Linear Dynamical Systems
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Week – 02
Stability
Lecture – 10
Proof of lyapunov stability theorem

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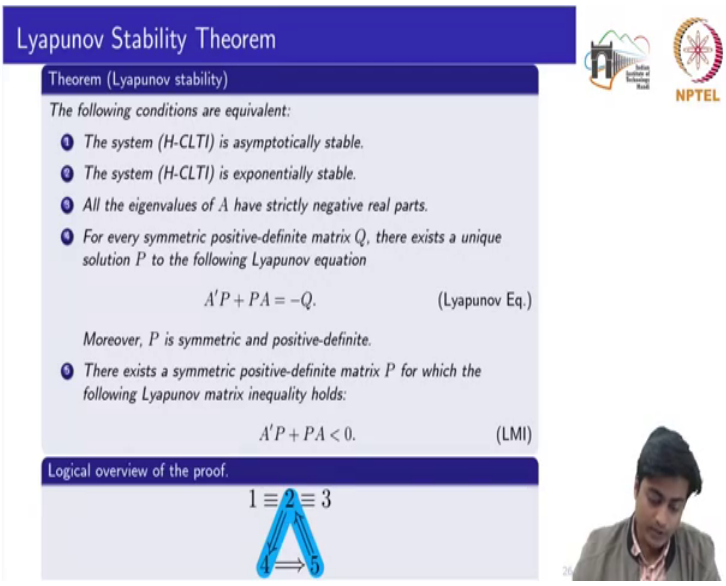

So, now, we will see the another implication that is 5 implies 2. The statement of 5 reach that we are already provided with the information that there are, there is there exists a matrix P already which is symmetric and positive definite, satisfying this equation. Now, what we need to prove that having P satisfying this equation implies that the system is asymptotically stable or exponentially stable, ok. So, let us see.

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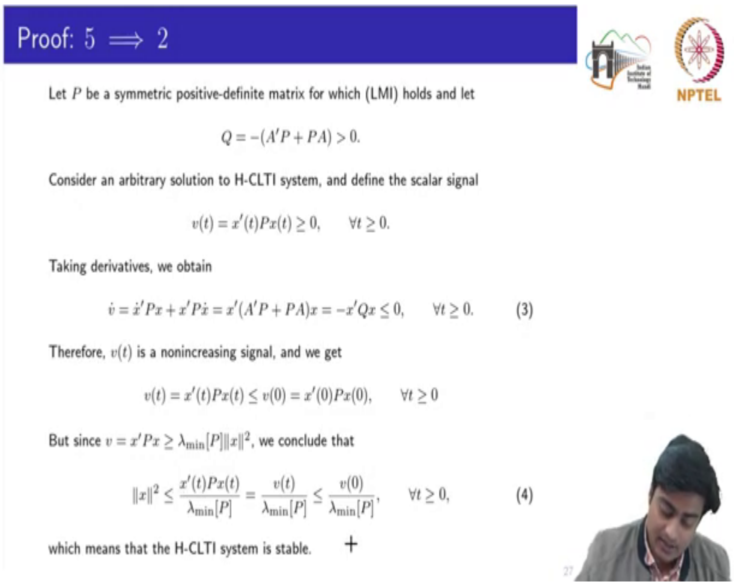

So, let P be a symmetric positive definite matrix for which the linear matrix inequality holds and let we define Q is equal to negative A transpose P plus PA which is greater than 0. I just rewritten this equation into another terms, ok. Because P and Q matrices are already given to us, and we already know that 4 implies P, sorry 4 implies 5, ok. So, consider an arbitrary solution to the system and define the scalar signal v of t which is another quadratic form with weight matrix P and the signal x which is given by x transpose P into x is greater than equal to 0, for all t greater than equal to 0, ok. So, here we are specifying basically that there is a signal v which again could be 0.

So, taking derivative of this signal we obtain this x dot transpose P x plus x transpose P x dot by putting x dot equal A x, here I can simplify this equation by this one, ok. So, the inside part is basically the matrix Q minus of Q because Q could be an identity matrix, and I know that v dot is less than equal to 0, right for all t greater than equal to 0 which means that the signal v of P is a non-increasing signal. Meaning to say that either the signal would remain constant or the signal would decay, ok. So, let us see.

In both the cases I could write that the value of the signal at some other time t then 0 would either be less than or equal to the signal at t is equal to 0, ok. Let us try to visualizing, that v of t could be a constant signal which is the time axis t and here we have v of t having said that v dot is equal to 0, this signal could be either a constant signal or this signal could be a diagonal signal. This signal would never increase. So, if I compute the value at any time t let us say t 1, this value would always be less than or equal to the value at t is equal to 0, right. And at t is equal to 0 the quadratic form I can also put t is equal to 0 in that quadratic form for all t greater than equal to 0.

But since now using the relationship we have introduced while discussing the quadratic forms or the positive definite that this quadratic form would have the minimal limiting value the smallest eigenvalue of the matrix P multiplied by the squared norm of the single x, which I could write norm of x square is less than equal to v which is x transpose P x divided by lambda min of P, ok. And this part is equal to v of t, and I know that v of t is less than or equal to v naught. So, v of t is less than equal to v naught which means that the system is stable it would never rise up.

But it only proves if the system is stable, it does not yet prove that the system is asymptotically stable system because for the stable system it must remain bounded which has been shown here that the norm is bounded now, ok. But it does not say that the norm would is bounded by an exponential matrix which defines the exponential stability, right. This is what we need to prove the next.

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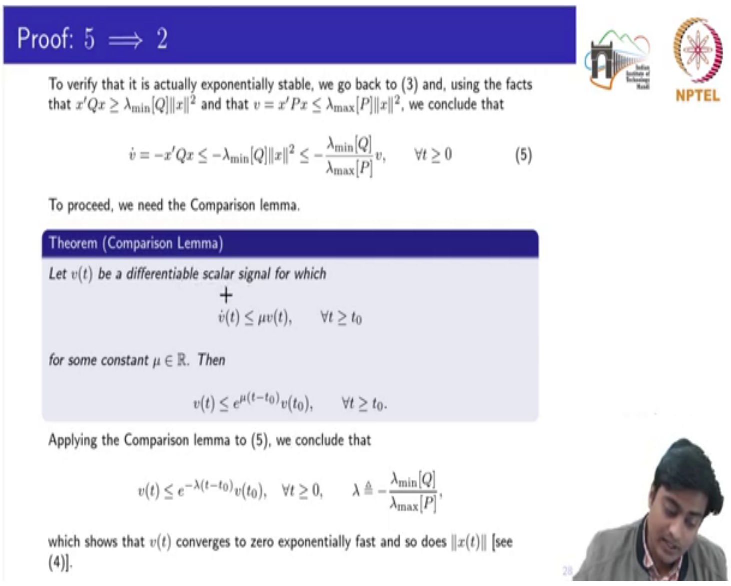

So, to verify that it is actually exponentially stable, we go back to the relationship 3. So, this relationship is basically this part which says that the signal v is a non-increasing signal, ok. We are go back to 3 and using the facts now we introduce another the same limiting values for the quadratic forms for both Q and P.

Here we define that this quadratic form is greater than equal to lambda min of Q and a square of the norm of x and that v which was defined as x transpose P x is less than or equal to lambda max of P into norm of x square. So, I can use either of the relation. Here in the previous proof we use this relationship which holds true similarly this relationship also holds true. So, let us see.

So, computing the v dot which is given by minus x transpose Q x from the relation 3 I can write that this part is less than equal to minus lambda min of Q norm of x square. Why?

Because both these quantities are positive. The norm of the vector is positive and since Q is positive definite, so in fact, the smallest eigenvalue of that matrix P would also be positive. So, I can reverse this relationship by putting a negative sign which I have done here, and this would be less than or equal to by using this equation that the norm of x square or the square of the norm of x is greater than or equal to v by lambda max of P, ok. So, I can write less than or equal to minus lambda min of Q comes from here and x square came from here lambda max of P v, for all t greater than equal to 0, ok.

So, I just applied this relationship and this relationship to the relationship that v dot is less than or equal to 0, ok. Basically, the relationship 3, v dot is less than equal to 0, ok.

So, to proceed further we give another lemma which is known as the comparison lemma. So, this lemma says that let v of t be a differentiable scalar signal which satisfies this relationship that v dot is less than or equal to some scalar mu, some constant mu into the signal itself for all t greater than equal to t naught. Then the signal v of t would always be less than equal to e to the power mu t minus t naught into the value of the signal at the initial time t naught, ok. Now, if you see the similarity between this lemma and this equation 5 we have v dot is less than equal to minus this part in to v and this part is some constant mu because I know that this part would always be a positive part. In fact, the smallest eigenvalue of the matrix Q and the largest eigenvalue of the matrix P they will always be positive and this part would always be negative, ok.

So, let us define by this part mu, so you could visualize now, that v dot is less than equal to mu of p. So, if v satisfy this then it should satisfy this, ok. So, I just write down this, ok. Now, if you see the similarity that v of t is less than this part which is nothing, but that definition of the exponentially stables systems, right which show that v t converges to 0 exponentially fast and so does norm of x t because first of all x t is bounded and now we have proved v which is also in terms of the signal x that it is exponentially stable. If v is exponentially stable then x should definitely be exponential stable. So, talking about the comparison lemma we could see a quick proof because this is a key lemma to prove that implication. So, this was the implication.

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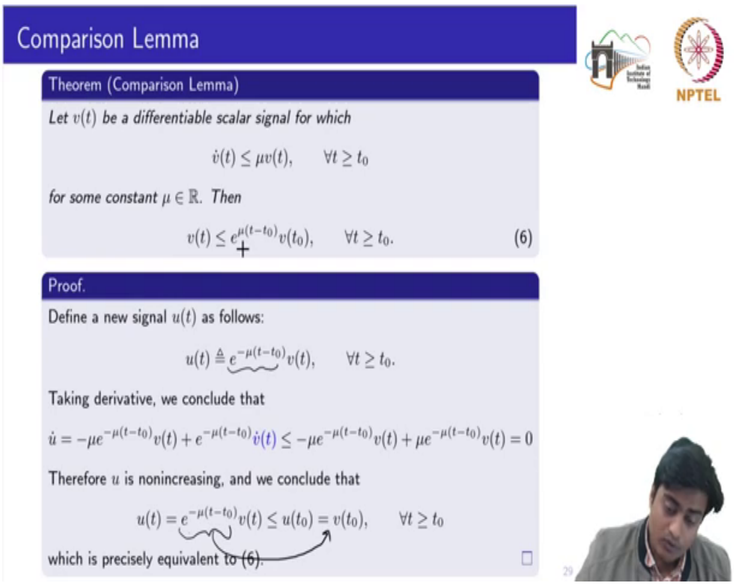

So, let us define a new signal u of t which is given by the exponential term mu t minus t naught v of t. Here we have no made no assumptions on the constant mu this mu could be positive or could be negative, ok.

So, taking derivative of the signal we obtain the rule chain for the derivatives. You could see the derivative of first into second plus first into the derivative of second. Now, here v dot t I know that this part is already given to us. So, I replaced this v dot from this and finally, obtained this relation which is in fact equal to 0, meaning to say that u dot equal to 0 that u is a non-increasing signal either it would be a constant or would decay, ok. So, we conclude that the value of u at some time t would always be less than equal to u at initial time t naught and u of t is basically given by this part.

So, taking this one here on the right hand side. So, what I would get? That v of t is always less than equal to this exponential into u of t naught and u of t naught if I put t is equal to t naught here this part would become equal to identity or 1, ok. So, u of t naught is actually equal to v of t naught. So, I can replace by v of t naught. So, finally, I have obtained this one for any mu, ok. This is how you can prove there is or you see the proof of this lemma, ok.

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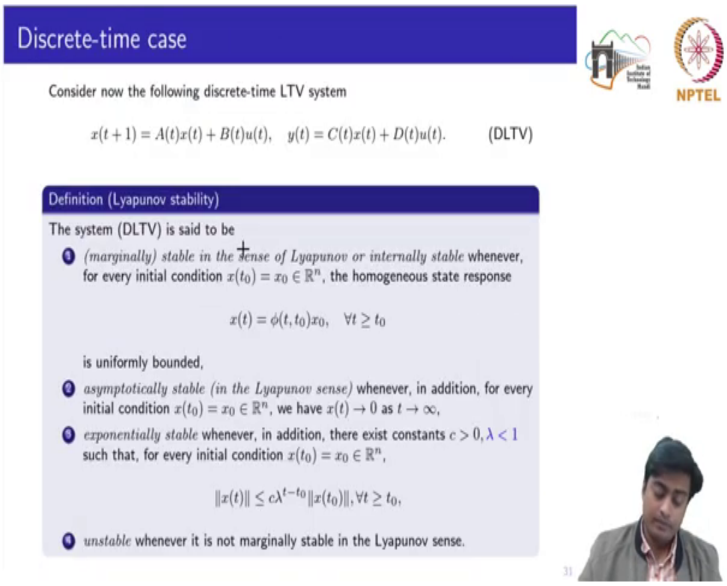

So, this is the overall proof of the key important theorem which speaks about that or which gives us the test for the stability without even computing first the solution, second without even computing the eigenvalues because for a higher order system or it would be a higher order polynomials for which there could be you need to solve a high order polynomial to compute the eigenvalues. Now, since the matrices gives us the compact form to analyze the systems we can easily compute the existence of the matrices which is here given by the P matrix, ok.

So, this completes the proof. We have already seen the equivalents between the 1, 2 and 3. We have discussed 2 implies 4 and 5 implies 2 and the implication between 4 implies 5 by could be easily obtained by putting Q as an identity matrix. So, this completes a proof.