discretization that is the second method which is equivalent to computing the discretize system by using the c2d command in the MATLAB which is the continuous to discrete time domain discretization. So, you will notice that that state matrix, whatever the state matrix we would obtain it would be stable for all the sample time .

So, this method, so, one of the consequences or one of the motivation behind using another method of discretization is that the Euler method, it is the least accurate result, provides the least accurate discretization discretize system and this is also visible in determining the stability also.

So, this condition provides a very conservative estimate of the sampling time. Now, for the more accurate discretize method which is equivalent to the c2d command we say that the or you would notice that the state matrix is always a stable matrix.

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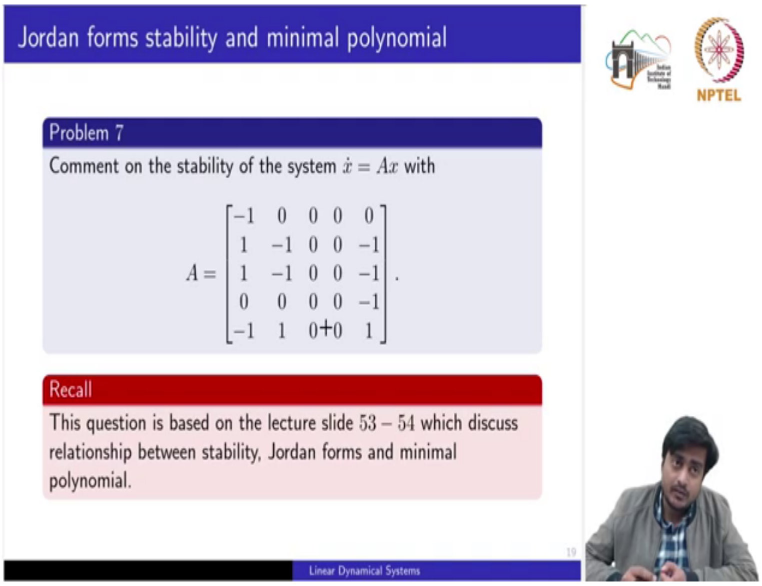

So, this is the last problem of this tutorial. So, we want to comment on the stability of the system x dot is equal to A x, where A is given by this 5 cross 5 matrix. So, here we would utilize the theoretical results, we had discussed onto the slide number 53 and 54 which basically discussed the relationship between the stability, the Jordan forms and the minimal polynomial.

So, if you recall that we had defined the or we have obtained one of the results of for stability that if the eigenvalues are on the left hand side and if some of the eigenvalues are on the imaginary axis or the origin and corresponding to those eigenvalues which has the zero real part, if the Jordan blocks are of 1 cross 1, then the system is marginally stable.

So, first of all we will compute the eigenvalues of this matrix, then convert them into another canonical form which is called the Jordan forms and we will see that whether if there are some eigenvalues on the having the zero real part if all the Jordan blocks are 1 cross 1, then the system is stable. If it is not, then the system going to be a marginally stable equivalently.

We have also defined in terms of computing the minimal polynomial which is basically computing the roots of a polynomial of a degree lesser than the characteristic polynomial if possible and there are some properties which needs to be satisfy for a minimal for the polynomial being the minimal polynomial.

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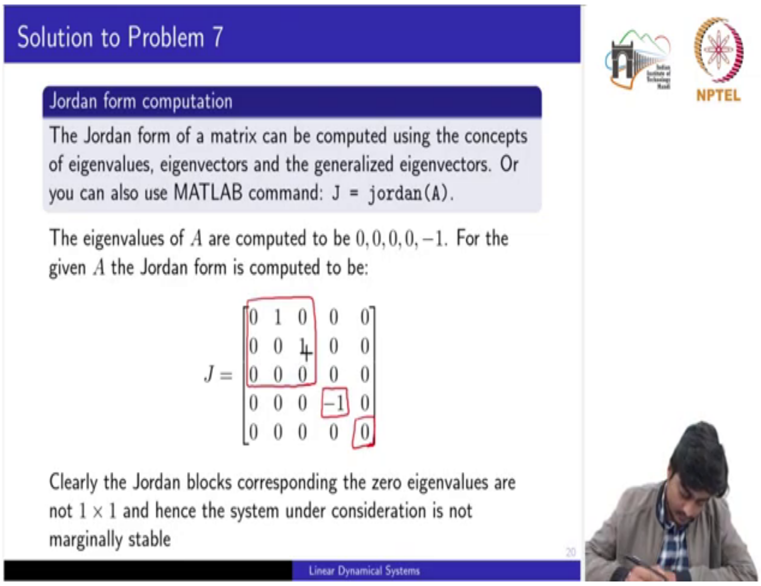

So, first of all we will compute the eigenvalues. The eigenvalues of this A matrix we have the repeated eigenvalues at the origin and the another eigenvalues on minus 1 ok. So, for computing the Jordan form, you can use this MATLAB command which is given by its name Jordan and by the given metric A. So, this J matrix you would obtain.

Now, if you notice here, we have 4 repeated eigenvalues at the origin and one eigenvalue at minus 1. So, start from; so, first of all we will start from the we will form the Jordan blocks.

So, the first block is this one which is corresponding to 3 repeated eigenvalues at 0; another Jordan block is with respect to minus 1 and the last Jordan block is with respect to the forth eigenvalue at 0. So, here we notice that at the repeated eigenvalues at the origin we have a Jordan block of 3 cross 3. So, this system is not a marginally stable system ok.

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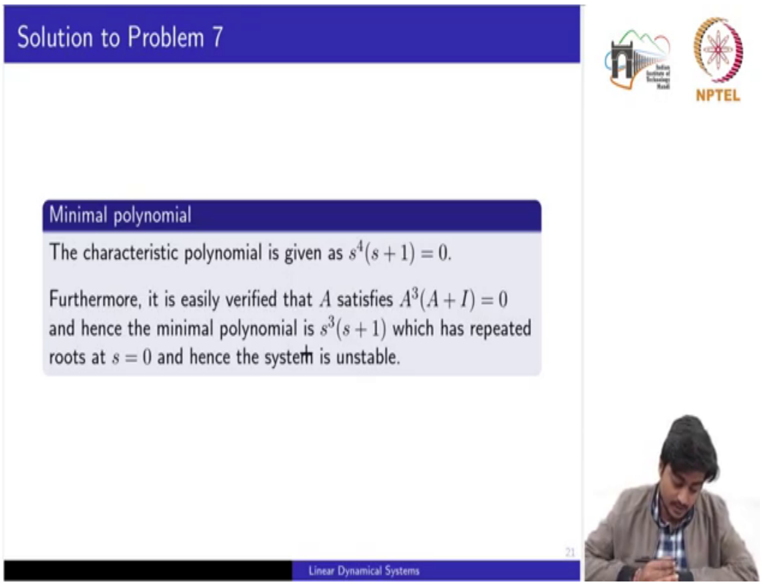

Now, computing the minimal polynomial, first we write the characteristic polynomial s to the power 4 s plus 1 is equal to 0. So, from here you would see directly we would obtain the repeated eigenvalues at located at 0 and another eigenvalue at minus 1.

Now, with the procedures introduced into the lecture slides, we compute the eigenvalues sorry, the minimal polynomial and the minimal and the polynomial which is minimal and satisfy all the properties we had introduce at that time. s given by s cube into s plus 1, the degree of this polynomial is four while the degree of the characteristic polynomial was 5.

So, this is a (Refer Time: 33:08) degree and this minimal polynomial again shows that we have 3 repeated eigenvalues at 0 which is equivalent to saying here into the Jordan blocks, since the Jordan block is a 3 cross 3. So, it is trivial or it is expected from the minimal polynomial that it would yield the not so simple eigenvalues located at the origin ok.