Linear Dynamical Systems
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Week - 01
State-space solutions and realizations
Lecture - 01
Response and state-space solution of Linear systems

So, hello everyone, now we would be starting with the first week of the Linear Dynamical System course. So, in the first week, we would be discussing about the State-space solution, realization and its equivalence.

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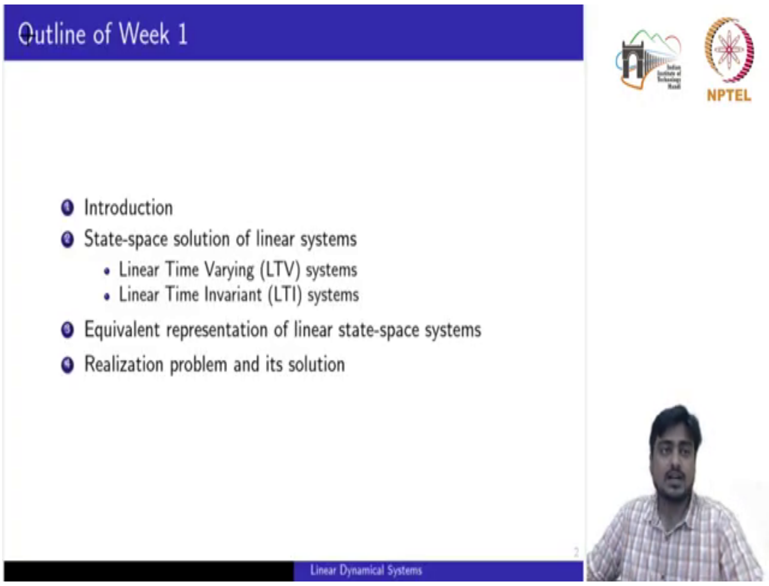

So, this is the outline of the first week, where we would be starting with the brief introduction about the input-output description, later on we would also introduce the notion of the states.

Then we would proceed with the solution of the state-space system particularly for the linear systems, and these linear systems we would be dealing both in the time variant case and in the time invariant case. The third module deals with the equivalent representation of the linear systems. And in the third and the final one we introduce the realization problem and its solution.

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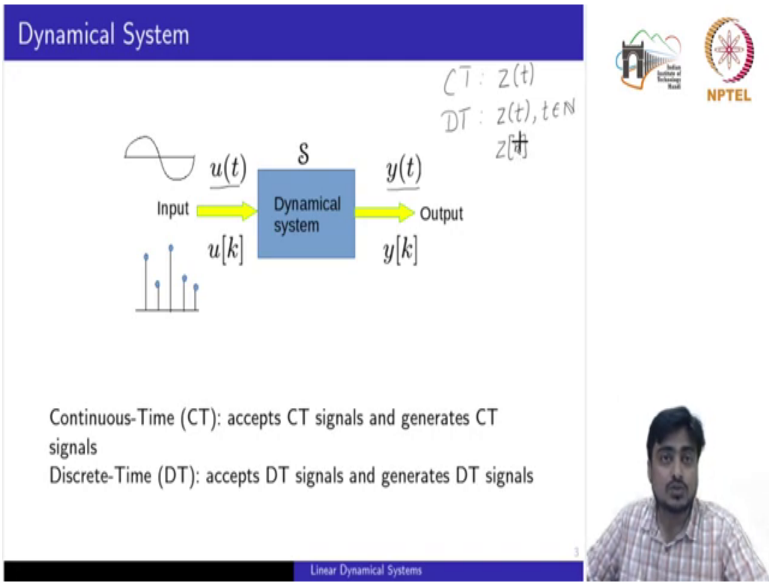

So, to proceed with the definition of the dynamical system; so, how do we define a dynamical system in the sense that dynamical system basically interacts with its environment through the input-output variables. So, here the input variable is defined by the u of t, while the output of the dynamical system is defined by the y of t. Now, note here that whenever we are discussing about the continuous-time systems, we use this representation where u of t of any signal.

So, if we are in the continuous-time, we would be denoting all the variables, let us see the variable is Z. So, this variable if I denote like this, it means that the system is it in the continuous-time. Now, if we are discussing about the discrete-time either we would be using the same notion Z of t, but with a specific definition of the t that it belongs to the integer or Z of k. So, here k is pre is implicit that it belongs to the set of integers ok.

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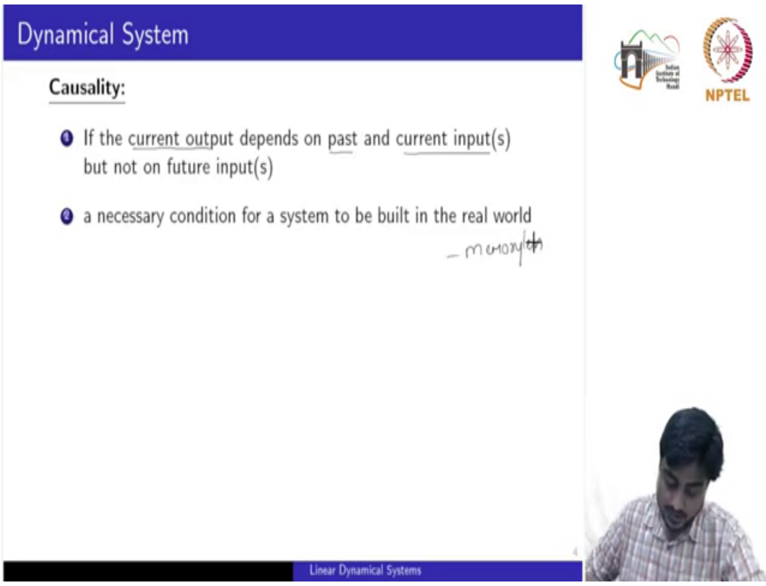

So, the first important property of a dynamical system is the causality. The causality is defined that in the sense if the current output depends on past and current inputs but not on the future inputs, then we say that the system is causal. This causality is basically a necessary condition for any system to be built or physically implementable in the real world. So, here two aspects are introduced that first is the current output either depend on the past input or on the current input.

So, one of the most important property of the dynamical system is the causality. We define the causality in the sense that if the current output depends on the past and current inputs but not on the future system, not on the future inputs, we will we say that the system is causal. Now, causality is basically a necessary condition for any system to be built or physically implementable or realizable in the real world.

Now, there are two aspects here that the current output either depends on the past input or on the current input. Here we would introduce two notions that if the current output depends only on the current and on only on the current inputs, then we say those systems are the memoryless systems.

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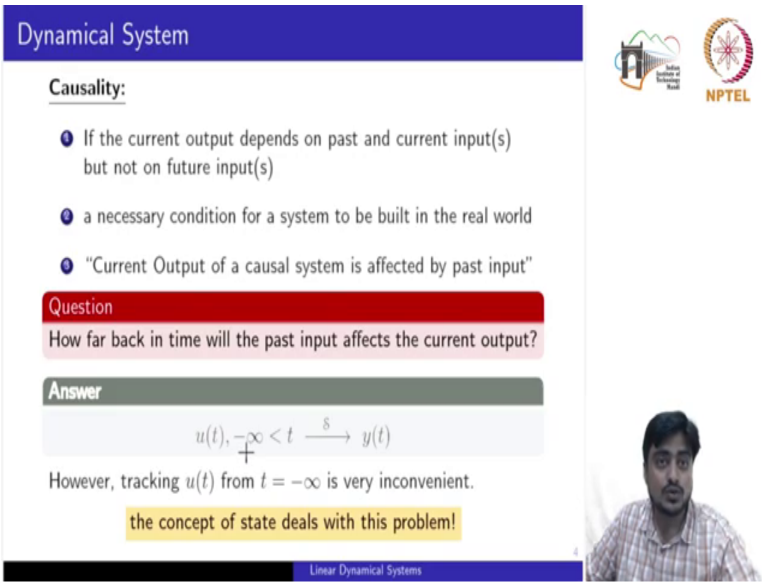

Now, if the current output depends on the past inputs, so how far the question arises that how far back in time will the past input affects the current output? The answer to this question is that we need the entire history of the input starting from minus infinity up to the current time t to actually compute the output y at time t. Now, this representation we would be using a number of time, this means that the u of t is applied to a dynamical system which yields the output y of t.

Now, here it says that we need to compute the output at time t, we need the information of the control input starting from minus infinity up to time t. Now, the question problem arises here is that tracking u t from t is equal to minus infinity is very inconvenient. So, how we can address this problem? To address this problem we would introduce the notion of the state, so that instead of tracking the entire history of the u starting from minus infinity, is there any answer that or is there any time, from which we need to know the control input to actually compute the output at time t.

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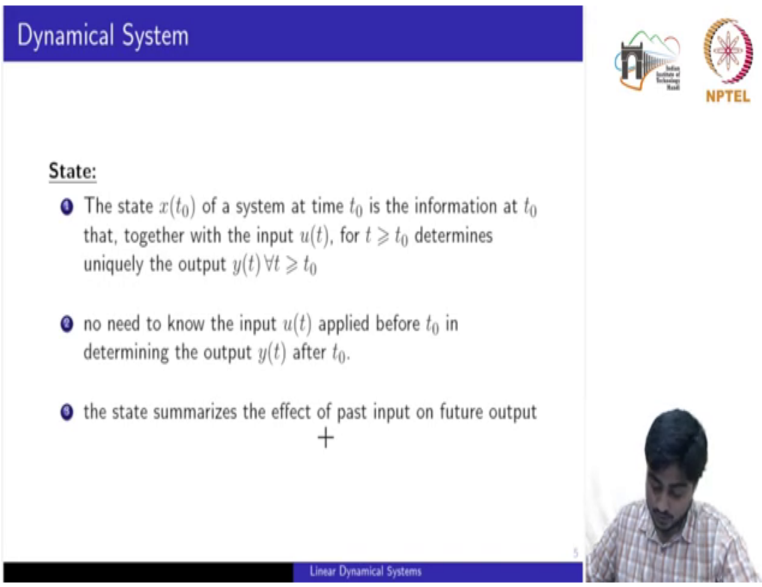

So, in this respect the state of a system at time t naught that is x of t naught is the information at t naught that, together with the input u of t for t greater than equal to t naught determines uniquely the output y of t. What does it mean that now if we have the information about the state at time t naught, then we do not need to know the u t applied before t naught. So, in fact, the state summarizes the effect of the past input on the future output.

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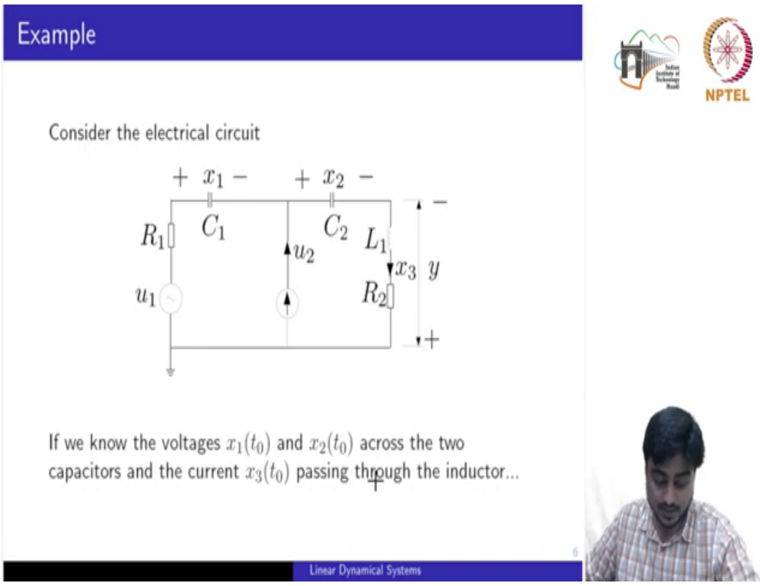

Say for example, consider this electrical circuit now if we know the voltages x 1 of t naught x 2 of t naught across the two capacitors and the current x 3 of t naught passing through the inductor.

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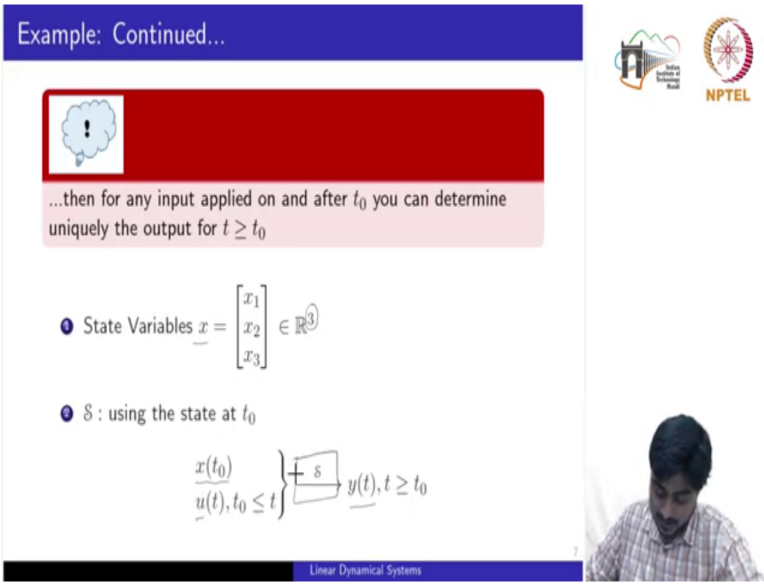

Then for any input applied on and after t naught, we can determine uniquely the output for t greater than equal to t naught. So, if we have three states in the previous example, we define a state variable as a vector of three state variables x 1, x 2, x 3 with an this x is basically of dimension three.

Now, if we revisit the input-output description of the system we introduced later or we introduced earlier by using the notion of the state, it says that if we have the information about the state at t naught and we have the information about u starting from that t naught up to time t, then only this information is require to compute uniquely the y of t for all t greater than equal to t naught. Again this is a dynamical system to which the input is applied and the output is obtained.

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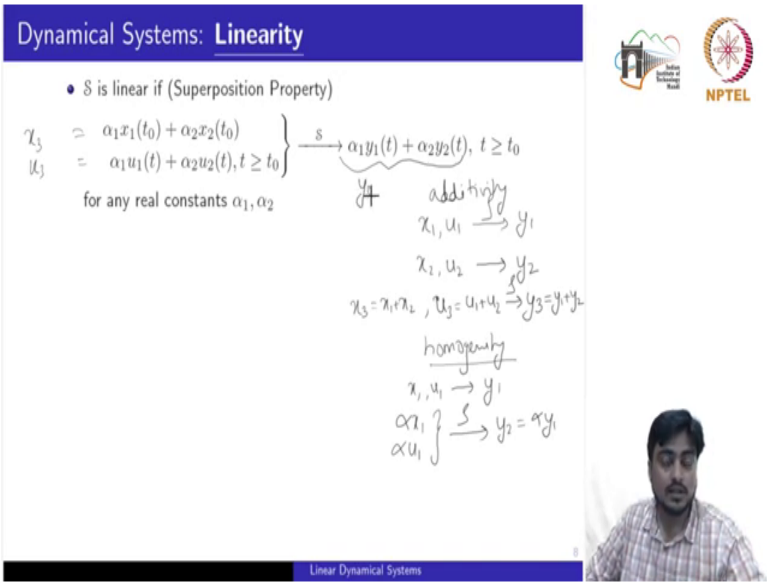

Linearity is another important property of a dynamical systems. So, we say that the system is linear if it satisfied the superposition property. Now, this superposition property is basically the combination of two properties; one is the additivity and another is homogenity. So, additivity says that if we apply x 1 to x 1 u 1 to the to any dynamical system S and we obtain the output y.

And if we apply x 2 and u 2 to the same dynamical system which gives us the output y 2, then it be we can take the linear combination of the state and the input that is to say we define x 3 as x 1 plus x 2, and u 3 as u 1 plus u 2 then if we apply this input to the system S, then the output which we obtain y 3 should be y 1 plus y 2.

Now, homogeneity says that if we have x 1 comma u 1 applied to the dynamical system S and we obtain the output y 1. Now, if we take any constant real constant, let us say alpha and multiply this x 1 and u 1 to the same dynamical system, then the output which we obtain let us say y 2 should be equal to alpha of y 1.

Now, we combine these two properties, it gives us these definition. For any real constants alpha 1 and alpha 2, if we take this input let input state or let us say the state vector is x 3 and the input signal is u 3, then this is the output we would obtain as y 3. If the system does not satisfy this property, we say that the system is non-linear.

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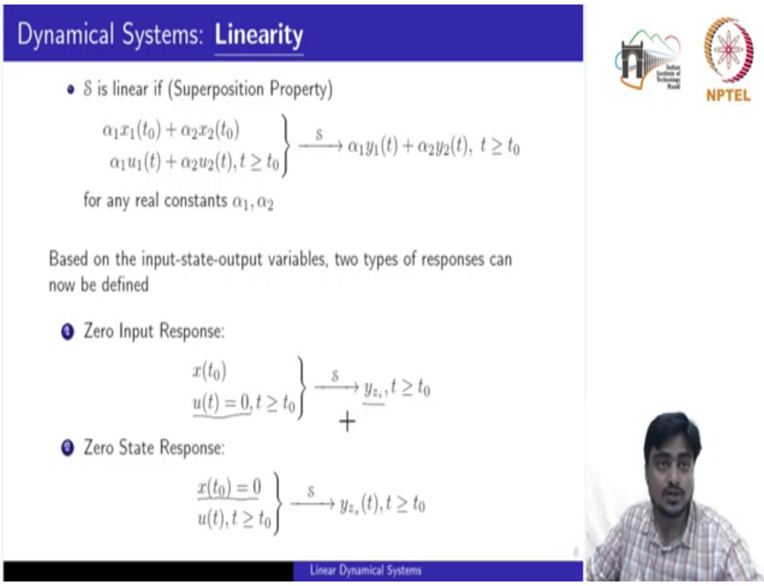

Now, based on the input state output variables, we did we will define two types of responses; first is the zero input response and second is the zero state response. So, as the name suggests in the zero input response we have u of we put u of t is equal to 0, for all t greater than equal to t naught. And for the zero state response we put the value at the state at time t naught equal to 0. And corresponding to these two responses, we denote y of z i that is the zero input response and the zero state response.

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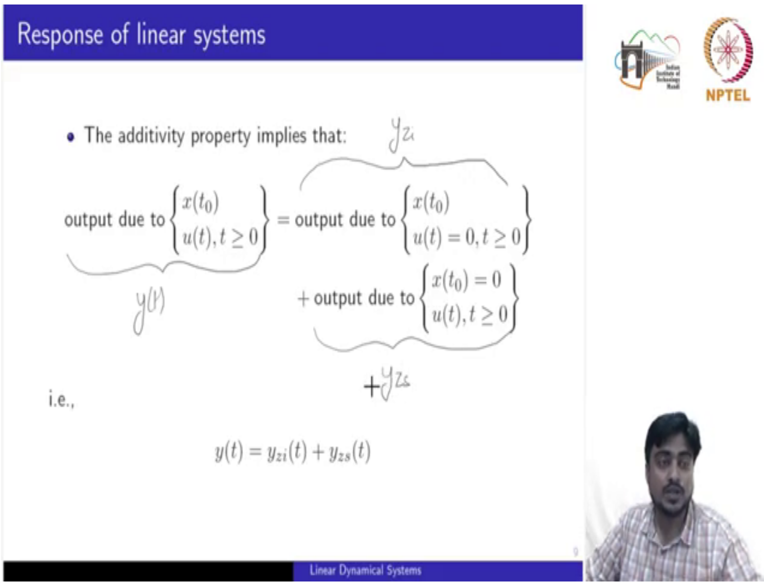

So, now if I combine these two responses owing to the superposition property then the complete output due to x t naught and u t which we define as the total response of the system. And, we take the output due to the zero input that is y z of i and the output due to when we put the state equal to 0, then the total response is basically the summation of y z i and y z s.

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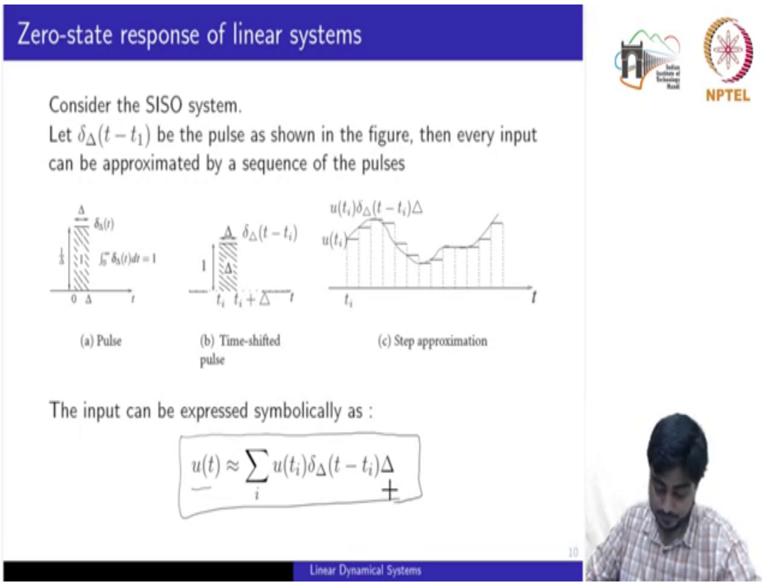

Now, first we will see the zero state response of the system and later we would combine both the responses into one through the state-space representation. Now, to proceed with let us consider the SISO system that is a single input single output system, both input-output variables are scalar variables. Now, we define an a pulse function which is of width delta and having a height 1 over delta.

So, if we if we take the property of the impulse function that the area under this pulse should be equal to 1; and it should be 0 other than the this width. In the figure b, we considered a time shifted version of this pulse where instead of starting with the t is equal to 0, we are starting with some t i by keeping the same width of delta.

But here notice that, we put the magnitude equal to 1 instead of 1 by delta. So, if we take the area inside this curve it is now delta. So, now, if we take any input signal which is a continuous-time signal and we multiply that signal with this pulse, and we need to know the value of the input signal at time t i, then this input variable is defined by this equation u of t i into delta of capital delta t minus t i into delta, and we know that this part is equal to 1 right.

So, we can approximate the input signal u of t by the summation of this value over all the i's right. So, this is expressed by this equation.

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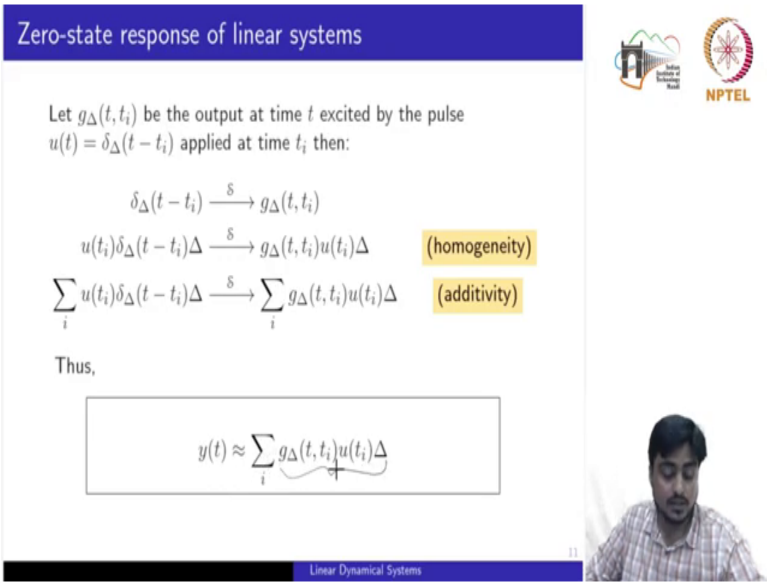

Now, suppose we define g of capital delta t comma t i is the output of the system at time t i which is excited by the pulse u t applied at time t i. So, pay attention to these two variable t comma t i, so it says that we are applying the input at t i and we are obtaining and we are observing the output at time t ok. So, this can be represented by using our earlier notation that when the input delta of capital delta is applied to the dynamical system, we obtained this response.

Now, if I multiply this input by u of t i and capital delta so seeing the homogeneity property, the output should also be multiplied by with this two variables. Now, introducing the additivity property, if I sum them up over all the i's, we obtain an approximate output of the system which is defined by this summation of this entire quantity.

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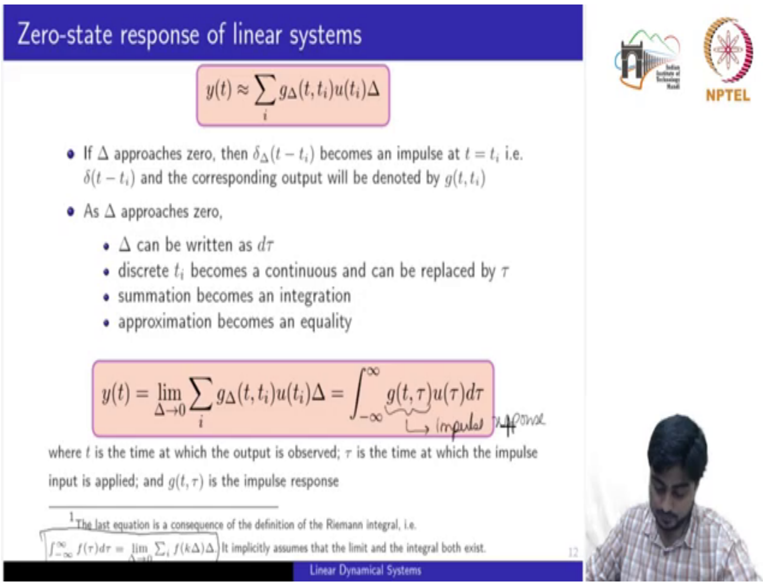

So, still let us pay bit of attention to the this equation. This equation is still an approximation of the output of the dynamical system. Now, if we want to have the equality, so we there are certain things which we need to take care of. Say, suppose, if with the capital delta approaches to 0, then we know that the pulse we introduced, in fact, would become an impulse function. So, we remove this capital delta and we represent this by a shifted version of the impulse function and the corresponding output will be denoted by g t comma t i, we also remove the capital delta from the subscript.

Further exploring the properties of the capital delta approaching towards to 0, that this capital delta can be written as d tau a small width. Now, all the discrete-time instance t i would become a continuous and can be replaced by variable tau. This summation would become an integral. And this approximation in that case would become an equality. So, basically this concept is from obtained from the Riemann integral going to this equation which defines the Riemann integral.

So, now, y of t is defined by an integral from minus infinity to infinity g of t comma tau u tau d tau. Now, it says that the input was applied at time t is equal to tau and we are obtaining the output at time t ok. And this g function is now defined as the impulse response of the system.

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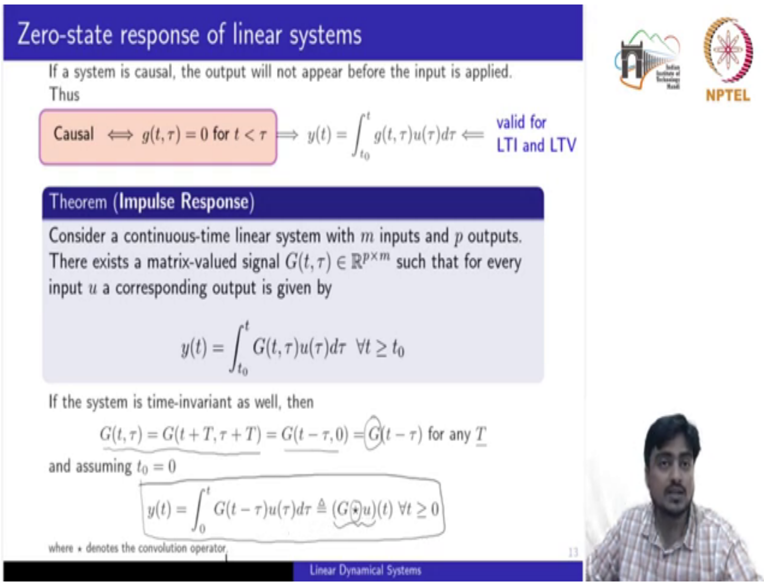

Now, if we apply the causality property on the impulse response of the system, we know that the output cannot appear before the input is applied. So, in that case, causality implies or its equivalent to gt comma tau is equal to 0 for all t greater than less than tau right. So, if we apply this into the last equation, now we start from t naught up to time t and the rest quantity would remain similar. So, instead of this minus infinity to infinity, we would have from t naught to t.

So, up to this point we have not introduced the notion of time variant and time invariant. So, the input-output description of the system given by this equation is in fact valid for both types of systems ok. We will further distinguish that how to do the analysis for the time invariant case and for the time variant case. So, we have this first result of the impulse response that consider a continuous-time linear system with m number of inputs and p number of outputs, there exist a matrix valued signal capital G t comma tau which is basically a matrix of dimension p cross n.

Now, note that here that whenever we are defining any function, we will be using the small letters. Now, if we are defining the relationship between the input and output by a function with a capital letter, it means that we are implicitly considering that that function is a matrix. So, such that for every input u, now here u is also vector a corresponding output is given by y of t by which is given by this equation, here u and y both are vector signals.

So, now, if we impose the time in variant property that if we shift the arguments of the function t comma tau by some constant capital T, then the behavior of the system should not change. So, this first equation basically says that that G t comma tau should be equal to G t plus T comma t plus capital tau plus capital T. Now, if since the system is time invariant, we this third equality is also satisfied.

Now, with the slight abuse of notation, we have used the same notation of the variable capital G by moving the second argument, because the second argument is basically 0. So, we can define the capital G as t comma tau for any term for any capital for any capital T. So, the important point to know here is that when we are dealing with the time invariant system, in that case we do not need two different arguments of the function capital G.

So, as and assuming that t naught is now we are starting with the t naught is equal to 0. We define the output y of t by this equation and the second equation is basically denotes the convolution between the capital G and the input signal u for all time t greater than equal to 0. And this star denotes the convolution operator. So, both of the representations are valid; we can use either of them.

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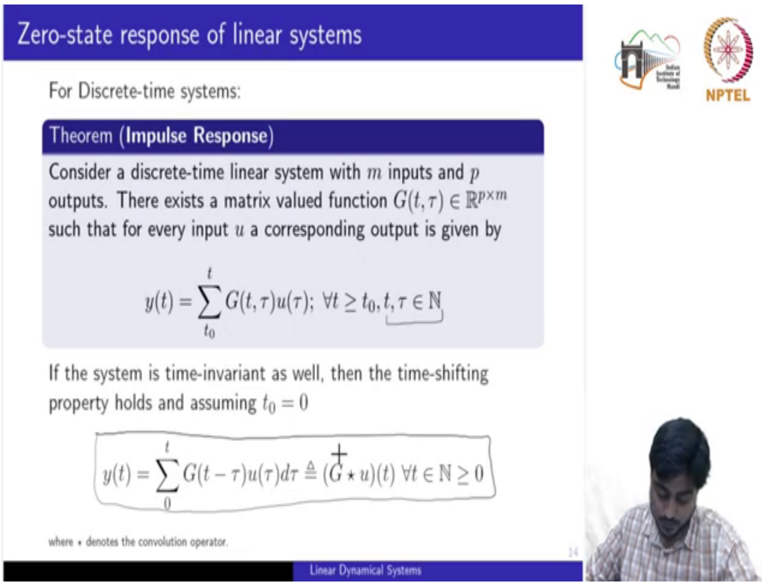

Now, going towards the discrete-time system, the similar theorem applies in the discrete-time systems as well that if the consider a discrete-time linear system with m inputs and p outputs. There exist a matrix valued function G comma tau of having the dimension p cross m, such that for every input u a corresponding output is given by y of t. Now, here you would notice that since we are dealing with the discrete-time system, we are having the summation from t naught to capital T. And we define both t and tau both are integers ok.

Now, if the system is time invariant as well, then the time shifting property holds as we had seen in the earlier slide. And assuming t naught is equal to 0, this is the response of the system when we assume the state is equal to 0 ok. Now, here we see two versions are, in fact, two equations are computing the output of the system in the continuous-time domain and in the discrete-time domain. In both the representations, we need to solve either the summation or the integral which might be a bit time consuming. So, in order to obtain the response of the system, we deal in the frequency domain.

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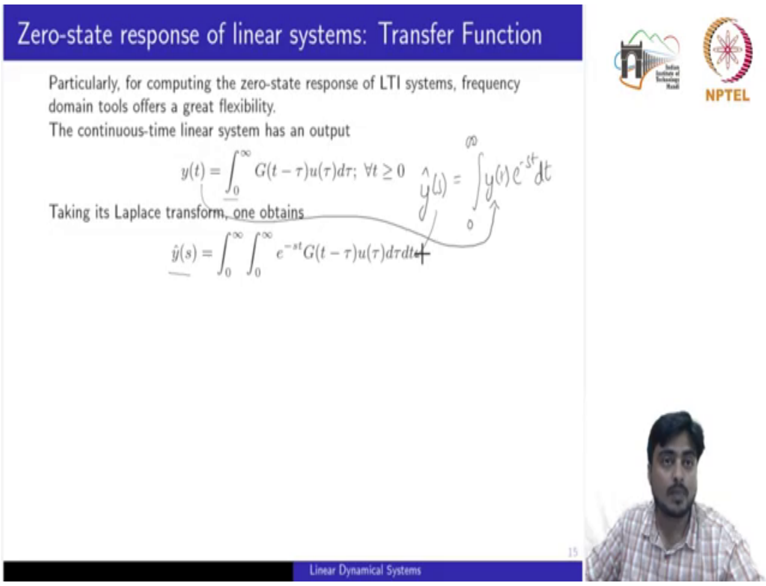

So, let us see that how what is the equivalence between the time domain representation and the frequency domain representation. So, let us see the first equation which is the basic equation what we had seen from the previous result. Now, if I take the Laplace transform of this part, we denote y hat of s. So, whenever we are using the frequency domain transformation of any time domain signal, we will introduce a hat over the same variable which is similar to what has been done here y hat of s.

So, if we take this integral 0 to infinity and 0 to infinity of this one, hence this given by e to the power minus s t G t comma minus tau u tau d tau and dt. So, if I write explicitly here we would have y hat of s and by 0 to infinity y of t e to the power minus st dt. So, we just put y t from here to here and then we obtain basically this one ok.

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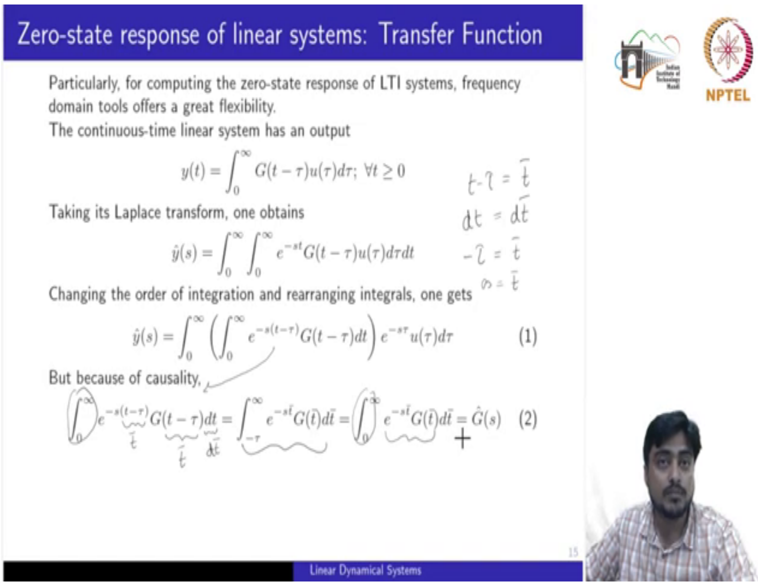

Now, if we change the order of the integration and rearrange the integrals by changing the order of the integration is we will put this order of integration, we interchange this d tau and dt. So, we club these terms e to the power minus s t minus tau. So, here in the above equation, we had e to the power minus s t. Since we have introduced e to the power s tau here, we need to cancel it out by e to the power minus s tau ok. So, we club these two terms by this one, and these two terms are clubbed into one.

Now, if we so of we see or if we pay a bit of close attention to the inner integral that this define basically the Laplace transform, but before the Laplace transform we need to see that how the causality property is playing its rule to finally say that is the Laplace transform. So, but because of the causality let us see we have this the same integral here.

Now, if we in this equation if we replace t minus tau by another variable t bar right, now this dt if we take the derivative of this one, it would be dt since tau is not a function of capital T, it would be 0, and here we would be having dt bar. So, dt would be replaced by dt bar, and t minus tau is replaced by t bar.

So, this t minus tau is t bar, and this also t bar, and dt is dt bar, this is what written in the second and the middle in this row. Now, if we see the limits if I put t is equal to 0, if we check the lower limit, if I put t is equal to 0, we would be having minus tau is tau bar right. And if I put the infinity, it would be infinity. So, now, this limit from 0 to infinity would change through minus tau to infinity if we are doing this integral over t bar right.

Now, since we know that my our system is causal, we cannot have this lower limit below 0. So, it has to start from 0, otherwise the system would not be causal. So, we start with 0 to infinity. So, this integral and this integral would now remain the same. And, this part can now be written as the Laplace transform of the impulse function which was defined in that time domain.

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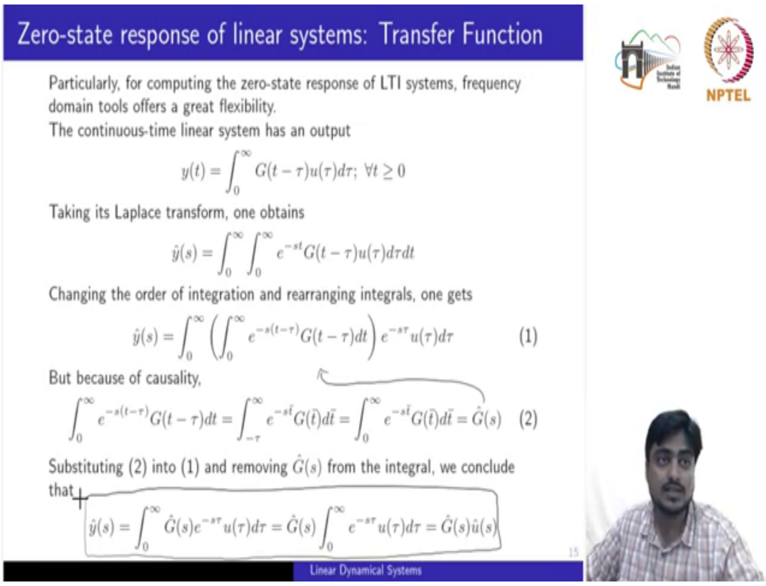

Now, substituting 2 into 1, and removing G hat of s because this so this G hat of s should go here. Now notice that we are taking the integral over tau and G hat of s would become a constant for this integral. So, we can take this outside and the inner integral it is basically the Laplace transform of that u hat.

So, our equation the final equation is that y hat of s in the frequency domain is given by the multiplication of the Laplace transform of the impulse function and the Laplace transform of the input signal. So, the convolution in the time domain is transformed into merely a multiplication of the two functions.

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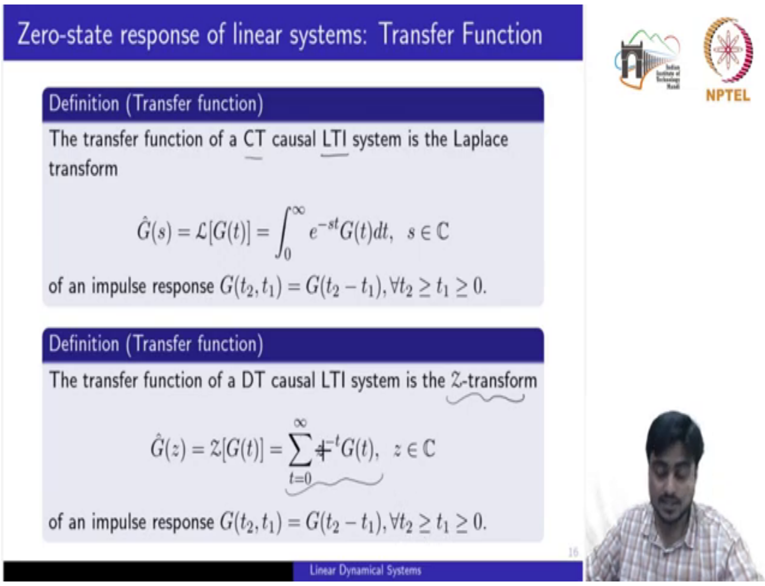

So, here we have these two definitions in the continuous-time domain and in the discrete-time domain that the transfer function of a continuous-time, this is a continuous-time causal linear time invariant system is the Laplace transform defined by this, where s is basically a complex number of an impulse response G t 2 comma t 1 and can be replaced by t 2 minus t 1 what we had seen in the earlier slides for all t 2 greater than equal to t 1 and greater than equal to 0.

Similarly, in the discrete-time domain, we would be taking the Z-transform and the transform function is defined by G hat of z by this summation, and where t 1, t 2 and t is basically the integer and z again is a complex number.

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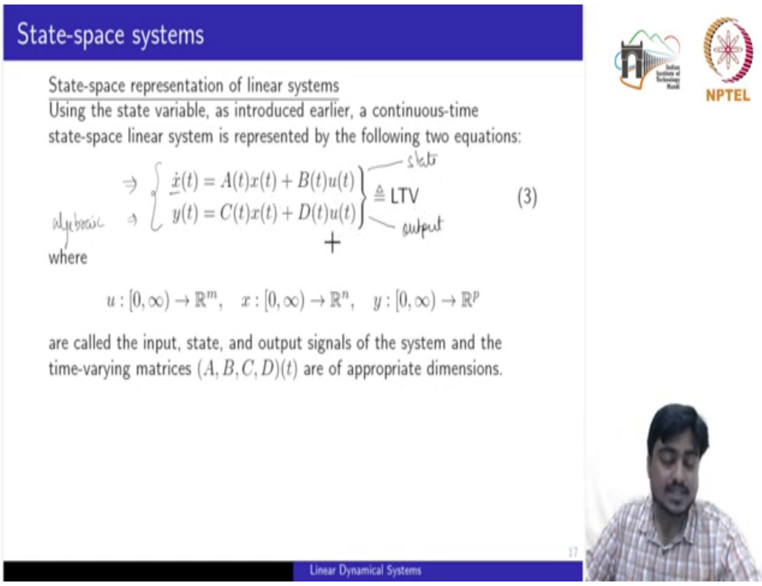

So, now introducing the state-space representation of a linear systems, the idea behind introducing the state-space representation is that whenever we are defining the output of the system, considering only the inputs and the outputs, then we define the zero state response. Now, if we need to compute a unique response combining the state and the output, we need some representation.

The earlier description we saw only in terms of the signals x, u and y. So, this state-space representation, we would be using for the linear systems that we introduce this x which is our state variable. So, by x dot is equal to A of t x of t plus B of t u of t. Now, the interesting thing of this state-space representation is that given any higher order differential equation in terms of u and y, you can parameterized that higher order differential equation into this first order differential equation by introducing the notion of the state x right.

So, the first equation, we call the state equation; the second equation we call the output equation. And note that here that all the dynamics are embedded into the state equation because of this derivative part, and the output equation is basically an algebraic equation, because here we do not have any derivative or integrals. So, this representation, we say is for the time varying system because all these parameters A, B, C, D are the functions of time. Now, if all these four parameters are constants, then we defined that system as an LTI system.

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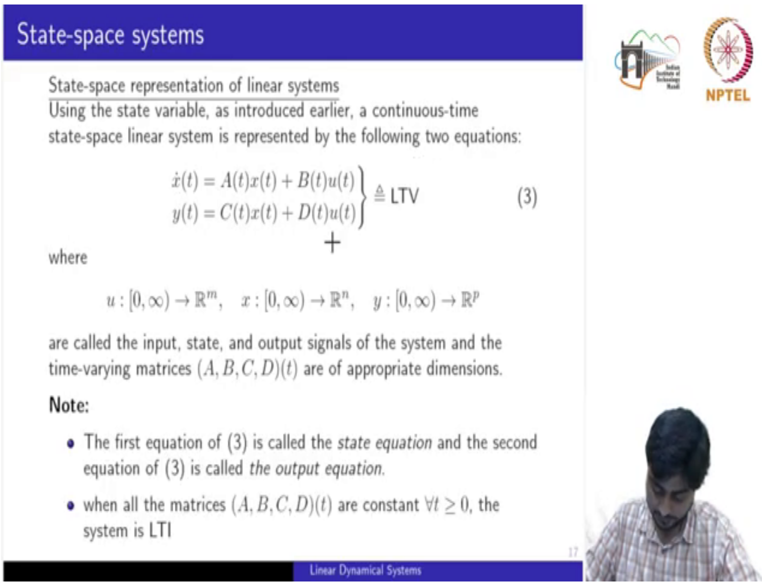

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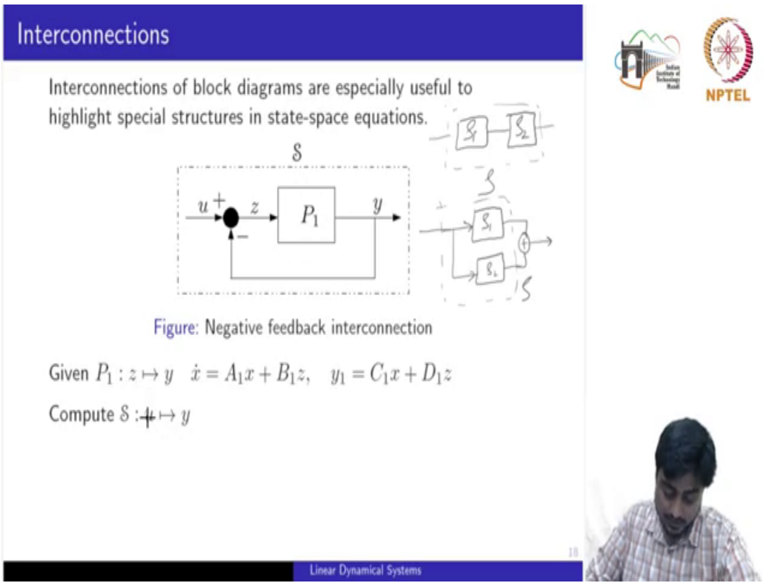

Let us see so what flexibility does state state-space representation offers towards. Now, let us say we have we could have one system or two system which might be interconnecting. Let us say we have this dynamical system S 1 either there could be an inter connection with the another dynamical system S 2 in this way which we define the cascaded one. So, if I club then S 1 and S 2 this would give me the overall system.

Another one we could have in as a parallel combination of S 1 and S 2 having the same input, then we could also write the state-space representation of the entire system. Here we would see the feedback system where we define the dynamical system by P 1 having an input-output variable as z and y. And this y is fed back to the plan p, P 1 right. Now, given this P 1 which is basically a mapping from z to y denoted by this state-space representation where we are assumed that A 1, B 1, C 1, D 1 all are constants. Now, we need to compute that the S, which is the mapping from u to y.

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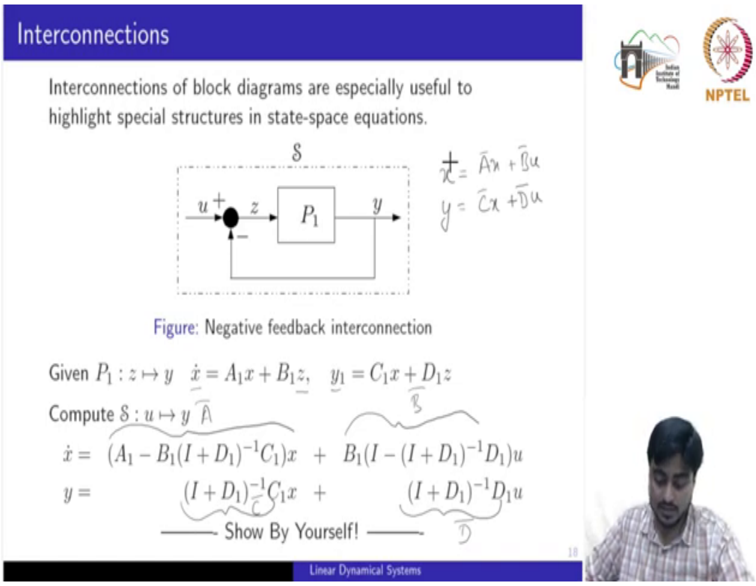

So, solving this state-space representation, we finally obtain this state-space representation of the entire system. Now, we would not be discussing in detail that how we have obtained these. So, you could define that this as A bar, this as B bar, let us say D bar and C bar. So, this you can write in a compact way as x dot is equal to A bar x plus B bar u, and y is equal to C bar x plus D bar u. So, note that here that when we have represented the P 1, the input is z one, but the state remains the same and the output we have the y 1 ok.

So, here is a mine at (Refer Time: 29:45), it should be basically y because we have denoted by y here. Now, when we compute the mapping from u to y, the input of the system is replaced by u from z, and all these represent all these matrices are now changed to these matrices by keeping the same output variable and the same state variable ok. So, this you can show by yourself that if we have any feedback system, then how to compute the overall state-space representation from u to y.

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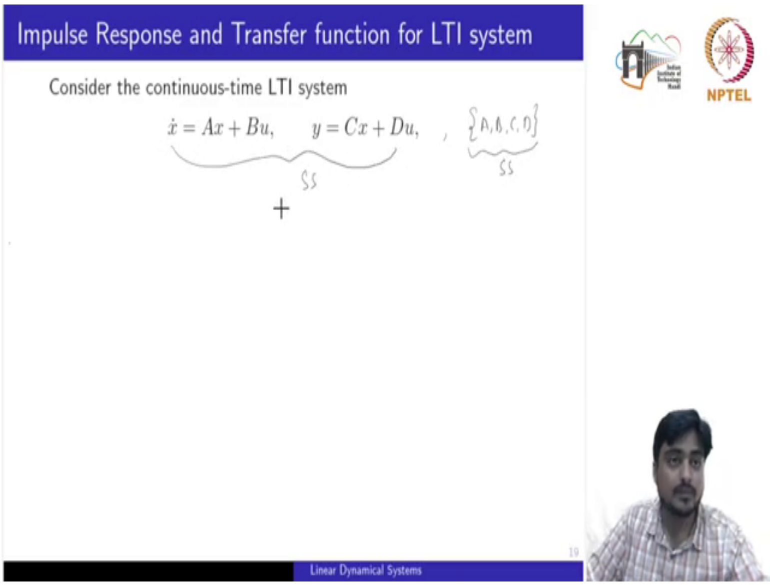

Now, introducing the impulse response and the transfer function for LTI system, what we are basically aiming at to see the relationship between the input-output description what we had introduced earlier and that state-space system we just introduced now. So, consider that continuous-time LTI system which is defined by A, B, C, D matrices.

So, there are two ways of representing a continuous-time LTI system, either we write these two complete equations or we write just that the A, B, C, D pair ok. So, this is also a state-space system, and this is also a state-space system.

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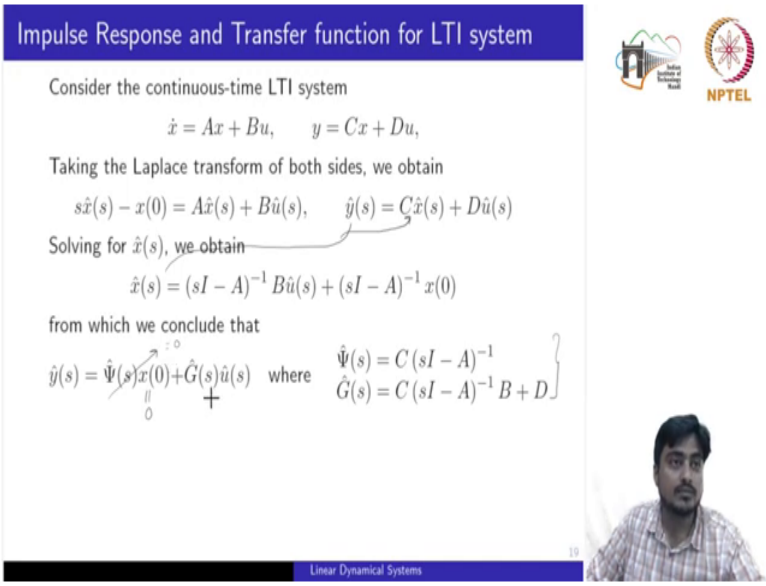

Now, if we take the Laplace transform of these two equations, we obtain this x x hat of x minus x of 0 which is the initial condition. If the initial condition is 0 at time t is equal to 0, then this part would be 0 right. This we would investigate later. And similarly the output equation because this may be a algebraic equation. Now, if we solve for x of hat of s from this equation, we obtain x hat of s we have s I minus A. If we take this A, here you obtain this s I minus A, and that inverse B u hat plus s I minus A inverse into x naught.

Now, the next step is to put this x hat of s here from which we conclude that if I put this x hat of s into the output equation, we obtain this representation which is y hat of s is equal to psi hat of s x naught plus G hat of s into u hat of s, where phi hat or psi hat, and G hat are defined by these two equations. So, then the interesting point to note here is that if we have x of 0 is equal to 0. So, this part would be equal to 0. And we obtain the same equation what we had obtained earlier before introducing the state-space representation right.

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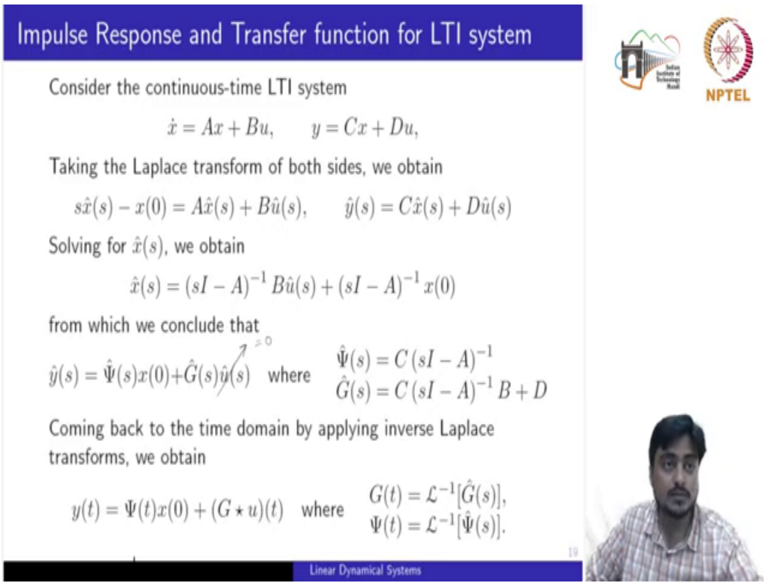

Now, if we have the u hat of s equal to 0 which implies that the zero input response, then the only response we are having because of the initial condition x naught. So, basically it is a linear combination of the responses due to the initial condition plus the input to the system. So, if we go back into the time domain by taking the Laplace inverse, we obtain the y of t is equal to psi t x naught plus G convolution with u into t, where we G of t and psi of t are basically the Laplace inverse of G hat and psi hat.

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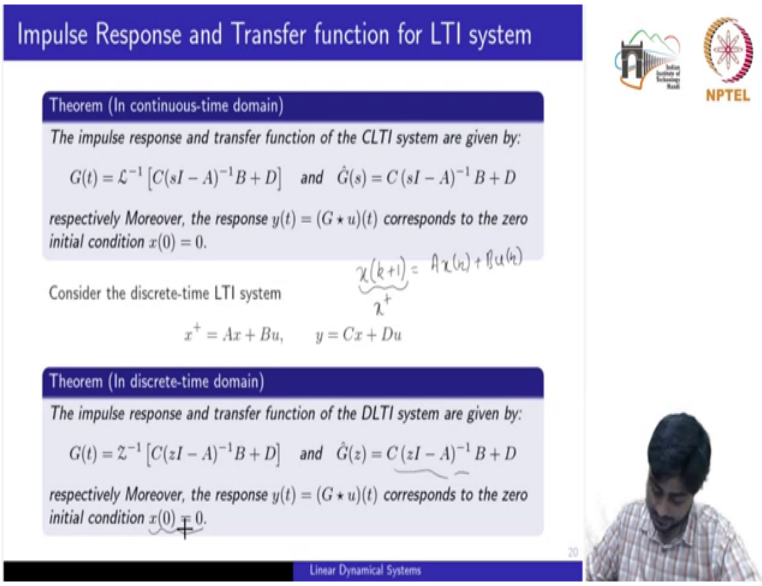

So, we have our next result in the continuous same domain that the impulse response and transfer function of the continuous time in the continuous-time linear time in variant system which are given by G of t as the Laplace inverse of A, B, C, D matrices this combination of A, B, C, D matrices and G hat of s is given by this respectively. More over the response y t it is equal to convolution of g u corresponds to the zero initial condition what we had seen in the last slide ok.

Now, consider the discrete-time LTI system which we are denoting by x plus by this x plus B mean to say that x k plus 1 is equal to A x k plus B u k. So, here we are using the short hand rotation that x k plus one is denoted by x plus. So, that if the important difference between the state-space representation of the continuous-time system and the discrete-time system is that now both the systems have become the algebraic equation, the state equation and the output equation.

So, in the discrete-time domain, the next results peaks, the impulse response and transfer function of the discrete LTI system are given by the Z-transform of the combination of the A, B, C, D matrices and the G hat of z is the defined by this part. More over the response the convolution between the G and u corresponds to the zero initial condition which is x 0 of g should be equal to 0.

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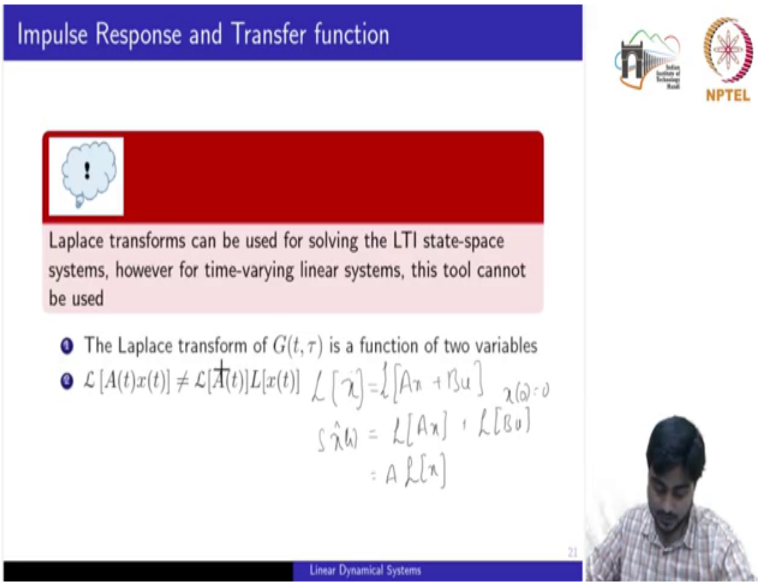

Now, here the idea behind introducing the Laplace transform is that instead of computing the convolution or the integral in the continuous-time domain and the summation in the district time domain, we dealt in the frequency domain by using the transformers either by using the Laplace or by using the Z-transform. So, but we cannot use these two frequency domain tools for the linear time varying systems.

Why, first of all that the Laplace transform of g t comma tau is now a function of two variables; for the LTI system instead of having these two variables, we use only one argument which is t minus tau right. Now, here we would be taking the Laplace transform of which is the function of two variables and now the second problem is that the Laplace of the multiplication of A t into x t is not equal to Laplace of A t and Laplace of x t.

Pay attention to the LTI case. When we have this x tau is equal to A x plus B u, where A and B matrices are the constant mattresses suppose we have the zero initial condition that is to say at 0 is equal to 0 ok. Now, if I take the Laplace transform of both side, then this I could write x hat of s and this part let us denote for the movement as A into x plus Laplace of B into u ok. Being A as a constant matrix, I could take it outside right which is not the case here for the time-invariant case.

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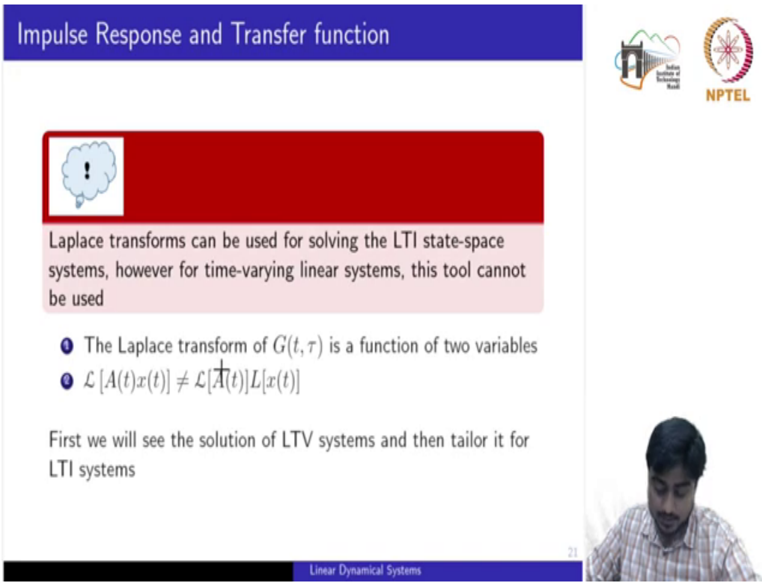

So, we cannot take the Laplace transform here. Because of these two basic problems we cannot use this Laplace transform or the Z-transform for the LTI case. So, in the next lecture, we would see the how to compute the solution of the LTV system, because for the LTI system we could use the frequency domain tools likewise we had seen. So, first we will see a pure solution in that time domain for the linear time varying systems and then we will tailor that solution for the LTI case.