Linear Dynamical Systems
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Week - 02
Stability
Lecture - 11
BIBO vs Lyapunov Stability

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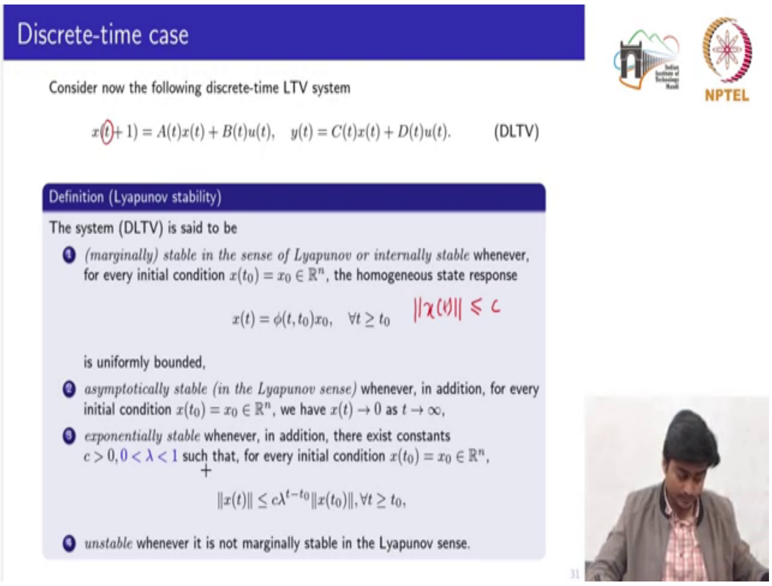

So, now we will see the Lyapunov Stability theorem results for the discrete time case. So, consider now the following discrete time, linear time varying system where now this t belongs to the set of integers. So, x t plus 1 is equal to A of t into x of t plus B of t u of t. So, here A B C D matrices are time varying matrices in the discrete time domain.

So, the result what we had discussed for the continuous time case, all those results almost remains the same for the discrete time case. So, the first result says is the system is the system is set to be marginally stable in the sense of the Lyaponuv or the system is internally stable whenever for every initial condition x naught the homogeneous state response x of t is uniformly bounded. So, the first definition we know already that the solution of the state based system must be bounded.

And when we speak about the uniform boundedness we actually mean to say that the norm of the signal x of t is always less than equal to some constant c, ok. Now, the second definition says the system is asymptotically stable occur in the sense of the Lyapunov, whenever in addition to the first statement for every initial condition x naught we have x of t approaching to 0 as t tends to infinity.

The third definition says that the system is exponentially stable whenever in addition to the above two statements there exists constants c and lambda both positive constants, but here with respect to the continuous time system, we have an additional condition on lambda that it should be less than 1. Such that for every initial condition x naught the norm of the signal x is less than equal to c into lambda to the power t minus t naught and the norm of the vector at t is equal to t naught. The system is set to be unstable whenever it is not marginally stable in the sense of Lyapunov.

So, almost all the definitions remains similar to their counterpart of the continuous time system, there is a minor difference only in the third statement. Again, here the B C D matrices played no role in this definition.

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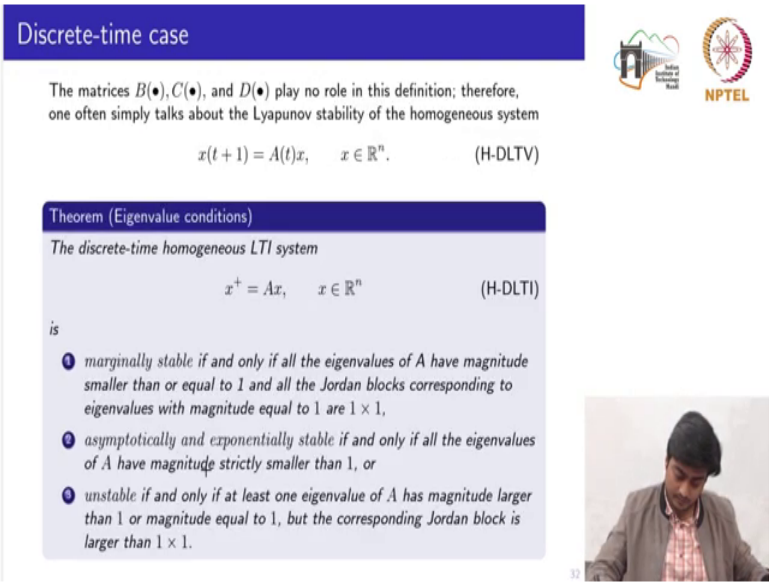

So, therefore, we will mostly speak about the stability of the homogeneous system that is without considering the input. So, for the continuous time system, we studied two conditions, basically 2 tests to determine the stability, one is the eigenvalue test and another is the Lyapunov test. So, the first eigenvalue test for the discrete time system or homogeneous LT I system; so, this x plus denotes the x t plus 1, ok.

So, this LTI system is marginally stable if and only if all the eigenvalues of the matrix A have magnitudes smaller than or equal to 1. Basically, all the eigenvalue should be inside a unit circle and all the Jordan blocks corresponding to the eigenvalues with magnitude equal to 1 are 1 cross 1, it is similar to what we had seen in the for the continuous time system. The LTI system is asymptotically and exponentially stable, because we already knew that the asymptotic stability and the exponential stability for the LTI system basically remains the same.

So, the system is stable if and only if all the eigenvalues of a have magnitude strictly smaller than 1 or the system is unstable if and only if at least 1 eigenvalue of the matrix A has magnitude larger than 1 or magnitude equal to 1, but the corresponding Jordan block is larger than 1 cross 1. So, we will not see the proof of this theorem since we had already gone through a detailed proof for the continuous time system, the Lyapunov test.

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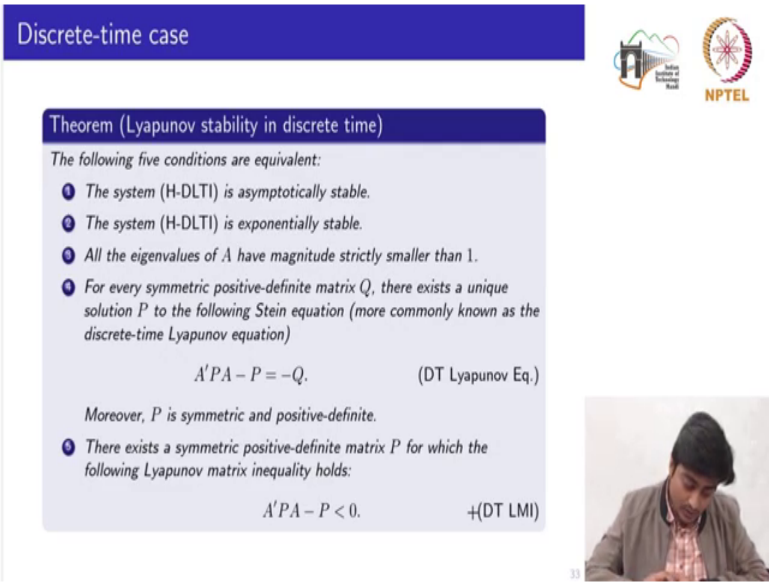

So, all these following five conditions are equivalent, the system homogeneous-discrete time linear time invariant system is asymptotically stable. Is equivalent to saying that the system is exponentially stable, is equivalent to saying that all the eigenvalues of the matrix A have magnitude strictly smaller than 1, is equivalent to say that for every symmetric positive definite matrix Q there exists a unique solution P to the following Stein equation.

So, this stein equation is also known as the discrete time Lyapunov equation. So, this Lyapunov equation reads A transpose P into a minus P is equal to minus Q. Moreover, P is symmetric and positive definite. Again, it is equivalent to say that there exists a symmetric positive definite matrix P for which the following Lyapunov matrix inequality holds. So, the change here is in the Lyapunov equation basically in the fourth and fifth statement with respect to the continuous time system, otherwise the first three statements are remains the same as we had seen there.

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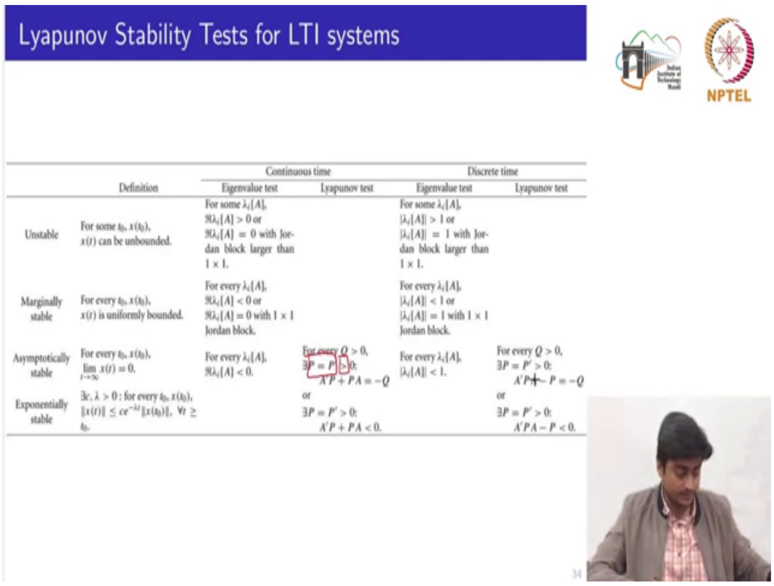

So, again you could do the proof of this theorem in accordance to what we had done for the continuous time system. So, if we summarize the overall results of the Lyapunov stability test for the LTI systems. So, here on the left column, we define all the definitions of the unstable, marginally stable, asymptotically stable and the exponentially stable systems. Here, we are giving a summary of the continuous time test and the discrete time test, again it is further divided into eigenvalue test and the Lyapunov test similarly for the discrete time.

So, let us see for the first for the unstable system for some t naught and the initial condition x of t naught, the signal x of t can be unbounded. It means, if we see the eigenvalue test in the continuous time that for some lambda i of A, the real part of the eigenvalue of the matrix A is greater than 0 or the real part of the eigenvalue is equal to 0 with Jordan block larger than 1 cross 1. For the discrete time case, the eigenvalue is outside the unit circle or on the unit circle itself, but with that Jordan block larger than 1 cross 1.

The system is marginally stable, if the signal x of t for a given initial condition is uniformly bounded which according to the eigenvalue test that all the real part of the eigenvalue should be strictly less than 0, and if it is lying on the 0 it should have a Jordan block of 1 cross 1 only. For the discrete time systems, the eigen value should be inside the unit circle or if it is lying on the unit circle, it should have a Jordan block of 1 cross 1.

The asymptotic stability defines that the signal itself should approach to 0 whenever t tends to infinity and for the eigenvalue test the real part of the eigenvalue of the matrix A should be strictly less than 0. For if we see the Lyapunov test so, for the given Q which is positive definite symmetric matrix Q, there exists a matrix P which is symmetric the symmetric condition is given by this one and the positive condition is given by this one.

So, there exists a matrix P which is symmetric and positive definite such that it satisfies this equation, the Lyapunov equation. For the discrete time case all the eigenvalue of the matrix A should lie inside the unit circle. Here, for the given positive definite matrix Q there again exists a matrix P which is symmetric and positive definite, such that it satisfies this discrete time Lyapunov equation.

The system is exponentially stable, if there exists a symmetric positive definite matrix P such that it satisfies this linear matrix inequalities, again similarly for the discrete time case. So, this table gives you the entire stability test specifically for the LTI systems both the eigenvalue test and the Lyapunov test.

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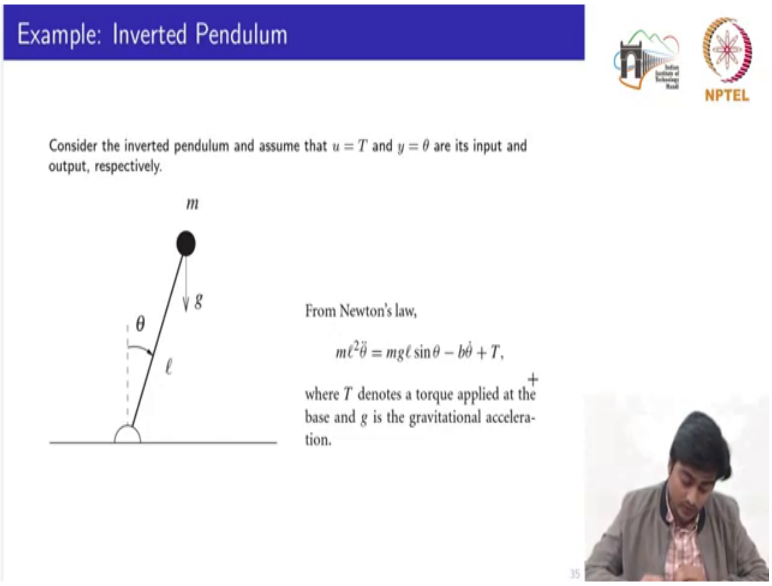

Let us consider one example with which we had started this lecture or this module. So, here we are considering the inverted pendulum. So, there we. So, initially we considered the simple pendulum, but you can also consider it as an inverted pendulum where theta is now the axis the sorry the angle made from the vertical axis, ok.

So, the control input to this inverted pendulum is the torque applied to the base and why or the state is the angle this pendulum makes from the vertical axis. So, we very it is well known that from the Newton's law, we obtain this particular equation which is you could imagine that it is a non-linear equation. So, while considering the case of the pendulum we identified 2 equilibrium points, one is at 0 and another is at pi. So, if we see the equilibrium point theta equals 0 means that the pendulum is standing still upwards, ok. So, let us see what happens.

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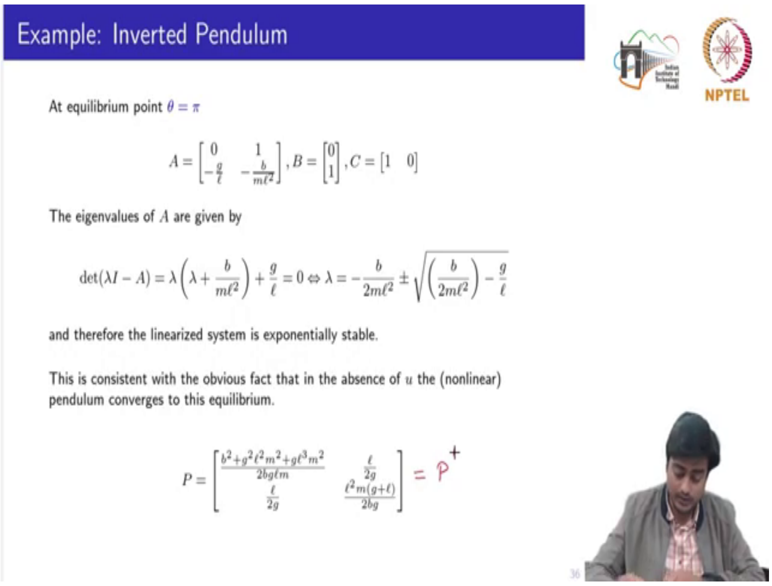

Now, if the equilibrium point theta is equal to pi is considered then linearizing this non-linear equation gives us this A B C matrices. Now, if I compute the eigenvalue of the matrix a by this formula determinant of lambda I minus A where lambda are the eigenvalues; so, after simplification I compute this two eigenvalues.

Now, notice here there that this part which is inside the under root part would always be less than or equal to this part. Meaning to say that all the eigenvalues would be strictly on the left hand side. So, it means that the system is asymptotically stable and also exponentially stable. Now, we can also compute the P matrix also. So, if we compute the you can do it by yourself for computing the P matrix. So, this is the P matrix for this system we have obtained.

So, now, if you compute or you can do the self test that if you compute the eigenvalues of this P matrix, you would see that all the eigenvalues of this matrix P are positive, the symmetric definite can be strictly seen. So, if you compute the transpose of this matrix P, you would see it is actually equal to the P transpose, ok.

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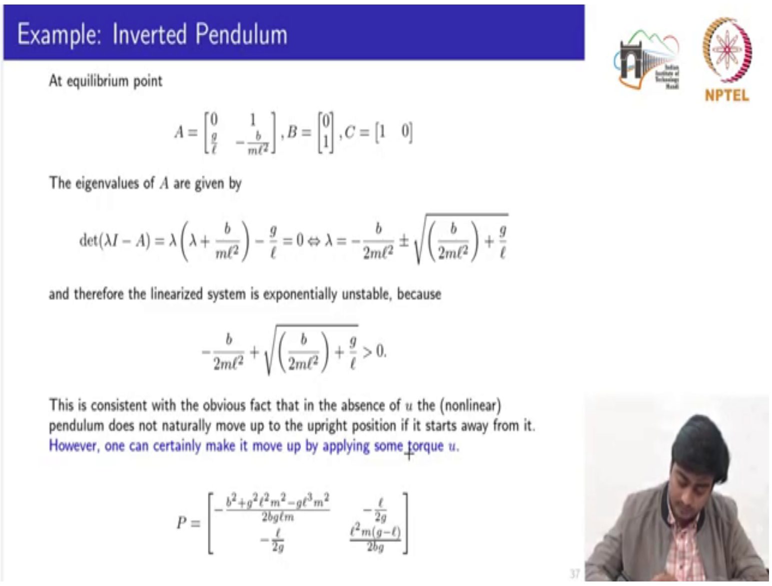

Now, considering at the equilibrium point theta is equal to 0, these are the set of state space matrices we have obtained. Now, the eigenvalues of this matrix A is given by this one and it is quite straightforward to see that these eigenvalues specifically with the positive sign is greater than 0. So, one of the eigenvalues is lying on the writing side meaning to say that the system is unstable and this we can also see while computing the P matrix.

So, if we compute the P matrix though it is symmetric, but it is not positive definite, ok. So, we see that the system or the homogeneous system without applying any control signal is a stable. However, one can certainly make it move up by applying some torque or some control input u, which we will study in the next module about the controllability and the state feedback, ok.