Nonlinear System Analysis
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Lecture - 09
Existence and Uniqueness Theorem of ODE Part-02

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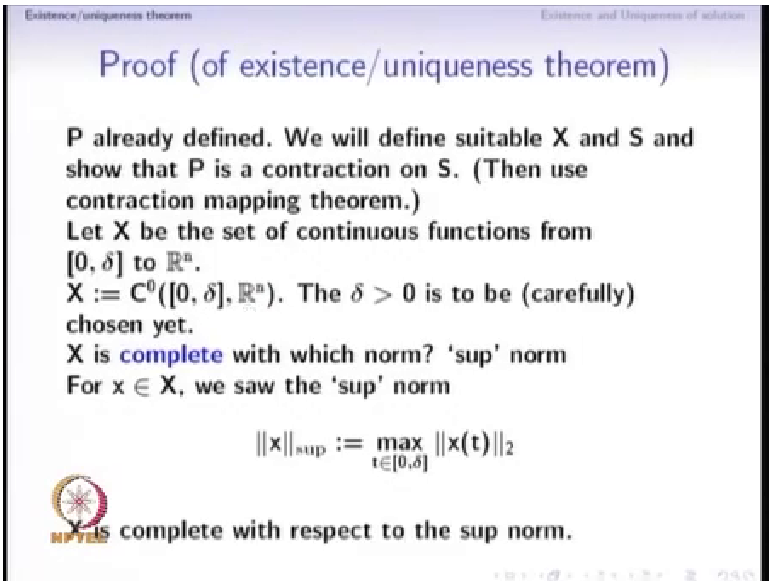

So, how do we use a contraction mapping theorem for the proof of existence and uniqueness? We already defined the operator P, we will now define a suitable X and S and the sub set S and we will show that this operator P we already defined is a contraction on S and then we will use the contraction mapping theorem. So, what is this X? So, X we will define is a set of all continuous functions from this interval 0 to delta to R n yeah.

So, the notation for X is C 0 from this domain to this co domain R n, this 0 means that it is required to be just continuous. It could be differentiable twice differentiable that is an extra property, but we are asking for all functions that are at least continuous and hence this 0 appears here. So, over what interval it is defined? From 0 to delta, the time duration that delta is to be carefully chosen yet. Now, we can ask the question is X complete with respect to some norm, after all for the contraction mapping theorem we require a Banach space X. So, with which norm is X complete?

So, we already saw that for a point X in capital X we saw the sup norm, for this space of functions for this space of continuous functions we define the sup norm as the maximum as t varies in the interval 0 to delta of the Euclidean norm of x of t at any time t, x of t is a vector in R n and we can take the conventional 2 norm, the Euclidean norm and this Euclidean norm itself is a function of time and we will see what is the maximum of that norm function as t varies from 0 to delta and that is called the sup norm. It is also called the max norm. So, we already saw that with respect to this sup norm this space of continuous functions on this interval to R n is a complete normed space.

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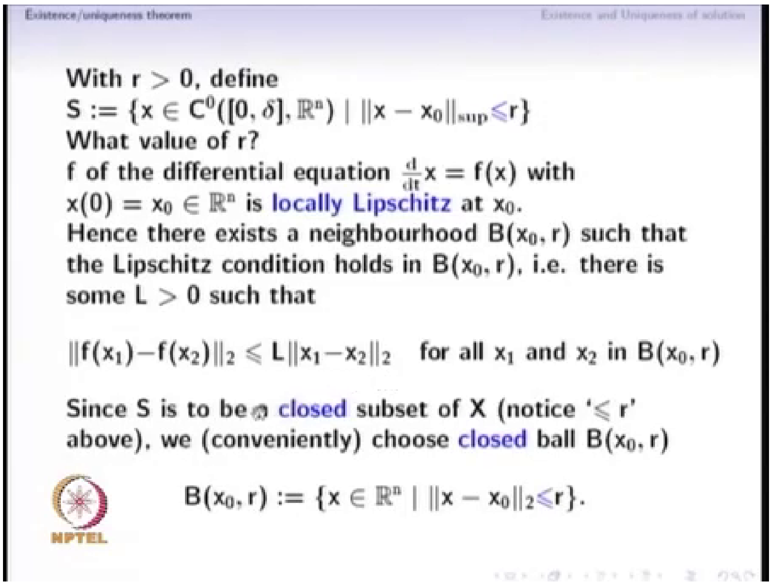

The next important property was the next important requirement was to define the closed subset S. So, we take some R greater than 0 and we define the set S of all those continuous functions in this particular set which satisfy the property that x minus x naught yeah x naught here is actually just a vector, but we also think of it as a function. We will see this in more detail, but the distance from this x naught the supremum of the distance of this as t various over the interval 0 to delta is at most r. So, we take all those continuous functions which satisfy the sup norm condition and we pick these functions and put them into the set S.

How do we choose the value of r for this definition of S? So, we have the differential equation f in the differential equation d by d t x is equal to f of x. We already are given that f is locally Lipschitz at the point x naught and what is the significance of x naught? x at time t equal to 0 is equal to x naught. So, because it is locally Lipschitz, we know there exists a neighbourhood B, the ball B centred at x naught and of distance and of radius equal to R. This closed ball we will very soon define it to be a closed ball.

We know that because f is locally Lipschitz at point x naught there exists such a ball such that the Lipschitz condition holds inside this ball to say that it the Lipschitz condition holds means that for all x 1 and x 2 inside this ball, these inequalities satisfied yeah. So, we pick this r from the locally Lipschitz property of the function f, also this S the way we have defined is a closed subset of x, it is a closed subset because of this in inequality being a non strict inequality yeah. So, notice the less than or equal to r above.

For the same reason, we will conveniently choose the ball B x naught comma r as the closed ball. So, B x naught comma r is defined to be set of all x, such that the distance from x naught is at most equal to r ok.

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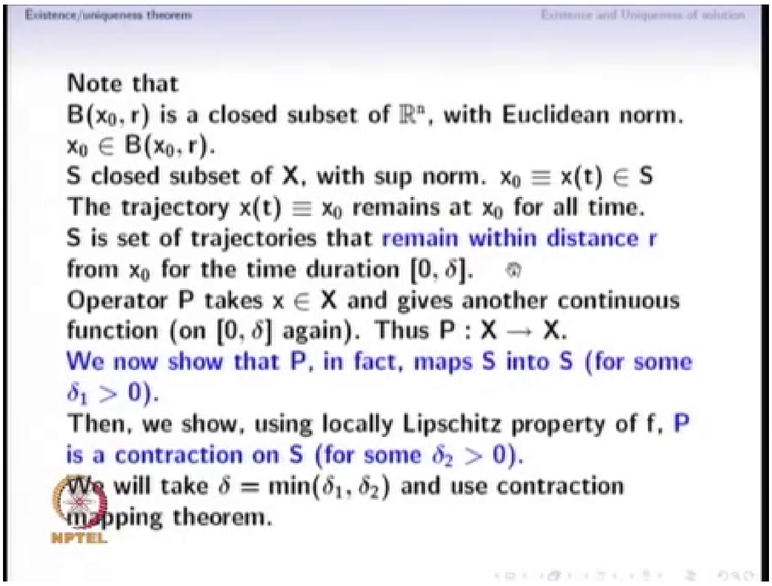

So, before we go further in the proof, let us quickly note that B x naught comma r is a closed subset of R n with the Euclidean norm and the point x naught is a element of this closed ball. It is infact the centre of this ball. On the other hand, S the subset S is a closed subset of X, the space of continuous functions over the interval 0 to delta and this space x has the sup norm yeah.

Because we are dealing with two types of norms here one a norm over R n the Euclidean norm and another a norm over X the sup norm because we are dealing with these two norms. It is very important to be careful about which norm at each place we use the norm function. So, for this subset S we have this particular function x of t which is always equal to x naught, always equal meaning as time t varies from 0 to delta x of t is the constant function, it is equal to x naught. So, this constant function is also an element of the set S. So, what is the meaning of that? The trajectory x of t is always equal to x naught, it remains at x naught for all time t.

So, what is the set S? S is S is a set of trajectories that remain within distance r from the point x naught for the time duration 0 to delta. So, what is the operator do, operator P do? It takes x small x and capital X and gives another function again on the interval 0 to delta. So, hence we see that P is a map from X to X. We now show that in fact, P maps S into S for some delta suitably small. Notice that we had delta as some number that was to be chosen yet.

So, for delta suitably small we will show that P maps not just X into X, but in fact, S into S. Then we will use the locally Lipschitz property of the function f to show that P is in fact, a contraction on S again for a sufficiently small delta. This we will call delta 2 greater than 0 and once we have these two conditions, delta 1 and delta 2, one which ensures that P maps S into S and another which ensures that P is a contraction on S, we will define the delta to be equal to the minimum of the 2 of delta 1 and delta 2 and since delta 1 delta 2 are both positive the minimum of the 2 will be a delta that needs the conditions in the theorem.

For this particular delta, we will use the contraction mapping theorem.

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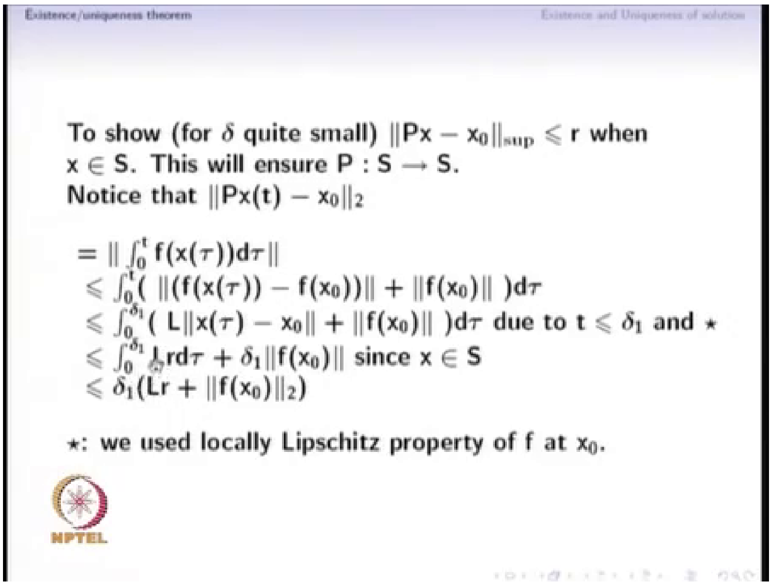

So, the first part of the statement was to show that for delta quite small, P x minus x naught sup norm is less than or equal to r, when S is in when x is in S. To show this particular inequality, we will imply that t takes an element x in capital S and gives you a function which is also in capital S. Why does it give a function again in capital S? Because the distance of P of x from x naught from the constant function x naught in the sup norm is at most r. If we show this will ensure that P is a map from S to S. So, in order to show this notice that P of x of t minus x naught in the two norm is equal to this.

So, once we take the norm function inside the integral sign it turns out that this right hand side will become larger. So, this inequality this norm of integral 0 to t P of x of tau d tau is less than or equal to integral 0 to t of this whole thing inside the brackets. So, notice that we have just subtracted and added f of x naught and while doing this particular quantity can increase because of the triangular inequality.

Now, what we will do? We will integrate not just from 0 to t, but 0 to delta 1, after all t is some number at most delta 1. So, if we integrate this positive quantity up to delta 1, it is only going to become larger and once we do this, we will also use the Lipschitz property of the function f and replace the first turn in the norm with x tau minus x naught times L yeah because f is locally Lipschitz at the point x naught this is satisfied for all x and x naught inside that ball, this other quantity we just leave as it is.

So, we have used the Lipschitz locally Lipschitz property of the function f. Since x of tau minus x naught is at most equal to R; why because the function x is inside S, hence this particular quantity is at most r. So, we have replaced x of tau minus x naught in the 2 norm by R that is a maximum distance it can be away from x naught and the second quantity because it is integral of a constant, we have removed f of x naught and replaced integral of d tau by delta 1.

Finally, we see that this particular quantity which we are integrating, this is also constant which is not varying as a function of tau and hence we called we equated that to delta 1 L r and after taking delta 1 common we obtain this expression.

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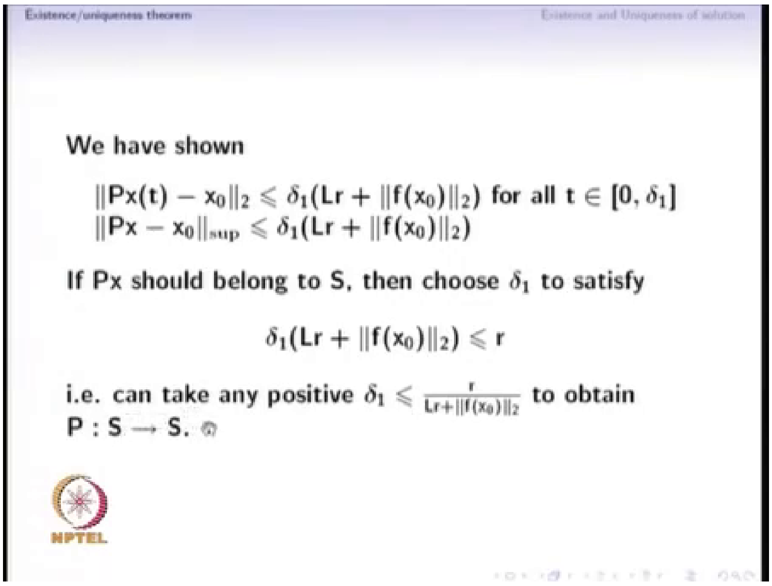

So, what have we shown? We have shown that the two norm of this particular function at any time t is bounded from above by this quantity and notice that on the right hand side there is no t. So, for all time t, the left hand side which depends on time t is bounded from above by this particular number which does not depend on t. So, in fact, if we take the supremum of the left hand side even the supremum will be bounded from above by the same quantity.

So, what does this show that, this particular P of x minus x naught in the sup norm is at most equal to this. So, now, we will choose delta 1 such that P of x belongs to S. If P of x should belongs to S then choose delta 1 to satisfy delta 1 times L r plus f of x naught 2 norm is at most equal to r. If we choose this delta 1 such that this is satisfied, then we see that P of x minus x naught in the sup norm is bounded from above by r and hence P of x goes into S.

So, we can take any positive delta 1 that is less than or equal to r times r divided by this quantity and we will then get the P maps S into S. So, we can take delta 1 equal to this.

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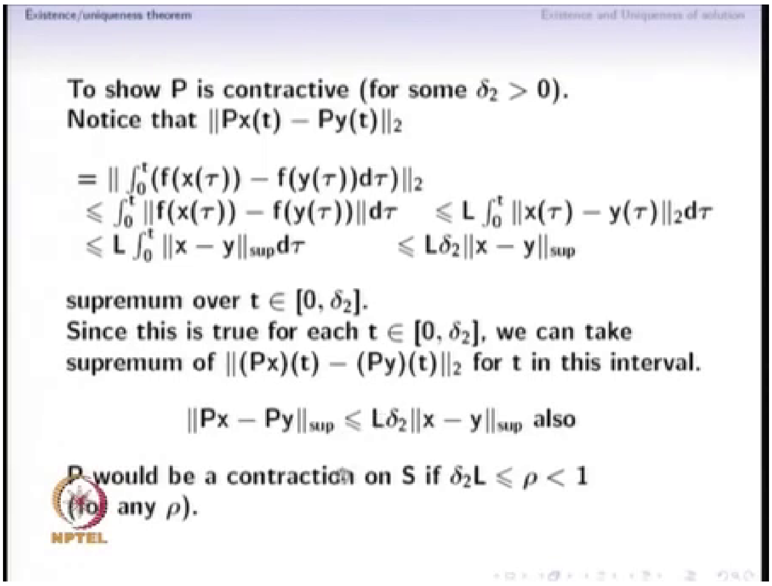

The next important step was to show that P is a contraction on S. So, for some delta 2 greater than 0 which we will carefully choose now we will show that P is contractive. So, for this purpose notice that P of x t minus P of y t in 2 norm is equal to the norm of this. From the definition of the operator P, we see that we obtain this and when we take the norm inside the integral sign then we get that this is at most equal to this.

By using the locally Lipschitz property of the function f inside the ball B of x naught comma r, we see that this quantity is bounded from above by this after taking the L outside this integral sign. Moreover this particular quantity we have written here is at most equal to this. Why because x and y both at any time t, we can take the difference between them in the 2 norm and integrate them, but instead of taking at anytime tau. We could also look at the maximum difference between them and this maximum difference is only going to be larger and hence we have obtained that this particular inequality is less than or equal to this particular quantity. By replacing the sup norm here, by replacing the two norm there with the sup norm here, this quantity can only become larger and hence this inequality less than or equal to.

Finally, this quantity which we are integrating it is over the interval 0 to t, but we could go ahead and integrate up to delta 2 this quantity because it is a norm, it cannot be negative and when we integrate further instead of only up to time t, but up to delta 2 then we see that we get L times delta 2 times sup norm of x minus y. So, here the supremum is being taken as t varies from 0 to delta 2 and here also the sup norm was being taken as t varies from 0 to delta 2.

So, what have you obtained? We have obtained that P of x of t minus P of y at time t the difference norm of that the 2 norm of that is bounded from above by some number by some quantity that is independent of time t. And this is true for each time t in the interval 0 to delta 2 and hence we can take the supremum of this quantity. Even the supremum will be bounded from above by the same number L times delta 2 times sup norm of x minus y. So, finally, we have obtained this inequality sup norm of P x minus P y is at most equal to L times delta 2 times sup norm of x minus y.

So, this should give us a hint as to how to choose delta 2 so that P is a contraction on S that was the objective of doing this inequality. So, when would this be a contraction? It would be a contraction if this particular quantity L times delta 2 is strictly less than 1. So, if you set delta 2 times L equal to some number rho and that number rho is strictly less than one then we will obtain that P is a contraction on S.

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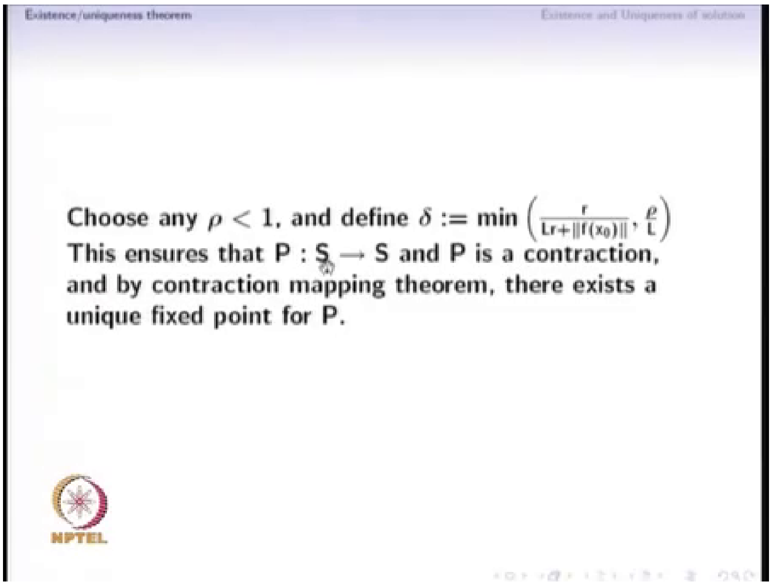

So, finally, we do the do as follows. Choose any rho strictly less than 1 and define delta to be the minimum of these 2 quantities. So, notice that this we had called as delta 1, the second one we had called as delta 2 and the minimum of these 2 quantities when we take that as delta it will ensure both. It will ensure that P is a map from S to S and it will ensure that t is a contraction. And once these two are guaranteed by the contraction mapping theorem we know that there exists a fixed point in S for the operator P and moreover there exists a unique fix point for this operator P, unique fix point inside the subset S.

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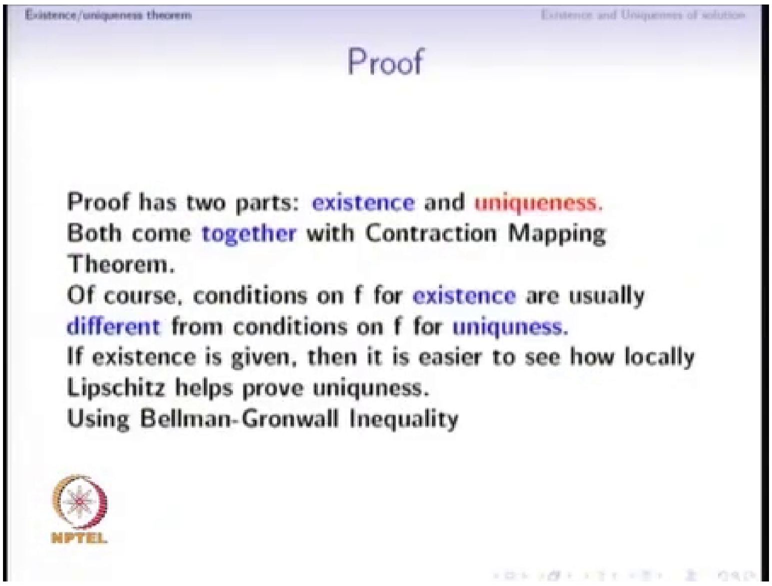

So, this proof has 2 parts. So, this completes a proof, just a small discussion about the proof it has 2 parts; one about the existence and one about uniqueness. So, notice that both come together with the contraction mapping theorem. The contraction mapping principle assures us both the existence and uniqueness. But of course, in general the conditions on f for existence of a solution to the differential equation are different from conditions on f for uniqueness of the solution to the differential equation, these conditions are usually different.

And suppose the existence is given, suppose due to some particular property on f; it turns out that we have a solution to the differential equation.